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On Nerves of Fine Coverings of Acyclic Spaces

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Abstract. The main results of this paper are: (1) If a space X can be embedded as a cellular subspace of $\mathbb{R}^n$ then X admits arbitrary fine open coverings whose nerves are homeomorphic to the n-dimensional cube D^n. (2) Every n-dimensional cell-like compactum can be embedded into $(2n+1)$-dimensional Euclidean space as a cellular subset. (3) There exists a locally compact planar set which is acyclic with respect to Čech homology and whose fine coverings are all nonacyclic.

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1. Introduction

In 1954 Borsuk [2] asked whether every compact absolute neighborhood retract is homotopy equivalent to some compact polyhedron? In 1977 his question was answered in the affirmative by West [20]. Much earlier, in 1928 Aleksandrov [1] had proved that every compact n-dimensional space X admits for any $\varepsilon > 0$, an ε-map onto some n-dimensional finite polyhedron P which is the nerve of some fine covering of X, whereas X does not admit any μ-map for some $\mu > 0$ onto a polyhedron of dimension less than n. These results motivated the classical Aleksandrov-Borsuk problem which remains open:

Problem 1.1. *Given an n-dimensional compact absolute neighborhood retract X and $\varepsilon > 0$, does there exist an ε-covering $\mathcal{U}$ of order $n+1$ such that the natural map of X onto the nerve $\mathcal{N}(\mathcal{U})$ of the covering $\mathcal{U}$ induces a homotopy equivalence?*

A special case is the following, also open problem:

Problem 1.2. *Does every n-dimensional compact absolute retract admit a fine covering $\mathcal{U}$ of order $n+1$ such that its nerve $\mathcal{N}(\mathcal{U})$ is contractible?*

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Note that for the class of cell-like cohomology locally conected compacta the answer to analogous question is negative, since there exists a 2-dimensional cell-like cohomology locally connected compactum whose fine coverings of order 3 are all nonacyclic [12,13].

In the present paper we shall investigate fine coverings of acyclic, cellular and cell-like spaces. A topological space X is called *acyclic with respect to Čech homology* or simply *acyclic* if Čech homology with integer coefficients of X is the same as of a point. A *cellular* subspace X of the n-dimensional Euclidean space $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ which is the intersections of a nested system of n-dimensional topological cubes D^n:

$$X = \bigcap_{i=1}^{\infty} D_i^n, \quad \text{where} \quad D_{i+1}^n \subset \mathrm{int} D_i^n.$$

Recall that it follows by the continuity property of Čech homology that every cellular space is acyclic.

Our first main result is the following:

Theorem 1.3. *If a space X can be embedded as cellular subspace into the n-dimensional Euclidean space $\mathbb{R}^n$ then X admits arbitrarily fine open coverings whose nerves are all homeomorphic to the n-dimensional cube D^n.*

It is well known that the class of all cellular spaces is quite large, for example all n-dimensional compact cell-like spaces X can be embedded as cellular subsets of $\mathbb{R}^m$, provided that $m \geq 2n + 2$ (see e.g. [9]).

We shall strengthen this fact as follows:

Theorem 1.4. *Every n-dimensional cell-like compact space X can be embedded into the $(2n + 1)$-dimensional Euclidean space $\mathbb{R}^{2n+1}$ as a cellular subset.*

Corollary 1.5. *Every n-dimensional cell-like (in particular, contractible) compact space X admits arbitrarily fine open coverings whose nerves are all homeomorphic to the $(2n + 1)$-dimensional cube D^{2n+1}.*

Note that there exist n-dimensional contractible compacta which are nonembeddable into $\mathbb{R}^{2n}$ (see e.g. [14]).

Finaly, we shall show that there exist acyclic planar spaces whose fine covering are all nonacyclic. These spaces are of course not compact because there is a classical result that any planar acyclic compactum is cellular (see e.g. [3,5,16]) and by our Theorem 1.3, they have fine coverings whose nerves are all homeomorphic to the 2-cell D^2.

2. Preliminaries

We shall begin by fixing some terminology and notations and we shall give some definitions which will be used in the sequel. All undefined terms can be found in [3,5,7,10,11,17,19].

By a *covering* $\mathcal{U}$ of X we mean a system of open subsets of a metric space X whose union is X. If the space X is compact then by a *covering* we mean a finite covering. We consider the standard metric ρ on the Euclidean space $\mathbb{R}^n$ and its subspaces. For a subspace A of the space X and for a positive number d we denote the d-neighborhood of the set A in X by $N(A, d)$, i.e.

$$N(A, d) = \{x : x \in X \text{ and } \rho(x, A) < d\}.$$

In particular, the open ball $B(a, d)$ in a metric space with center at the point a and radius d is the set $N(\{a\}, d)$. By the *mesh* of the covering $\mathcal{U}$, $\operatorname{mesh}(\mathcal{U})$, we mean the supremum of the diameters of all elements of the covering $\mathcal{U}$. We say that the space admits *fine acyclic* coverings if for every open covering $\mathcal{U}$ there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that homology of the nerve $\mathcal{N}(\mathcal{V})$ is trivial, i.e. homology of $\mathcal{N}(\mathcal{V})$ is the same as homology of a point. For compact spaces this is equivalent to existence of acyclic coverings $\mathcal{V}$ with $\operatorname{mesh}(\mathcal{V}) < \varepsilon$, for every positive number ε.

We consider only Čech homology with integer coefficients.

Definition 2.1. *A* kernel U_i^0 *of the element* U_i *of the covering* $\mathcal{U} = \{U_i\}_{i=\overline{1,n}}$ *of the space* X *is an open non-empty subset of* U_i *such that it does not intersect with other elements* $U_j, j \neq i$, *of the covering* $\mathcal{U}$.

Definition 2.2. *A covering* $\mathcal{U} = \{U_i\}_{i=\overline{1,n}}$ *is called* canonical *on the subspace* $A \subset X$ *if for every* i *such that* $U_i \cap A \neq \emptyset$ *it follows that* $U_i^0 \cap A \neq \emptyset$.

Definition 2.3. *A canonical covering* $\mathcal{U} = \{U_i\}_{i=\overline{1,n}}$ *on the subspace* A *is called a* canonization *of the covering* $\mathcal{V} = \{V_i\}_{i=\overline{1,n}}$ *if* $U_i \subset V_i$ *for every* i, *and this refinement induces a simplicial isomorphism between the nerves* $\mathcal{N}(\mathcal{U})$ *and* $\mathcal{N}(\mathcal{V})$.

Lemma 2.4. *For every finite covering* $\mathcal{V} = \{V_i\}_{i=\overline{1,n}}$ *of a metric space* X *and its subset* A *without isolated points there exists a canonization* $\mathcal{U}$ *of the covering* $\mathcal{V}$ *on the subspace* A.

Proof. In every nonempty intersection $V_{i_0} \cap V_{i_0} \cap \cdots V_{i_k} \cap A$ let us choose a point $a_{i_0 i_1 \ldots i_k}$ such that to different systems of open sets there correspond different points (this is possible because the set A does not contain isolated points). We get a finite set of points.

Let d be any positive number less than the minimum of the distances between the chosen points and such that if the intersection $V_i \cap A \neq \emptyset$ then $B(a_i, d) \subset V_i$. Let $U_i = V_i \backslash \bigcup \overline{B\left(a_j, \frac{d}{2}\right)}$ (the union is over all j, $j \neq i$ for which the point a_j is defined, i.e. $V_j \cap A \neq \emptyset$). The nerves of the coverings $\mathcal{U}$ and $\mathcal{V}$ are isomorphic since we did not remove the points $a_{i_0 i_1 \ldots i_k}$ from the space X and since the balls $B\left(a_i, \frac{d}{2}\right)$ lie in the kernel of U_i.

Therefore the covering $\mathcal{U}$ is a canonization of the covering $\mathcal{V}$ of the subspace A. □

Definition 2.5. *By a* chain connecting the element U *of the covering* $\mathcal{U}$ *with subset* A *along the connected subset* M *of the space* X *we mean a system* $\{U_{i_1}, U_{i_2}, \ldots U_{i_m}\}$ *of elements of* $\mathcal{U}$ *such that* $U_{i_1} = U$, $U_{i_k} \cap A = \emptyset$ *for* $k < m$, $U_{i_m} \cap A \neq \emptyset$ *and* $U_{i_t} \cap U_{i_{t+1}} \cap M \neq \emptyset$ *for* $t = \overline{1, (m-1)}$.

Next, we shall need the following construction. Consider the covering $\mathcal{U}$ of a topological space X and the 4-tuple $\{U, x, \varepsilon, m\}$ in which $U \in \mathcal{U}$, $x \in U^0$ (U^0 is kernel of the U), ε is a positive number such that $B(x, \varepsilon) \subset U^0$, and m

is any natural number. Consider the covering $\mathcal{U}'$ which consists of all elements of the covering $\mathcal{U}$ except the element U. Instead of U we choose the following m elements for $m > 1$:

- $U(x, \varepsilon, m, 1) = U \setminus \overline{B(x, \frac{\varepsilon}{2})}$,
- $U(x, \varepsilon, m, k) = B(x, \frac{\varepsilon}{k-1}) \setminus \overline{B(x, \frac{\varepsilon}{k+1})}$, for $k > 1, k < m$,
- $U(x, \varepsilon, m, m) = B(x, \frac{\varepsilon}{m-1})$.

If $m = 1$, then $U(x, \varepsilon, 1, 1) = U$. The covering $\mathcal{U}'$ is called the *grating* of $\mathcal{U}$ with respect to the 4-tuple $\{U, x, \varepsilon, m\}$.

If the element U of the covering $\mathcal{U}$ is connected then $\mathcal{N}(\mathcal{U}') = \mathcal{N}(\mathcal{U}) \cup P$, where P is a segment subdivided into $m-1$ parts if $m > 1$ and it is the empty set if $m = 1$.

Let $\mathcal{U}$ be any covering of the space X which refines a covering $\mathcal{W}$ and let $\varphi : \mathcal{N}(\mathcal{U}) \to \mathcal{N}(\mathcal{W})$ be a simplicial mapping induced by this refinement. Suppose that $\{U_{i_0}, U_{i_1}, \ldots, U_{i_m}\}$ are subsets of the covering $\mathcal{U}$ having empty intersection.

Definition 2.6. *The covering $\mathcal{W}$ is called an* extension *of the covering $\mathcal{U}$ with respect to the set $\{U_{i_0}, U_{i_1}, \ldots, U_{i_m}\}$ if the mapping φ is injective and the complex $\mathcal{N}(\mathcal{W})$ is the union of the complex $\mathcal{N}(\mathcal{U})$ with an m-dimensional simplex corresponding to $\{U_{i_0}, U_{i_1}, \ldots, U_{i_m}\}$ and possibly some of its faces.*

Lemma 2.7. *For every covering $\mathcal{U}$ canonical on the set A and for every system of its elements $\{U_{i_0}, U_{i_1}, \ldots U_{i_m}\}$ such that $\bigcap_{t=0}^{m} U_{i_t} = \emptyset$ and $A \cap U_{i_t} \neq \emptyset$ there exists for every t, a covering canonical on A which is an extension of the covering $\mathcal{U}$ with respect to $\{U_{i_0}, U_{i_1}, \ldots U_{i_m}\}$.*

Proof. In the kernel of one of the sets $U_{i_0}, U_{i_1}, \ldots U_{i_m}$ choose a ball $B(a, d)$ and replace U_{i_t} by $U_{i_t} \cup B(a, d)$ for every $t = \overline{0, m}$. We obviously get the desired extension. □

Subspace X of $\mathbb{R}^n$ is cellular if and only the quotient space $\mathbb{R}^n/X$ is homeomorphic to $\mathbb{R}^n$ (see e.g. [9]). Therefore we can assume that for a cellular subset X, $X = \bigcap_{i=1}^{\infty} D_i^n$, where $D_{i+1}^n \subset \mathrm{int} D_i^n$, there exists a retraction $r_i : D_i^n \to D_{i+1}^n$ such that preimage of every point x of the boundary ∂D_{i+1}^n is homeomorphic to the segment $[0, 1]$.

Definition 2.8. ([9,10]). *A polyhedral neighborhood N of the polyhedron $P \subset \mathbb{R}^n$ is called* regular *if there exists a piecewise linear mapping $\varphi : N \times [0, 1] \to N$, such that $\varphi(x, 0) = x$, $\varphi(x, 1) \in P$ for all $x \in N$ and $\varphi(x, t) = x$ for $x \in P$ and $t \in [0, 1]$ or in other words, P is a strong deformation retract of N under a piecewise linear homotopy φ.*

Definition 2.9. ([9,10]). *An ε* push *of the pair $(\mathbb{R}^n, X)$ is a homeomorphism h of $\mathbb{R}^n$ to itself for which there exists a homotopy $\varphi : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ such that*

1. $\varphi(x, 0) = x$, $\varphi(x, 1) = h(x)$;
2. $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ *is a homeomorphism for every $t \in [0, 1]$ and $\rho(x, \varphi_t(x)) < \varepsilon$ for all $x \in \mathbb{R}^n$;*
3. $\varphi(x, t) = x$ *for every $t \in [0, 1]$ and all x such that $\rho(x, X) \geq \varepsilon$.*

Definition 2.10. ([9,10]). *Let P be a compact subpolyhedron of $\mathbb{R}^n$ and let ε be a positive real number. An ε-regular neighborhood of P in $\mathbb{R}^n$ is a regular neighborhood N of P such that for any compact subset Y of $\mathbb{R}^n \backslash P$, there is an ε-push h of $(\mathbb{R}^n, P)$ such that $h(Y) \cap N = \emptyset$.*

We note that it follows by definition that every ε-regular neighborhood N of subpolyhedron P is a proper subset of $N(P, \varepsilon)$.

The following lemma follows by the regular neighborhood theory (see e.g. [9,10]).

Lemma 2.11. *For any finite subpolyhedron P of $\mathbb{R}^n$ and any $\varepsilon > 0$ there exists an ε-regular neighborhood N of P.*

3. Proof of Theorem 1.3

Since X is a cellular subset of $\mathbb{R}^n$ we have

$$X = \bigcap_{i=1}^{\infty} D_i^n, \quad \text{where } D_{i+1}^n \subset \text{int} D^n$$

and there are natural retractions $r_i : D_i^n \to D_{i+1}^n$.

Fix a positive number ε and some natural number K which will be specified later. Since $X \subset$ int D_1 and X is a compact space there exists a finite system of open balls of radius $\varepsilon' < \frac{\varepsilon}{K}$ in $\mathbb{R}^n$ which cover X, i.e. $X \subset \cup_{x \in F} B(x, \varepsilon')$, F is a finite subset of X for which $\cup_{x \in F} B(x, \varepsilon') \subset D_1^n$. There exists an index i_0 such that $D_{i_0}^n \subset \cup_{x \in F} B(x, \varepsilon')$. Let r be a natural retraction of D_1^n on $D_{i_0}^n$. Since the mapping r is uniformly continuous there exists a positive number $\delta < \varepsilon'$, such that $\varrho(r(x), r(y)) < \varepsilon'$ whenever $\varrho(x, y) < \delta$.

Consider a triangulation of D_1^n with the diameters of simplices less than $\frac{\delta}{2}$. Consider a covering of D_1^n by open stars of the vertices of this triangulation. According to Lemma 2.4 there exists a refinement $\mathcal{U} = \{U_i\}_{i=\overline{1,m}}$ of this covering which is canonical on X. Note that the nerve $\mathcal{N}(\mathcal{U})$ is homeomorphic to D^n.

We wish to associate to every open set U_i of the $\mathcal{U}$ some open subset of the space X. If the intersection $U_i \cap X$ is nonempty then we associate to U_i the open set $U_i \cap X$ in X. If $U_i \cap X = \emptyset$ then we choose a point $y_i \in U_i$. The point $r(y_i)$ belongs to some ball $B(x_i, \varepsilon')$, $x_i \in F$, and the subspace $r^{-1}(r(y_i))$ is homeomorphic to a segment if $y_i \notin D_{i_0}$ or is a point if $y_i \in D_{i_0}$. So the union $M_i = r^{-1}(r(y_i)) \cup B(x_i, \varepsilon')$ is a connected set.

Since M_i is connected there obviously exists a chain $\{U_{i_1}, U_{i_2}, \dots U_{i_{m(i)}}\}$ connecting U_i with X along M_i. Since the covering $\mathcal{U}$ is canonical on X, the intersection of the kernel of $U_{i_{m(i)}}$ and X is nonempty, and we can find a point $z_i \in U^0_{i_{m(i)}} \cap X$. Let ε_i be a positive number such that $B(z_i, \varepsilon_i) \cap X \subset U^0_{i_{m(i)}} \cap X$. So we have a 4-tuple $\{U_{i_{m(i)}} \cap X, z_i, \varepsilon_i, m(i)\}$ and we can take a grating of $\mathcal{U}|_X = \{U_i \cap X\}_{i=\overline{1,m}}$ with respect to this 4-tuple.

Repeat this procedure for all i. We get some canonical covering $\mathcal{U}'$ of X. There is a simplicial mapping $\mathcal{J} : \mathcal{N}(\mathcal{U}') \to \mathcal{N}(\mathcal{U})$ which maps the vertices $U(z_i, \varepsilon_i, m(i), k)$ to the vertices U_{i_k}. This mapping is in general not injective because some element $U_k \in \mathcal{U}$ can be the element of several chains of the type $\{U_{i_1}, U_{i_2}, \ldots U_{i_{m(i)}}\}$.

Consider a new covering $\mathcal{W}$ whose elements are unions of all elements of $\mathcal{U}'$ which correspond to the elements U_k under a mapping $\mathcal{T}$.

Let us estimate the diameters of the elements of $\mathcal{W}$. Take two points a_1 and a_2 from any $W \in \mathcal{W}$. By construction, there must exist two sets M_{i_1} and M_{i_2} which intersect with U_k. The distance between the points $r(y_{i_1})$ and $r(y_{i_2}) < \varepsilon'$ since the diameter of $U_k < \delta$. Diameters of balls $B(x_{i_1}, \varepsilon')$ and $B(x_{i_2}, \varepsilon')$ are less than or equal to $2\varepsilon'$. The diameters of the elements of the covering $\mathcal{U}$ are also less than ε' since $\delta < \varepsilon'$.

By the Triangle Inequality it follows that $\rho(a_1, a_2) < 7\varepsilon'$ therefore $\operatorname{diam}(\mathcal{W}) \leq 7\varepsilon'$. We now have the injective mapping $\mathcal{J}' : \mathcal{N}(\mathcal{W}) \to \mathcal{N}(\mathcal{U})$ which maps vertices of $\mathcal{N}(\mathcal{W})$ bijectively onto the vertices of $\mathcal{N}(\mathcal{U})$. Suppose that $\mathcal{J}'$ is not surjective. Then there exists a system of elements $\{W_{i_1}, W_{i_2}, \ldots W_{i_{m(i)}}\}$ of the covering $\mathcal{W}$ with empty intersection.

Let us apply the operation of the extension of the covering $\mathcal{W}$ (see Definition 2.6). We get a new covering. Since all coverings are finite, after few applications of this operation we finally get the covering $\mathcal{W}'$ of the space X and a bijective mapping $\mathcal{J}'' : \mathcal{N}(\mathcal{W}') \to \mathcal{N}(\mathcal{U})$.

Let us estimate the distance between the points of the sets W_{i_k} and W_{i_l} which are the vertices of same simplex of the polyhedron $\mathcal{N}(\mathcal{W}')$. For the mapping $\mathcal{T}'$ to W_{i_k} and W_{i_l} there correspond two elements U_{i_k} and U_{i_l} which intersect. We have two points $y_{i_k} \in U_{i_k}$ and $y_{i_l} \in U_{i_l}$. Since $U_{i_k} \cap U_{i_l} \neq \emptyset$, we have $\rho(y_{i_k}, y_{i_l}) < 2\delta$ and $\rho(r(y_{i_k}), r(y_{i_l})) < 2\varepsilon'$. By construction $\rho(r(y_{i_k}), W_{i_l}) < 7\varepsilon'$ for every k.

It follows that the distance between any points of W_{i_k} and W_{i_l} is less than $16\varepsilon'$. So the diameters of the elements of the covering $\mathcal{W}'$ are no more than $16\varepsilon'$. Since the number K was arbitrary we can put $K > 16$ and get that $\operatorname{diam}(\mathcal{W}') < \varepsilon$. So we have a fine covering whose nerve is homeomorphic to D^n.

4. Proof of Theorem 1.4

We shall need the following lemmas.

Lemma 4.1. (Freudenthal [7,8,18]). *Every compact metrizable space X is homeomorphic to the inverse limit of the inverse sequence $\{P_i \xleftarrow{f_i} P_{i+1}\}_{i\in\mathbb{N}}$ of finite polyhedra P_i with piecewise linear (i.e. quasi-simplicial [7, pp.148, 153]) surjective projections f_i. If* $\dim X \leq n$ *then* $\dim P_i \leq n$.

Lemma 4.2. (see [9, Theorem 3.3]). *Let* $\dim X \leq n$ *and suppose that X is homeomorphic to the inverse limit of the inverse sequence $\{P_i \xleftarrow{f_i} P_{i+1}\}_{i\in\mathbb{N}}$ of finite polyhedra P_i with piecewise linear surjective projections f_i and* $\dim P_i \leq n$. *Then for every i, P_i can be embedded as subpolyhedron R_i in $\mathbb{R}^{2n+1}$ so that:*

- *For every i there exists a q_i-regular neighborhood N_i of R_i in $\mathbb{R}^{2n+1}$, $q_i < \frac{1}{i}$ and $\overline{N}_{i+1} \subset N_i$;*
- *X is homeomorphic to $\cap_{i=1}^{\infty} N_i$.*

Let us give a brief sketch of the proof of this lemma, see also [6, Exercise 3.4.5].

Proof. Let K_1 be the number of vertices of the polyhedron P_1 for some fixed triangulation. Choose points $\mathcal{P}_1 = \{p_{1,1}, p_{1,2}, \ldots p_{1,K_1}\}$ in general position in the space $\mathbb{R}^{2n+1}$ and embed simplicially the polyhedron P_1 in $\mathbb{R}^{2n+1}$ in such a way that to the vertices of P_1 there correspond the points $\mathcal{P}_1$ (see e.g. [11,17]). Denote by Q_1 the image of polyhedron P_1 in $\mathbb{R}^{2n+1}$ with a given triangulation.

Since f_1 is a quasi-simplicial mapping there exists barycentric triangulations of P_1 and P_2 such that f_1 becomes simplicial mapping. Let L_1 and K_2 be the number of vertices of the polyhedra P_1 and P_2 after these triangulations, respectively. We have points $\{q_{1,1}, q_{1,2}, \ldots q_{1,L_1}\}$ of the polyhedron Q_1 which correspond to the vertices of the polyhedra P_1 for this triangulation.

These points are not in general position but we can move them in such a way that we get points $\mathcal{R}_1 = \{r_{1,1}, r_{1,2}, \ldots r_{1,L_1}\}$ which are in general position and we get a new subpolyhedron R_1 of $\mathbb{R}^n$ piecewise homeomorphic to Q_1 generated by these points with the simplicial mapping $f_1 : P_2 \to R_1$ (here, and in the sequel, we shall use the same symbol for mappings if the domain/range are the same and if the corresponding diagram is commutative).

Let N_1 be 1-regular neighborhood of R_1, see Lemma 2.11. Let r_1 be the distance between R_1 and $\mathbb{R}^{2n+1}\backslash N_1$.

Let d_1 be any positive number less than 1 and the maximum of the diameters of the simplices of the polyhedron R_1. Choose points $\mathcal{P}_2 = \{p_{2,1}, p_{2,2}, \ldots p_{2,K_2}\}$ in $\mathbb{R}^{2n+1}$ which satisfy the following conditions:

(1) All points $\mathcal{R}_1 \cup \mathcal{P}_2$ are in general position, see e.g. [7, p. 102, Theorem 1.10.2];
(2) If the vertex corresponding to the point $p_{2,i}$ is mapped by f_1 to the point $r_{1,j}$ then $p_{2,i} \in B\big(r_{1,j},\ \min\big\{\frac{r_1}{3}, \frac{d_1}{3}\big\}\big)$.

Since the points $\mathcal{P}_2$ are in general position we can simplicially embed the polyhedron P_2 with respect to these vertices. Denote by Q_2 the image of P_2 in $\mathbb{R}^{2n+1}$. Let us estimate the distance between the points $x \in Q_2$ and $f_1(x) \in R_1$. Take any point $x \in P_2$. Then we have for some λ_i, $\sum \lambda_i = 1$, $\lambda_i \geq 0$ and for some $p_{2,i}$ that $x = \sum \lambda_i p_{2,i}$ where $p_{2,i}$ are vertices of some simplex of the polyhedron P_2 which contains x. Then $f(x) = \sum \lambda_i f(p_{2,i})$.

Furthermore,

$$\rho(x, f(x)) = ||x - f(x)|| = \big|\big|\sum \lambda_i (p_{2,i} - f(p_{2,i}))\big|\big|$$
$$< \sum \lambda_i \cdot \min\left\{\frac{r_1}{3}, \frac{d_1}{3}\right\} = \min\left\{\frac{r_1}{3}, \frac{d_1}{3}\right\} = \delta_1.$$

It now follows that $N(Q_2, \delta_1) \subset N_1$.

So we have a triad $\{R_1, N_1, Q_2\}$ of subpolyhedra of $\mathbb{R}^{2n+1}$ such that R_1 is piecewise homeomorphic to P_1, polyhedron N_1 is a 1-regular neighborhood of R_1, and Q_2 is homeomorphic to P_2. There is a natural mapping $f_1 : Q_2 \to R_1$ which is associated to $f_1 : P_2 \to P_1$ and for any $x \in Q_2$ we have

$$\rho(x, f_1(x)) \leq \delta_1 = \min\left\{\frac{r_1}{3}, \frac{d_1}{3}\right\}$$

where r_1 is the distance between R_1 and $\mathbb{R}^{2n+1} \backslash N_1$, and d_1 is the maximum of the diameters of all simplices of the polyhedron R_1.

Let us suppose that we are given for some index i a triad $\{R_i, N_i, Q_{i+1}\}$ of subpolyhedra of $\mathbb{R}^{2n+1}$ such that:

- R_i is piecewise homeomorphic to P_i;
- N_i is a q_i-regular neighborhood of R_i, $q_i < \min\{\frac{1}{i}, d_i\}$ and d_i is the maximum of the diameters of all simplices of the polyhedron R_i;
- Q_{i+1} is homeomorphic to P_{i+1} and there is a natural mapping $f_i : Q_{i+1} \to R_i$ which is associated to $f_i : P_{i+1} \to P_i$;
- for any $x \in Q_{i+1}$ we have

$$\rho(x, f_i(x)) \leq \delta_i = \min\left\{q_i, \frac{r_i}{3}, \frac{d_i}{3}\right\}$$

  where r_i is the distance between R_i and $\mathbb{R}^{2n+1} \backslash N_i$.

We call the triad with these properties a *special triad.*

Now we construct a special triad $\{R_{i+1}, N_{i+1}, Q_{i+2}\}$ in the following way. We have a piecewise linear mapping $f_{i+1} : P_{i+2} \to P_{i+1} = Q_{i+1}$ therefore there exist barycentric subdivisions of P_{i+2} and Q_{i+1} such that f_{i+1} becomes a simplicial mapping. Let L_{i+1} and K_{i+2} be the number of vertices of the polyhedra P_{i+1} and P_{i+2} after these triangulations, respectively. We have points $\{q_{i+1,1}, q_{i+1,2}, \ldots q_{i+1,L_{i+1}}\}$ of the polyhedron Q_{i+1} which correspond to the vertices of the polyhedron P_{i+1} for this triangulation.

We move these points in such a way that we obtain points

$$\mathcal{R}_{i+1} = \{r_{i+1,1}, r_{i+1,2}, \ldots r_{i+1,L_{i+1}}\}$$

which are in general position and $\rho(q_{i+1,i}, r_{i+1,i}) < \frac{r_i}{3}$. We get a new polyhedron R_{i+1} which lies in the neighborhood $N(Q_{i+1}, \frac{r_i}{3})$ and for which we have a simplicial mapping $f_{i+1} : P_{i+2} \to R_{i+1}$. Let d_{i+1} be the maximum of the diameters of all simplices of the polyhedron R_{i+1}. Let $q_{i+1} < \min\{\frac{1}{i+1}, d_{i+1}\}$ be such a number that the q_{i+1}-regular neighborhood N_{i+1} of R_{i+1} is a subset of $N(Q_{i+1}, \frac{r_i}{3})$.

Let r_{i+1} be the distance between R_{i+1} and $\mathbb{R}^{2n+1} \backslash N_{i+1}$, and let d_{i+1} be the maximum of the diameters of all simplices of the polyhedron R_{i+1}. Choose points $\mathcal{P}_{i+2} = \{p_{i+2,1}, p_{i+2,2}, \ldots p_{i+2,K_i+2}\}$ in $\mathbb{R}^{2n+1}$ satisfying the following conditions:

(1) All points $\cup_{i=1}^{i+1} \mathcal{R}_i \cup \mathcal{P}_{i+2}$ are in general position;
(2) If the vertex corresponding to the point $p_{i+2,i}$ is mapped by f_{i+1} to the point $r_{i+1,j}$ then $p_{i+2,i} \in B\left(r_{i+1,j}, \min\left\{q_{i+1}, \frac{r_{i+1}}{3}, \frac{d_{i+1}}{3}\right\}\right)$.

Since the points $\mathcal{P}_{i+2}$ are in general position we can simplicially embed the polyhedron P_{i+2} with respect to these vertices. Denote by Q_{i+2} the image of P_{i+2} in $\mathbb{R}^{2n+1}$. It is easy to see that $N(Q_{i+2}, \frac{r_{i+1}}{3}) \subset N_{i+1}$. We now have a special triad $\{R_{i+1}, N_{i+1}, Q_{i+2}\}$. By induction we can thus obtain a special triad $\{R_k, N_k, Q_{k+1}\}$ for every $k \in \mathbb{N}$.

If we consider two different sequences of points $x_i \in P_i, f_i(x_{i+1}) = x_i$ and $x'_i \in P_i, f_i(x'_{i+1}) = x'_i$, then obviously there exists an index i_0 such that x_i and x'_i belong to different simplices of R_{i_0}. It follows by our choice of numbers d_i that the limit points x and x' of the sequences $\{x_i\}$ and $\{x'_i\}$ are different.

Therefore the space X is homeomorphic to the intersection $\bigcap_1^\infty \overline{N_i}$ and it is the limit of the sequence of polyhedra $\{R_i\}_{i\in\mathbb{N}}$. □

Now we can prove Theorem 1.4. First, let us prove Theorem 1.4 in the case $n = 1$, i.e. let us prove that every 1-dimensional cell-like compactum X can be embedded as a cellular subspace into $\mathbb{R}^3$.

According to the classical Case-Chamberlin theorem, every 1-dimensional cell-like continuum is tree-like, i.e. any of its open coverings has a tree-like refinement (a refinement whose nerve is a 1-dimensional finite contractible complex) [4]. By the proof of the Freudenthal Theorem [18] it follows that $X = \varprojlim \left(P_i \xleftarrow{f_i} P_{i+1}\right)$, where each P_i is a contractible 1-dimensional polyhedron and all projections f_i are piecewise linear mappings.

It follows by Lemma 4.2 that X can be embedded in $\mathbb{R}^3$ so that its image has arbitrary fine neighborhoods N_i with contractible spines $R_i \approx P_i$, i.e. N_i is homeomorphic to D^3 and the embedding of X in $\mathbb{R}^3$ is cellular in this case.

Let now $n \geq 2$. Then we embed space X in a regular way in $\mathbb{R}^{2n+1}$ according to Lemma 4.2. Let us show that such an embedded space X satisfies the cellularity criterion in $\mathbb{R}^{2n+1}$ (see [6]).

Consider any neighborhood U of X in $\mathbb{R}^{2n+1}$. Since the space X is cell-like there exists a neighborhood V of X in U such that the embedding $V \subset U$ is homotopic to the constant mapping. Consider any mapping f of ∂D^2 to $V\backslash X$. Since the embedding $V \hookrightarrow U$ is homotopic to the constant mapping there exists an extension $\overline{f} : D^2 \to U$. By the Simpicial Approximation Theorem we can suppose that f and $\overline{f}$ are simplicial mappings and the image of $\overline{f}$ is a 2-dimensional polyhedron in U.

Let ε be a positive number such that $N(X, \varepsilon) \subset V$ and choose the index i so that the q_i-regular neighborhood N_i of R_i, is a subset of V (it suffices to require $q_i < \varepsilon$). We may assume that $\mathrm{Im}\overline{f}$ and R_i are in general position and that the spaces $\mathrm{Im}\overline{f}$ and P_i do not intersect, $\mathrm{Im}\overline{f} \cap P_i = \emptyset$, since $2+n < 2n+1$.

By Lemmas 2.11 or 4.2 there exist for N_i, a q_i-push h_i of the pair $(\mathbb{R}^{2n+1}, R_i)$ such that $h_i\overline{f}(D^2) \cap N_i = \emptyset$. It follows that $h_i\overline{f}(D^2) \cap X = \emptyset$. Therefore $f : \partial D^2 \to U\backslash X$ is inessential and we obtain an embedding of the cell-like space X into $\mathbb{R}^{2n+1}$, $2n + 1 \geq 5$ which satisfies the Cellularity Criterion of McMillan (see [15] or [6, Theorem 3.2.3]). It follows that X is cellular.

5. Acyclic Subspaces of the Plane Whose Fine Coverings are all Nonacyclic

We shall present two examples of locally compact planar acyclic with respect to Čech homology spaces whose fine coverings are all nonacyclic.

Example 5.1. Consider in the plane $\mathbb{R}^2$ a countable bouquet of circles S^1_i with a base point A and with a common tangent line, whose diameters tend to infinity. From every circle S^1_i remove a small open arc AA_i such that the diameters of these arcs tend to 0. We get the desired space X_1, see Fig. 1.

Obviously, X_1 is a locally compact space. Consider the following cofinite system of coverings of X_1. Triangulate the segments $S^1_i \backslash AA_i$ and take their coverings by open stars of all vertices of the triangulations except the stars of the vertex A. For the point A consider the open set $B(A,\varepsilon) \cap X_1$. Obviously, the coverings of such type are cofinal in the set of all coverings of X_1.

The nerves of these coverings are homeomorphic to the countable bouquet

$$(\vee_1^n I_i) \bigvee_A (\vee_{n+1}^{\infty} S^1_i)$$

of circles and a finite number of segments with respect to the point A. Therefore their 1-dimensional homology groups are isomorphic to the direct sums $\sum_{i=n+1}^{\infty} \mathbb{Z}$ and we have the following inverse system:

$$\sum_1^{\infty} \mathbb{Z} \hookleftarrow \sum_2^{\infty} \mathbb{Z} \hookleftarrow \sum_3^{\infty} \mathbb{Z} \hookleftarrow \cdots .$$

The inverse limit of this system is zero and the space X_1 is acyclic. However, since all homomorphisms in this system are nonzero monomorphisms it follows that all fine coverings are nonacyclic.

Example 5.2. Consider the "compressed sinusoid" CS as a subspace of the rectangle $[0,1] \times [-1,1] \subset \mathbb{R}^2$:

$$CS = \left\{ (x,y) |\ y = \sin \frac{1}{x} \ \text{ if } x \in (0,1],\ \text{ and } y \in [-1,1] \ \text{ if } x = 0 \right\}.$$

Remove the continuum $\{0\} \times [0,1]$ and obtain a locally compact space $X_2 = CS \backslash (\{0\} \times [0,1])$.

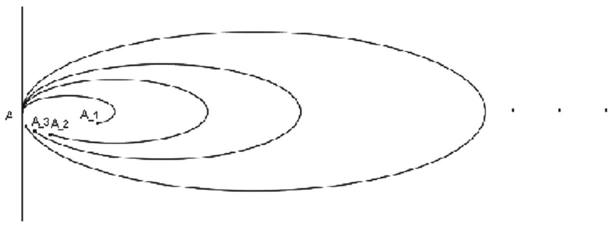

Figure 1. Locally compact acyclic space whose fine coverings are all nonacyclic

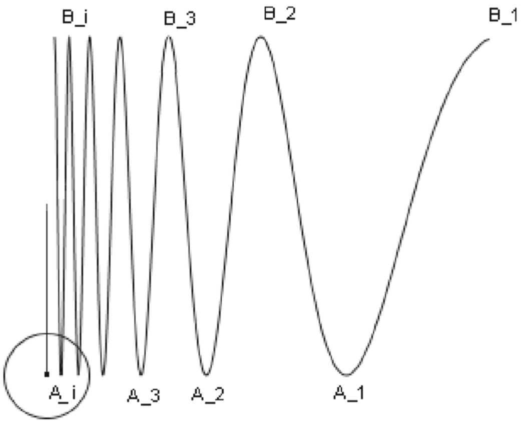

Figure 2. Locally compact acyclic subspace of CS without acyclic fine coverings

To prove the acyclicity consider the following strong deformation retract T of $CS\backslash(\{0\} \times [0,1])$, see Fig. 2:

$$T = \left\{(x, y)|\ y = \sin\frac{1}{x}\ \text{ if } x \in (0,1],\ \text{ and } y = -1\ \text{ if } x = 0\right\}.$$

The space T has the same homotopy type as X_2 and it consists of the curve $y = \sin\frac{1}{x}$ which is homeomorphic to $(0,1]$ and the point $(0;-1)$. There exists a cofinite system of open coverings of this space T. Indeed, on this line consider the standard triangulation and its cover by open stars of its vertices. For the point $(0;-1)$ consider the open subspace $B((0;-1),\varepsilon) \cap T$ of T. The nerves of this cofinite system of coverings are homeomorphic to a countable bouquet of circles and a segment.

We have (as in the first example) the following cofinite inverse system of homology groups and homomorphisms:

$$\sum_{1}^{\infty}\mathbb{Z} \hookleftarrow \sum_{2}^{\infty}\mathbb{Z} \hookleftarrow \sum_{3}^{\infty}\mathbb{Z} \hookleftarrow \cdots.$$

The inverse limit of this system is trivial, therefore $\check{H}_1(T) = 0$ and hence space X is acyclic with respect to Čech homology. However, all fine covering of X_2 are nonacyclic.

6. Epilogue

It follows by Corollory 1.5 that every n-dimensional contractible compactum has arbitrary fine coverings of order $2n+1$ whose nerves are all contractible. The following question is a special case of Problem 1.2:

Question 6.1. *Does there exist an n-dimensional contractible compactum whose fine coverings of order $n+1$ are all nonacyclic?*

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References

[1] Aleksandroff, P.S.: Über den allgemeinen Dimensionsbegriff und seine Beziehungen zur elementaren geometrischen Anschauung. Math. Ann. **98**, 617–636 (1928)

[2] Borsuk, K.: Sur l' élimination de phenomènes paradoxaux en topologie générale. In: Proceedings International Congr. Math., Amsterdam, pp. 197–208, (1954)

[3] Borsuk, K.: Theory of Shape, Monografie Mat., vol. 59. PWN, Warsaw (1975)

[4] Case, J.H., Chamberlin, R.E.: Characterization of tree-like continua. Pacific. J. Math. **10**, 73–84 (1960)

[5] Daverman, R.J.: Decompositions of Manifolds, Academic Press, Orlando (1986)

[6] Daverman, R.J., Venema G.A.: Embeddings in Manifolds, Graduate Studies in Mathematics, vol. 106. American Mathematical Society, Providence (2009)

[7] Engelking, R.: Dimension Theory. Polish Scientific Publishers, Warsaw (1978)

[8] Freudenthal, H.: Über die Entwicklung von Raumen und Gruppen. Compositio. Math. **4**, 145–234 (1937)

[9] Geoghegan, R., Summerhill, R.: Concerning the shapes of finite-dimensional compacta. Trans. Amer. Math. Soc. **179**, 281–292 (1973)

[10] Hudson, J.F.R.: Piecewise Linear Topology. Benjamin, New York (1969)

[11] Hurewicz, W., Wallman, H.: Dimension Theory. Princeton University Press, Princeton (1948)

[12] Karimov, U.H.: An example of a space of trivial shape, all fine coverings of which are cyclic. Soviet. Math. Dokl. **33**, 113–117 (1986)

[13] Karimov, U.H.: Three lemmas of combinatorial group theory. Dokl. Akad. Nauk. Tadzhik. SSR. **29**, 191–192 (1986)

[14] Karimov, U.H., Repovš, D.: On embeddability of contractible k-dimensional compacta into $\mathbb{R}^{2k}$. Topology Appl. **113**, 81–85 (2001)

[15] McMillan, D.R. Jr.: A criterion for cellularity in a manifold. Ann. Math. **79**, 327–337 (1964)

[16] Moore, R.L.: Concerning upper semicontinuous collection of compacta. Trans. Amer. Math. Soc. **27**, 416–428 (1925)

[17] Seifert, H., Threlfall, W.: A Textbook of Topology Pure Appl. Math., vol. 89. Academic Press Inc., New York (1980)

[18] Sklyarenko, E.G.: Uniqueness theorems in homology theory. Math. USSR. Sb. **14**, 199–218 (1971)

[19] Steenrod, N., Eilenberg, S.: Foundations of Algebraic Topology, Princeton Univ. Press, Princeton (1952)

[20] West, J.E.: Mapping Hilbert cube manifolds to ANR's: a solution to a conjecture of Borsuk. Ann. Math. **106**, 1–18 (1977)

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