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On two-dimensional planar compacta not homotopically equivalent to any one-dimensional compactum

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Abstract

The paper provides examples of planar "homotopically two-dimensional" compacta, (i.e., of compact subsets of the plane that are not homotopy equivalent to any one-dimensional set) that have different additional properties than the first such constructed examples (amongst them cell-like, trivial π_1, and "everywhere" homotopically two-dimensional). It also points out that open subsets of the plane are never homotopically two-dimensional and that some homotopically two-dimensional sets cannot be in such a way decomposed into homotopically at most one-dimensional sets that the Mayer–Vietoris Theorem could be straightforwardly applied.

MSC: primary 54F15, 55M10; secondary 54D05

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1. Introduction

The well-known example of Barratt and Milnor [1] shows that there exists a Peano continuum X in $\mathbb{R}^3$ whose singular homology groups with integer coefficients $H_n(X)$ are nontrivial for every $n > 1$.

Planar sets behave more naturally with respect to their algebraic topology: By [13,6] they are all aspherical, and by [14] also acyclic *in higher dimensions* with respect to singular homology, i.e., $H_n(M)$ of any planar set M, $M \subset \mathbb{R}^2$, is trivial for all $n > 1$. The proof in [14] is very delicate and complicated. However, for certain kinds of spaces there exist simpler proofs, e.g., for the spaces which are homotopy equivalent to a 1-dimensional separable metric space [7]. Not all planar sets are homotopy equivalent to 1-dimensional spaces: Most recently Cannon and Conner [5] characterized when a discrete subset of the plane satisfies that its complement has this property, but the first (not codiscrete) examples with this property have already appeared before ([6, §5], [13, A.4.13]). The fundamental group of these spaces are uncountable.

This is one of the aspects in which we improve upon these results by providing the following three examples:

Brief description of our main examples

All these spaces are planar continua that are not homotopy equivalent to any one-dimensional space. In addition

Example 1 (see Fig. 1(a)) *shows that it is not necessary to have fundamental group to obtain this effect*. Indeed this space, which is constructed by wedging a null-sequence of two comb-spaces to the boundary of a disk, is path-connected, simply connected and cellular (cf. 3.1–3.2).

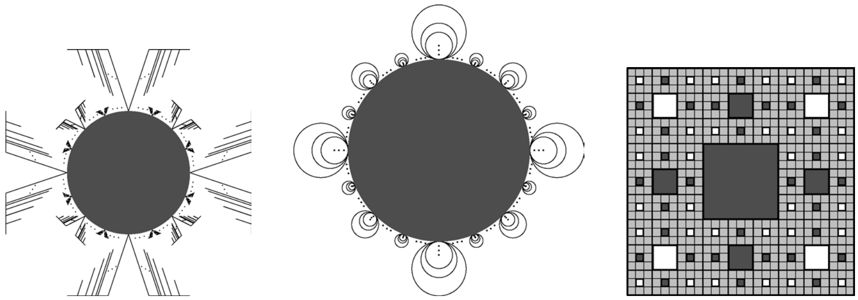
Example 2 (see Fig. 1(b)) *shows that this effect can also be achieved for a Peano continuum that is a disjoint union of a disk and open intervals*. It is built by similarly wedging countably many Hawaiian Earrings to the boundary of a disk (cf. 3.1–3.2).

Example 3 (see Fig. 1(c)) *shows that one can also construct planar Peano continua "no part" of which is homotopy equivalent to any one-dimensional set,* more precisely the continuum is everywhere homotopically 2-dimensional (cf. Definition 2.1(ii)). Here the construction is based on the Sierpiński Carpet, by filling (instead of removing) an appropriate subset of all holes (cf. 3.3–3.4).

More precise definitions of these examples and proofs of these claims will follow in Section 3. Section 4 will then be devoted to proving the following facts:

Proposition 1.1. *A subset of the plane that is not homotopy equivalent to a one-dimensional space can neither be*

(i) *simply connected and a Peano continuum* (*cf. Proposition* 4.1), *nor be*
(ii) *open* (*cf. Remark* 4.2).



(a) (b) (c)

Fig. 1. Pictures of our main examples.

Towards the end we will return to the question that was raised at the very beginning of this paper: We will point out that, although Examples 1 and 2 are not homotopy equivalent to any one-dimensional space, the proof of their higher acyclicity can be given in a simple way by reducing the situation to one-dimensional spaces. However, we will also point out that this way of reduction is not available for our Example 3.

2. Preliminaries

The (*Triangular*) *Comb space* is the following subspace of the plane:

$$\mathcal{C} := \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x \in \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right\},\ 0 \leqslant y \leqslant (1-x) \text{ or } x \in [0,1],\ y = 0\right\}.$$

The *Hawaiian earring* is the planar space defined as:

$$\mathcal{H} := \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2,\ n \in \mathbb{N}\right\}.$$

Definition 2.1.

(i) A space X is said to be *homotopically n-dimensional* (or to have *homotopy dimension* n, abbrev. hdim $X = n$), if it is homotopy equivalent to some n-dimensional space and is not homotopy equivalent to any $(n-1)$-dimensional space (cf. [3, p. 111]). Here by dimension of space we mean covering dimension.

(ii) We say that a space X is *everywhere homotopically n-dimensional* if for every open subset U of X and every id-homotopy $H : X \times I \to X$ which is *stable* on $X \setminus U$, i.e., $H(x,t) = x$ for every $x \in X \setminus U$ and for every t, the intersection $U \cap H(X, 1)$ is n-dimensional. By an *id-homotopy* we mean any mapping $H : X \times I \to X$ such that $H(x, 0) = x$ for all $x \in X$.

(iii) A subset X of the plane $\mathbb{R}^2$ is called *simply connected* if it is path-connected and its fundamental group is trivial, and it is called *cellular in* $\mathbb{R}^2$ if it is the intersection of a decreasing sequence of topological disks, i.e., $X = \bigcap_{i=1}^{\infty} D_i^2$, where $D_i^2 \subset \mathbb{R}^2$ are homeomorphic to the standard disk $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$

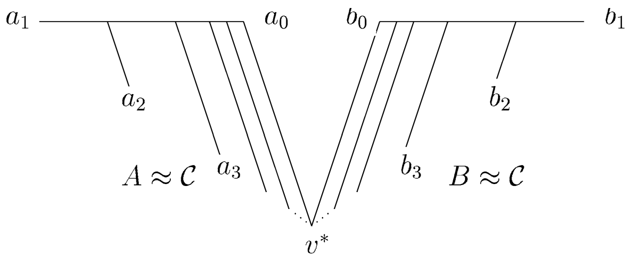


Fig. 2. A wedge of two comb spaces, as used in the construction of Example 1.

and $D^2_{i+1} \subset \mathrm{Int}(D^2_i)$. See [11], [8, §III.15] for the relevance of cellular sets and their generalization, cell-like sets.

Definition 2.2. A point x_0 of a space X is called a *homotopically fixed point* (*hf-point*) if, for any id-homotopy $H : X \times I \to X$ and for every $t \in I$, the point x_0 is a fixed point of the mapping $H(\cdot, t)$, i.e., $H(x_0, t) = x_0$, for all $t \in I$.

Lemma 2.3. *The wedge $\mathcal{C} \vee \mathcal{C}$, if constructed as in Fig.* 2, *is a space where the wedge-point is homotopically fixed.*

Proof. Take two copies of comb-spaces A and B, embedded into the plane and denoted as shown in Fig. 2, and call the resulting space X.

The point v^* is homotopically fixed in X. Indeed, suppose that it is not so. Then there exists an id-homotopy $H : X \times I \to X$ and a number $t_0 \in [0, 1]$ such that $H(v^*, t_0) = x^* \neq v^*$. Since the space X is path connected we can suppose that $x^* = b_0$ and that t_b is the minimal number such that

$$H(v^*, t_b) = b_0. \tag{$*$}$$

Let U and V be disjoint neighborhoods of b_0 and v^*, respectively, and suppose that $U \cap A = \emptyset$. Let $W = V \cap H^{-1}(\cdot, t_b)(U)$. This is an open nonempty set. Since v^* is the limit point of the a_n, there exists a number n_0 such that $a_n \in W \cap A$ if $n > n_0$. Consider the path $H(a_n, \cdot) : [0, t_b] \to X$ for some $n \geqslant n_0$. Since $H(a_n, 0) \in W$ and $H(a_n, t_b) \in U$, there exists $t_n \in [0, t_b]$ such that $H(a_n, t_n) = a_0$. Because I is sequentially compact and $a_n \to v^*$, there exists a number t_a, $t_a < t_b$ such that $H(v^*, t_a) = a_0$. Since the condition $H(v^*, t_a) = a_0 \neq v^*$ is similar to $(*)$, there exist $t'_b < t_a < t_b$ such that $H(v^*, t'_b) = b_0$. This contradicts the minimality of the number t_b. So v^* is a homotopically fixed point of the space X. $\square$

Remarks 2.4. The proofs of our claims that come with Examples 1–3 all use the following principles:

(i) They are all based on having one or more two-dimensional disks D^2, whose boundary is homotopically fixed at a dense set (and hence at all its points). This also was the main idea of the examples in [6, §5], [13, A.4.13]. If then X is one of our examples,

Y is 1-dimensional and $X \xrightarrow{f} Y \xrightarrow{g} X$ is a composition of a homotopy equivalence and its inverse, the properties of the one-dimensional Y make it impossible that $g \circ f|_{\partial D^2}$ would be the identity contradicting the homotopy fixedness of ∂D^2.

(ii) An argument why $g \circ f|_{\partial D^2} \neq \mathrm{id}$ was given for metric Y in [6, Theorem 5.2(1)] by quoting [4]. The condition "metric" is not necessary when instead working with Čech-cohomology: Since it is defined via the same coverings as covering dimension, the second Čech-cohomology groups of a one-dimensional space are trivial. Therefore, when combined with the natural inclusion i into and the retraction r onto D^2 of X, the map $r \circ g \circ f \circ i$ cannot induce the identity of $\check{H}^2(D^2, \partial D^2) \cong \mathbb{Z}$.

(iii) If $X \supset Z \ni z$ and z is homotopically fixed in Z, we have to ask whether it is also homotopically fixed in X. The following two observations (iv) and (v) suffice to deal with all cases that come up in discussing our examples.

(iv) *Let X be a topological space, A a retract of X and a_0 an hf-point of A, and let $r : X \to A$ be a retraction such that $r^{-1}(a_0) = a_0$. Then a_0 is an hf-point of X.* This follows, since an id-homotopy H of X that moves a_0 can via $r \circ H$ be turned into an id-homotopy of A with this property.

(v) *Let a metric space X be the union of two closed subspaces X_1 and X_2, let $X_1 \cap X_2$ be an AR and let $x_0 \in X_1$ be an hf-point of the path-connected nontrivial space X_1. Then x_0 is an hf-point of the space $X_1 \cup X_2$.* The proof is, similarly as for the preceding item, based on turning an id-homotopy H of X that moves x_0 to a into one of X_1 that moves x_0. This task is only not trivial, if $H(x_0, (0, 1]) \subset X_2 \setminus X_1$ and $x_0 \in X_1 \cap X_2$. Connect x_0 and a by an arc $A \subset H(x_0, [0, 1])$, and then find another nontrivial arc B that connects x_0 to some point b. If possible, choose B inside $X_1 \cap X_2$, but in any case inside X_1. As an AR, $X_1 \cap X_2$ is either a singleton, or it contains nontrivial arcs. The AR-property of $X_1 \cap X_2$ and of B gives that a homeomorphism between arcs, $f : A \to B$, $x_0 \mapsto x_0$, $a \mapsto b$ extends to a continuous mapping $g : X_2 \to X_1$ with $g|_{X_1 \cap X_2} = \mathrm{id}$. Therefore g can via the identity of X_1 be extended to a map $h : X \to X_1$, and $h \circ H$ is the desired homotopy that moves x_0 to b.

3. Discussion of our main examples

Construction 3.1. For constructing Examples 1 and 2 consider in the plane $\mathbb{R}^2$ the standard disk D^2 and a countable dense subset $M = \{m_i\}_{i=1}^{\infty}$ of the boundary of D^2. Attach to each point m_i a space X_m with an *hf*-point at this point such that the spaces do not intersect each other and the diameters of these spaces tend to zero, $\lim_{m\to\infty}(\mathrm{diam}\, X_m) = 0$. In case of Example 1 choose each X_m to be homeomorphic to $\mathcal{C} \vee \mathcal{C}$, in case of Example 2 $X_m \approx \mathcal{H}$ for each m. Use X to denote the resulting space.

Proof 3.2. The proof that $\mathrm{hdim}(X) = 2$ for both, Examples 1 and 2, follows 2.4, using 2.3 and 2.4(ii) as basis for getting the points on ∂D^2 homotopically fixed. Example 1 is cellular, as can be seen by explicitly constructing disks as required by Definition 2.1(iii) in the spirit of that one that we painted in grey in Fig. 4(b). The verification of the remaining claims is elementary. □

Construction 3.3. Example 3 (see Fig. 1(c)) is constructed according the following inductive process that is analogous to the construction of the Sierpiński Carpet: As for the Sierpiński Carpet, the construction is based on iteratively subdividing a square into nine congruent subsquares, one "central square" c, four "corner squares" a_i, while the remaining four squares will be denoted by b_i $(i = 1, \ldots, 4)$. The 9^n squares on n iteratively subdivision steps are bijectively associated to the words of length n over the alphabet $\{a_1, b_1, \ldots, a_4, b_4, c\}$, where the jth letter describes the relative position of the square in the jth subdivision step. The space $\mathcal{S}$ now, our Example 3, is defined as the complement of the interior regions of those squares that can be described by a word of the form "$\omega a_i c$", where ω is a word over the reduced alphabet $\{a_1, b_1, \ldots, a_4, b_4\}$. In Fig. 1(c) those areas which have such words of length at most three are white. In black we painted those areas which entirely belong to $\mathcal{S}$, since after three subdivisions it is already clear that no subregion can match the condition for not belonging to $\mathcal{S}$, while in grey we painted those regions which will split up into white and black areas on forthcoming subdivision steps.

Remark. Note that neither the open, nor the closed disk is everywhere homotopically two-dimensional in the sense of Definition 2.1(ii), since the condition is not satisfied, e.g., for $U = D^2$. In contrast to this we prove:

Proposition 3.4. *$\mathcal{S}$ is an everywhere homotopically 2-dimensional planar Peano continuum.*

Proof. This proof essentially rests on two observations:

(i) the fact that the black squares of Fig. 1(c) are dense in $\mathcal{S}$ after infinitely many iterates. This in particular ensures that all points of $\mathcal{S}$ have local dimension 2.

(ii) Conversely, the white squares satisfy an analogous density-property which, in particular, ensures that a dense subset of boundary points of each black square consists of singular points of subspaces of $\mathcal{S}$ that are homotopy equivalent to Hawaiian Earrings. Therefore all boundary points of all black squares will be homotopically fixed.

The square I_i^2 in Fig. 3(a) represents a black square and Fig. 3(b) represents the interior structure of each of the smaller squares in Fig. 3(a). Essentially Fig. 3 shows one way of how retractions can be defined that reduce the problem of showing the homotopic fixedness of the boundary of black squares to situations where Propositions 2.4(ii)–(v) can be applied.

Having observed that the black squares are dense with homotopically fixed boundaries, it follows that $H(\mathcal{S}, 1) = \mathcal{S}$ for every id-homotopy H; for the interior points of the black squares the proof is analogous as for Examples 1 and 2. Since this in particularly holds for those homotopies that, in the spirit of Definition 2.1(ii), fix the outside of some neighborhood, the proof of this proposition is complete. □

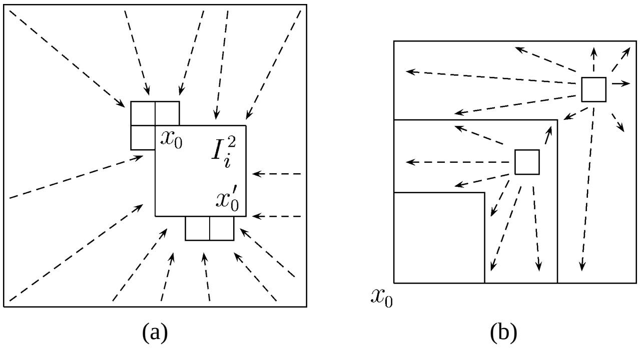


Fig. 3.

4. Proofs of additional propositions

The proof of Proposition 1.1(i) will follow from

Proposition 4.1. *Every simply connected planar Peano continuum is an AR space and therefore it is homotopy equivalent to a point.*

Proof. If the complement of a Peano space in $\mathbb{R}^2$ is not connected, then there exists a Jordan curve J in this space which is its retract, and therefore the space is not simply connected (see [10, Chapter 10, §61, II, Theorem 5]). Vice versa, if the complement of a continuum in $\mathbb{R}^2$ is connected, then it follows by the Alexander duality that this continuum is acyclic with respect to the Čech cohomology. By our assumption, the continuum X is simply connected, therefore its complement is connected and X acyclic. By [2, Chapter V, Theorem 13.1], every planar acyclic Peano continuum is an AR. So X is an AR and therefore homotopy equivalent to a point. □

Remark 4.2. Proposition 1.1(ii) seems to be folklore knowledge since Bing's time, including its generalization to triangulable open m-dimensional manifolds. On the other hand, we are not aware of a written version of this general claim apart from [12]. However, in the two-dimensional case that is of interest for us, we can provide the following simpler argument, and for that argument we are indebted to the referee:

Proof of Proposition 1.1(ii). The following construction is to be performed separately for each of the at most countable many connected components, therefore we may assume that our open set is a *region*, i.e., an open connected subset of $\mathbb{R}^2$. Each simply connected region in $\mathbb{R}^2$ is an open disk and therefore homotopy equivalent to a point. Any nonsimply connected region in $\mathbb{R}^2$ is homeomorphic to the complement of a compact 0-dimensional subset of $\mathbb{R}^2$ (see, e.g., [10, Chapter 10, 61, IV, Corollary 9]. Every 0-dimensional compact subset of $\mathbb{R}^2$ is tame [10, Chapter 10, 61, V, Theorem 4], i.e., there exists an autohomeomorphism of the plane which maps this 0-dimensional set to a compact subset $\mathcal{F}$ of $\mathbb{R}^1 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \subset \mathbb{R}^2$. So it is sufficient to prove that $\mathrm{hdim}(\mathbb{R}^2 \setminus \mathcal{F}) < 2$. The fundamental groups of such sets have been investigated by Eda [9].

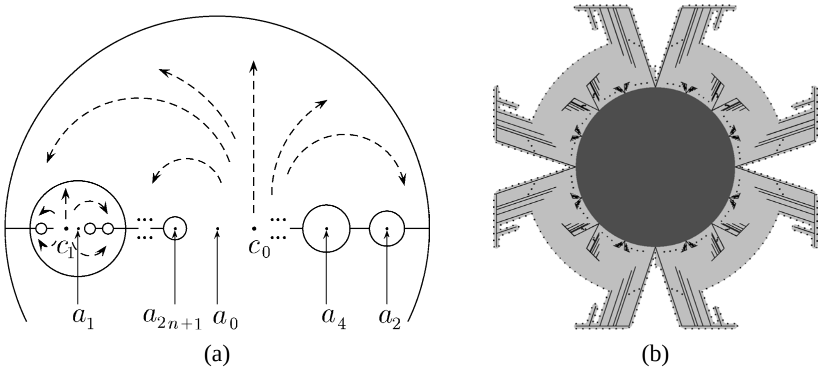


Fig. 4.

Let $\mathcal{F}$ be a compact zero-dimensional subset of $\mathbb{R}^1$. Since it is compact, there exist a_0 and ε_0 such that $(a_0 - \varepsilon_0, a_0 + \varepsilon_0) \supset \mathcal{F}$. Choose c_0 as (one of) the nearest point(s) of $\mathcal{F}$ with respect to a_0. Since $\mathcal{F}$ is zero-dimensional, it cannot contain an entire interval, hence $\{c_0\}$ already is a connected component of $\mathcal{F}$. For similar reasons it is always possible, if $a_0 = c_0$ and if $(c_0 - \varepsilon, c_0)$ and $(c_0, c_0 + \varepsilon)$ intersect for all $\varepsilon > 0$ with $\mathcal{F}$, to find two monotone sequences of closed disjoint intervals that have empty intersection with $\mathcal{F}$ and converge from different sides in such a way to c_0, that the complementary intervals all intersect with $\mathcal{F}$. Choose $(a_i)_{i \in \mathbb{N}}$ as the sequence of midpoints of these complementary intervals and ε_i as the corresponding sequence of radii. As a result of this construction, we have that $\bigcup_{i=1}^{\infty} U(a_i, \varepsilon_i)$ covers the entire $\mathcal{F} \setminus \{c_0\}$ while $a_0 - \varepsilon_0 < a_1 < a_3 < a_5 < \cdots < c_0 < \cdots < a_6 < a_4 < a_2 < a_0 + \varepsilon_0$. Let $\mathbb{R}^1 \supset \mathcal{L} := (a_0 - \varepsilon_0, a_1 - \varepsilon_1) \cup (a_2 + \varepsilon_2, a_0 + \varepsilon_0) \cup \bigcup_{j=1}^{\infty} (a_{2\cdot j-1} + \varepsilon_{2\cdot j-1}, a_{2\cdot j+1} - \varepsilon_{2\cdot j+1}) \cup (a_{2\cdot j+2} + \varepsilon_{2\cdot j+2}, a_{2\cdot j} - \varepsilon_{2\cdot j})$. Observe that by construction $\mathcal{L} \cup \bigcup_{i=0}^{\infty} \partial U(a_i, \varepsilon_i)$ has empty intersection with $\mathcal{F}$. Furthermore, obviously $\overline{U(a_0, \varepsilon_0)} - (\{c_0\} \cup \bigcup_{i=1}^{\infty} U(a_i, \varepsilon_i))$ retracts to this one-dimensional set (see Fig. 4(a)).

We then iteratively continue to build this one-dimensional set $\mathcal{L} \cup \bigcup_{i=0}^{\infty} \partial U(a_i, \varepsilon_i)$ into the interior of each of the disks $U(a_i, \varepsilon_i)$ by applying the same scheme of construction as before to the disk $U(a_0, \varepsilon_0)$. Finally, from combining the retraction constructed above with the analogous retractions inside $U(a_i, \varepsilon_i)$ $(i > 1)$, and from simultaneously retracting the outside of $U(a_0, \varepsilon_0)$ to its boundary, the desired result follows. Observe that the case, where any of the sets $(c_0, c_0 + \varepsilon) \cap \mathcal{F}$ or $(c_0 - \varepsilon, c_0) \cap \mathcal{F}$ is empty for sufficiently small ε, is not markedly different: In this case just a finite (or empty) set of neighborhoods $U(a_i, \varepsilon_i)$ suffices to cover the corresponding segment of $\mathcal{F}$. $\square$

Remark. Since the vertices of the above system of circles and intervals on $\mathbb{R}^1$ that we constructed as a retract for $\mathbb{R}^2 \setminus \mathcal{F}$ do only accumulate at points of $\mathcal{F}$, this line-system can be regarded as a locally tame graph in $\mathbb{R}^2 \setminus \mathcal{F}$, and hence it has the topology of a one-dimensional CW-complex.

Definition 4.3. A planar space X is said to be *homotopically decomposable* if there exist open subsets X_1 and X_2 such that $X = X_1 \cup X_2$, $\mathrm{hdim}(X_1) = \mathrm{hdim}(X_2) = \mathrm{hdim}(X_1 \cap X_2) \leqslant 1$ and $H_1(X_1 \cap X_2) \to H_1(X_1) \oplus H_1(X_2)$ is a monomorphism. In this case it follows from

Mayer–Vietoris exact sequence and from Curtis' and Fort's result [7] that all decomposable spaces are acyclic in higher dimensions.

Note that our Examples 1 and 2 and the examples from [6, §5] and [13, A.4.13] are homotopically decomposable, e.g., $X_1 =$ complement of the mid-point, $X_2 =$ a small disk-neighborhood of the mid-point. Such decompositions should also exist for all spaces where, similarly as in the proof of Proposition 1.1(ii), all holes can be reduced to a closed set that is arranged along a line. However, planar Peano continua without such a decomposition do exist, e.g., Examples 3 or 4.4 below. The proofs are analogous, we restrict ourselves to exposing the proof in the simpler case of Example 4.4.

Example 4.4. Let X be our Example 2, $\{X_i\}_{i\in\mathbb{N}}$ be a countable number of copies of X and let $\mathcal{W} = \bigvee_{i=1}^{\infty} X_i$ be the compact wedge with respect any base point lying on the boundary of D^2. This gives a Peano continuum that can be embedded into the plane.

Proposition 4.5. *Example* 4.4 *is an indecomposable* 2*-dimensional planar Peano continuum.*

Proof. Every open subset U of $\mathcal{W}$ which contains the base point of $\mathcal{W}$ should contain one of the factors $X_i \approx X$. By Remark 2.4(v) all hf-points of X_i are hf-points of U and hence U is homotopically 2-dimensional. It follows that the space $\bigvee_{i=1}^{\infty} X_i$ is nondecomposable. □

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