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On the homology of the Harmonic Archipelago

Research Article

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Abstract: We calculate the singular homology and Čech cohomology groups of the Harmonic Archipelago. As a corollary, we prove that this space is not homotopy equivalent to the Griffiths space. This is interesting in view of Eda's proof that the first singular homology groups of these spaces are isomorphic.

MSC: 54F15, 55N15, 54G20, 57M05

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1. Introduction

The following interesting problem from contemporary theory of Peano continua and combinatorial group theory has been widely discussed and investigated (most recently at the 2011 workshop on wild topology in Strobl, Austria [3]), because it concerns two well-known and important 2-dimensional spaces, namely the *Griffiths space* $\mathcal{G}$ and the *Harmonic Archipelago* $\mathcal{HA}$, cf. [3]:

Problem 1.1.

Are the fundamental groups of the Griffiths space $\mathcal{G}$ and the Harmonic Archipelago $\mathcal{HA}$ isomorphic?

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This difficult problem remains open. Its solution will require a deep understanding of the structure of the fundamental groups of these spaces. In the present paper, which is a step in this direction, we shall investigate the *abelianization* of the fundamental group of the Harmonic Archipelago $\mathcal{HA}$.

It is well known, cf. e.g. [10, Theorem 2A.1], that the 1-dimensional singular homology group with integer coefficients $H_1(X;\mathbb{Z})$ of a path-connected space X is isomorphic to the abelianization of the fundamental group $\pi_1(X)$ of X:

$$H_1(X;\mathbb{Z}) \cong \pi_1(X)/[\pi_1(X),\pi_1(X)].$$

Our first result is based on the structure of the homology groups of the Hawaiian Earring $\mathbb{H}$ (an alternative proof, using infinitary words, was given by Eda [5]):

Theorem 1.2.

Let $\mathcal{HA}$ denote the Harmonic Archipelago. Then

$$H_1(\mathcal{HA};\mathbb{Z}) \cong \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right),$$

whereas $H_n(\mathcal{HA};\mathbb{Z}) \cong 0$ for all $n \geq 2$.

Eda proved [5] that the Griffiths space $\mathcal{G}$ and the Harmonic Archipelago $\mathcal{HA}$ have isomorphic 1-dimensional singular homology groups,

$$H_1(\mathcal{G};\mathbb{Z}) \cong H_1(\mathcal{HA};\mathbb{Z}).$$

Now, it is well known that the Griffiths space $\mathcal{G}$ is cell-like and therefore it has trivial Čech cohomology groups, $\check{H}^*(\mathcal{G};\mathbb{Z}) \cong \check{H}^*(\mathrm{pt};\mathbb{Z})$. On the other hand, by our second main result stated below, the Čech cohomology of the Harmonic Archipelago $\mathcal{HA}$ does not vanish:

Theorem 1.3.

Let $\mathcal{HA}$ denote the Harmonic Archipelago. Then

$$\check{H}^2(\mathcal{HA};\mathbb{Z}) \cong \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right),$$

whereas $\check{H}^n(\mathcal{HA};\mathbb{Z}) \cong 0$ for all $n \neq 0, 2$.

As an immediate consequence we obtain the following important corollary:

Corollary 1.4.

The Griffiths space $\mathcal{G}$ and the Harmonic Archipelago $\mathcal{HA}$ are not homotopy equivalent.

2. Preliminaries

The constructions of the Griffiths space $\mathcal{G}$ and the Harmonic Archipelago $\mathcal{HA}$ are based on the *Hawaiian Earring* $\mathbb{H}$, a classical 1-dimensional planar Peano continuum:

$$\mathbb{H} = \bigcup_{n\in\mathbb{N}} \left\{ (x,y)\in\mathbb{R}^2 \,:\, x^2 + \left(y-\frac{1}{n}\right)^2 = \left(\frac{1}{n}\right)^2 \right\}.$$

The *Griffiths space* $\mathcal{G}$ is a one-point union of two cones over the Hawaiian Earring $\mathbb{H}$, cf. [8]. Consider two copies of the Hawaiian Earring $\mathbb{H}$ in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$:

$$\mathbb{H}_+ = \bigcup_{n\in\mathbb{N}} \left\{ (x,y,0) \in \mathbb{R}^3 \,:\, x^2 + \left(y - \frac{1}{n}\right)^2 = \left(\frac{1}{n}\right)^2 \right\}, \qquad \mathbb{H}_- = \bigcup_{n\in\mathbb{N}} \left\{ (x,y,0) \in \mathbb{R}^3 \,:\, x^2 + \left(y + \frac{1}{n}\right)^2 = \left(\frac{1}{n}\right)^2 \right\}.$$

Let $C(\mathbb{H}_+, (0,0,1))$ and $C(\mathbb{H}_-, (0,0,-1))$ be two cones on the spaces $\mathbb{H}_+$ and $\mathbb{H}_-$ with vertices at the points $(0,0,1)$ and $(0,0,-1)$, respectively. The Griffiths space $\mathcal{G}$ is then defined as the following subspace of $\mathbb{R}^3$:

$$\mathcal{G} = C(\mathbb{H}_+, (0,0,1)) \cup C(\mathbb{H}_-, (0,0,-1)).$$

Griffiths [8] proved that the fundamental group of this one-point union of contractible spaces is nontrivial.

The *Harmonic Archipelago* $\mathcal{HA}$ was introduced by Bogley and Sieradski [2]. It can be simply described as follows: $\mathcal{HA}$ is a noncompact space which is obtained by adjoining a sequence of 'tall' disks between consecutive loops of the Hawaiian Earring, cf. [11].

The following proposition will be useful in the sequel, cf. e.g. [10, Corollary 0.21] or [12, Theorem 1.4.13].

Proposition 2.1.

Spaces X and Y are homotopy equivalent if and only if there is a space Z, containing both X and Y as deformation retracts.

Let

$$C_n = \left\{ (x,y,0) \in \mathbb{R}^3 \,:\, x^2 + \left(y - \frac{1}{n}\right)^2 = \left(\frac{1}{3n(n+1)}\right)^2 \right\}, \qquad n \in \mathbb{N},$$

be a countable number of circles and $\theta = (0,0,0)$ the origin of $\mathbb{R}^3$. It follows by Proposition 2.1 that the Harmonic Archipelago $\mathcal{HA}$ is homotopy equivalent to the subspace of $\mathbb{R}^3$ consisting of all cones $C(C_n, (0,1/n,1))$ over the circles C_n, with the vertices at the points $(0,1/n,1)$, $n \in \mathbb{N}$, connected by the segments and the point $\{\theta\}$, which we shall denote by HA and call the *Formal Harmonic Archipelago*:

$$HA = \bigcup_{n=1}^{\infty} C\left(C_n, \left(0, \frac{1}{n}, 1\right)\right) \cup \bigcup_{n=1}^{\infty} \left\{ (0,y,0) \,:\, y \in \left[\frac{3n+7}{3(n+1)(n+2)}, \frac{3n+2}{3n(n+1)}\right] \right\} \cup \{\theta\}.$$

The *Modified Hawaiian Earring* $\mathcal{MH}$ is defined as follows:

$$\mathcal{MH} = \bigcup_{n=1}^{\infty} C_n \cup \bigcup_{n=1}^{\infty} \left\{ (0,y,0) \,:\, y \in \left[\frac{3n+7}{3(n+1)(n+2)}, \frac{3n+2}{3n(n+1)}\right] \right\} \cup \{\theta\}.$$

The Modified Hawaiian Earring $\mathcal{MH}$ is homotopy equivalent to the Hawaiian Earring, $\mathcal{MH} \simeq \mathbb{H}$. Indeed, both of these spaces are deformation retracts of the third one, as indicated in the middle of Figure 1 (all points a_n, b_n, c_n, d_n, e_n, and f_n converge to the point o). The piecewise linear deformation which moves the points c_n and e_n to the point o, and fixes the points b_n and d_n, yields a space homeomorphic to the Hawaiian Earring $\mathbb{H}$.

The piecewise linear deformation which moves sequentially the points a_n to the points d_n, the segments $[f_n, d_n]$ to $[e_n, d_n]$, and the segments $[o, f_n]$ to the line $[o, b_n] \cup [b_n, e_n]$, with fixed points o, b_n, c_n, d_n, yields the space $\mathcal{MH}$, therefore by Proposition 2.1, U is homotopy equivalent to the Hawaiian earrring $\mathbb{H}$.

Let us define the *Modified Griffiths space* $\mathcal{MG}$. To this end let us introduce some new spaces. Let HA_- be the space symmetric in $\mathbb{R}^3$ to HA, with respect to the point θ:

$$HA_- = \bigcup_{n=1}^{\infty} C\left(C_n^-, \left(0, \frac{-1}{n}, -1\right)\right) \cup \bigcup_{n=1}^{\infty} \left\{ (0,y,0) \,:\, -y \in \left[\frac{3n+7}{3(n+1)(n+2)}, \frac{3n+2}{3(n)(n+1)}\right] \right\} \cup \{\theta\},$$

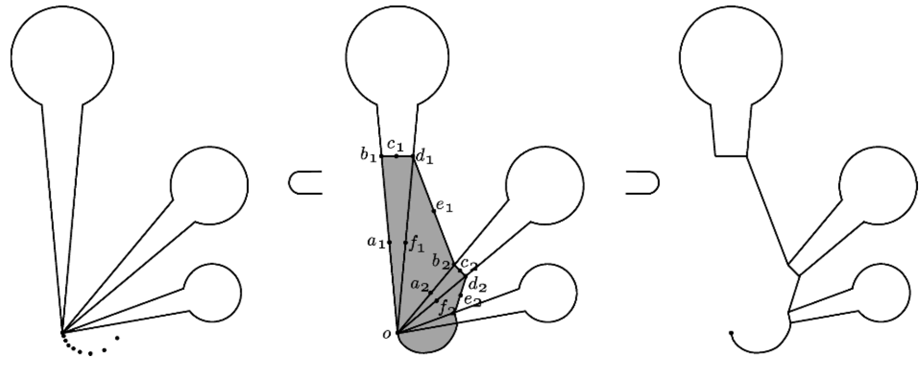

Figure 1. Homotopy representatives of the Hawaiian Earring $\mathbb{H}$.

where

$$C_n^- = \left\{ (x, y, 0) \in \mathbb{R}^3 \,:\, x^2 + \left(y + \frac{1}{n} \right)^2 = \left(\frac{1}{3n(n+1)} \right)^2 \right\}, \qquad n \in \mathbb{N}.$$

Define the *convex hull* $L(M)$ of a subset M of $\mathbb{R}^3$ as the intersection of all convex sets in $\mathbb{R}^3$ containing the set M. For $n \in \mathbb{N}$, the sets F_n^+ and F_n^- are defined as the convex hulls of the quadruples of points of $\mathbb{R}^3$ as follows:

$$F_n^+ = L\left(\left\{ \left(0, \frac{3n+7}{3(n+1)(n+2)}, 0\right), \left(0, \frac{3n+2}{3n(n+1)}, 0\right), \left(0, \frac{1}{n+1}, 1\right), \left(0, \frac{1}{n}, 1\right) \right\} \right)$$

and

$$F_n^- = L\left(\left\{ \left(0, \frac{-3n-7}{3(n+1)(n+2)}, 0\right), \left(0, \frac{-3n-2}{3n(n+1)}, 0\right), \left(0, \frac{-1}{n+1}, -1\right), \left(0, \frac{-1}{n}, -1\right) \right\} \right),$$

respectively. The Modified Griffiths space $\mathcal{MG}$ is then defined as the following subspace of $\mathbb{R}^3$:

$$\mathcal{MG} = HA \cup HA_- \cup F_n^+ \cup F_n^- \cup L\big(\{(0,0,-1), (0,0,1)\}\big),$$

where $L\big(\{(0,0,-1),(0,0,1)\}\big)$ is a compact segment in $\mathbb{R}^3$ with end points at $(0,0,1)$ and $(0,0,-1)$, see Figure 2.

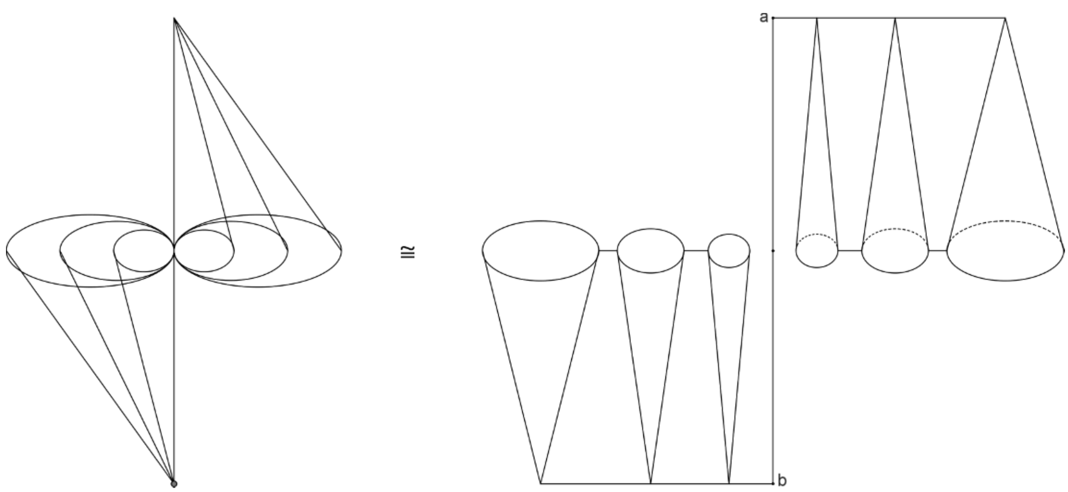

Figure 2. The Griffiths space $\mathcal{G}$ and the Modified Griffiths space $\mathcal{MG}$.

Now, let us also define the *Modified Harmonic Archipelago* $\mathcal{MHA}$. Let $a = (0,0,1)$ and $b = (0,0,-1)$ be two points of the $\mathcal{MG}$ and set

$$\mathcal{MHA} = \mathcal{MG} \setminus \{a,b\}.$$

Proposition 2.2.

Suppose that in the short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{$*$}$$

there exists a projection $p\colon B \to A$ (i.e. p is a homomorphism such that $p \circ \alpha\colon A \to A$ is the identity mapping 1_A). Then there exists an isomorphism $\varphi\colon B \to A \oplus C$ such that $\varphi(b) = (p(b), \beta(b))$ for every $b \in B$.

In this case, the short exact sequence ($*$) is said *to split* [10, Splitting Lemma, p. 147] or is *splitting*, cf. e.g. [7, Lemma 9.1 and p. 38]. The following statement is well known, see e.g. [7].

Proposition 2.3.

If the group A is algebraically compact and C is torsion-free then every exact sequence ($*$) splits.

Throughout this paper only singular homology H_* and Čech cohomology $\check{H}^*$ with integer coefficients will be used. The following statement is a reformulation of a theorem of Eda and Kawamura [6] (they used the notion of p-adic completion).

Proposition 2.4.

For the 1-dimensional singular homology group of the Hawaiian Earrings $H_1(\mathbb{H})$ there exists the following exact sequences which splits:

$$0 \to \left(\prod_{i=1}^{\infty} \mathbb{Z}\right) \bigg/ \left(\sum_{i=1}^{\infty} \mathbb{Z}\right) \to H_1(\mathbb{H}) \xrightarrow{\sigma} \prod_{i=1}^{\infty} \mathbb{Z} \to 0$$

or, equivalently

$$0 \to \left(\sum_{i\in c} \mathbb{Q}\right) \oplus \left(\prod_{p\ \text{prime}} \prod_{i=1}^{\infty} \mathbb{J}_p\right) \to H_1(\mathbb{H}) \xrightarrow{\sigma} \prod_{i=1}^{\infty} \mathbb{Z} \to 0.$$

Proof. It was proved in [6] that there exists the following exact sequence which is splitting:

$$0 \to \left(\prod_{p\ \text{prime}} A_p\right) \oplus \left(\sum_{i\in c} \mathbb{Q}\right) \to H_1(\mathbb{H}) \xrightarrow{\sigma} \prod_{i=1}^{\infty} \mathbb{Z} \to 0,$$

where A_p is the p-adic completion of the direct sum of p-adic integers $\bigoplus_c \mathbb{J}_p$, and c is the continuum cardinal. According to a theorem of Balcerzyk [1], [7, VII.42, Exercise 7], we have

$$\left(\prod_{i=1}^{\infty} \mathbb{Z}\right) \bigg/ \left(\sum_{i=1}^{\infty} \mathbb{Z}\right) \cong \left(\prod_{p\ \text{prime}} A_p\right) \oplus \left(\sum_{i\in c} \mathbb{Q}\right).$$

Therefore the first desired isomorphism follows. The second isomorphism again follows from [6, Theorem 3.1] and [7, Theorem 40.2]. □

Proposition 2.5 (cf. [10]).

Let the space X be a countable union of an increasing system of open sets $\{U_i\}_{i=1}^{\infty}$. Then for the Čech cohomology $\check{H}^*$ there exists the following exact sequence:

$$0 \to \varprojlim{}^{(1)} \check{H}^{n-1}(U_i) \to \check{H}^n(X) \to \varprojlim \check{H}^n(U_i) \to 0,$$

where $\varprojlim^{(1)}$ is the first derived functor of the inverse limit functor $\varprojlim$.

3. Proofs of the main theorems

3.1. Proof of Theorem 1.2

Since the Harmonic Archipelago $\mathcal{HA}$ is homotopy equivalent to the Formal Harmonic Archipelago HA, it suffices to calculate the group $H_1(HA)$. Let U and V be the open sets defined as follows:

$$U = HA \cap \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, z \in \left[0, \frac{2}{3}\right) \right\}, \qquad V = HA \cap \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, z > \frac{1}{3} \right\}.$$

Consider the following part of the Mayer–Vietoris sequences of the triad (HA, U, V):

$$H_2(HA) \to H_1(U\cap V) \xrightarrow{i} H_1(U)\oplus H_1(V) \xrightarrow{j} H_1(HA) \xrightarrow{\delta} H_0(U\cap V) \to H_0(U)\oplus H_0(V).$$

Obviously, V is homotopy equivalent to a countable discrete union of points and the space $U\cap V$ is homotopy equivalent to a discrete countable union of circles.

The homomorphism $H_0(U\cap V) \to H_0(V)$ is an isomorphism since the 0-dimensional homology group is isomorphic to the direct sum of Z cardinality of path-connectedness components [12, Theorem 4.4.5]. The space U is homotopy equivalent to $\mathcal{MH}$ since $\mathcal{MH}$ is a deformation retract of U and $\mathcal{MH} \simeq \mathbb{H}$, therefore $U \simeq \mathbb{H}$.

The singular homology groups are homology groups with compact support, therefore $H_n(HA) \cong \varinjlim H_n(P)$, where P are Peano subcontinua of HA, cf. [12, Theorem 4.4.6]. Obviously, there exists a confinal sequence of Peano continua such that every P is homotopy equivalent to the Hawaiian Earring $\mathbb{H}$ and $H_n(P) \cong 0$ for all $n > 1$ (by [4], H_n are trivial for one-dimensional spaces for all $n > 1$), therefore $H_n(HA) \cong 0$ for all $n > 1$. In particular, $H_2(HA) \cong 0$.

Therefore we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \operatorname{Ker}\sigma \arrow[r, "\cong"] \arrow[d, "\varphi_1"] & \operatorname{Ker}\sigma \arrow[d, "\varphi_2"] & \\
0 \arrow[r] & \sum_{i=1}^{\infty}\mathbb{Z} \arrow[r, "\varphi_3"] \arrow[d, "\cong"] & H_1(H) \arrow[r, "\varphi_4"] \arrow[d, "\sigma"] & H_1(HA) \arrow[r] \arrow[d, "\varphi_5"] & 0 \\
0 \arrow[r] & \sum_{i=1}^{\infty}\mathbb{Z} \arrow[r, "\varphi_6"] & \prod_{i=1}^{\infty}\mathbb{Z} \arrow[r, "\varphi_7"] \arrow[d] & \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

in which homomorphisms φ_3 and φ_4 correspond to i and j, respectively. The homomorphism σ is defined for any element $[l]$ of the $H_1(\mathbb{H})$ as $(l_1, l_2, l_3, \dots)$, where l_i is the winding number of the loop l around the i-th circle S_i, cf. [6, p. 310]. It follows that the composition $\sigma\varphi_3$, which we can identify with φ_6, is a monomorphism and $\operatorname{Im}\varphi_3 \cap \operatorname{Ker}\sigma = 0$. Then the composition $\varphi_1\varphi_4$ is a monomorphism which we shall identify with φ_2.

The homomorphism φ_7 is the quotient mapping. The homomorphism φ_5 is defined as follows. Take any element $a \in H_1(HA)$. Due to exactness of the middle row there exists an element $b \in H_1(H)$ such that $\varphi_4(b) = a$. Define $\varphi_5(a) \equiv$

$\varphi_7\sigma(b)$. Let us show that this mapping is well-defined. Suppose that $\varphi_4(b') = a$. Then the difference $b - b'$ belongs to $\operatorname{Im}\varphi_3$ and $\sigma(b-b') \in \operatorname{Im}\varphi_6$, therefore $\varphi_7(b-b') = 0$. This means that $\varphi_7(b) = \varphi_7(b')$ and the mapping φ_5 is well-defined.

If $a = a_1 + a_2$ then there exist b_1 and b_2 such that $\varphi_4(b_1) = a_1$ and $\varphi_4(b_2) = a_2$. Since σ and φ_7 are homomorphisms we have $\varphi_5(a) = \varphi_7\sigma(b_1 + b_2) = \varphi_7\sigma(b_1) + \varphi_7\sigma(b_2) = \varphi_5(a_1) + \varphi_5(a_2)$ and φ_5 is indeed a homomorphism. Since σ and φ_7 are epimorphisms it follows that φ_5 is an epimorphism. The composition $\varphi_5\varphi_2$ is trivial since the composition $\sigma\varphi_1$ is the zero homomorphism. If $a \in H_1(HA)$ is such that $\varphi_5(a) = 0$ then $\varphi_7\sigma(b) = 0$ for any b for the corresponding a.

Choose one of these elements b. It follows that there exists c such that $\varphi_6(c) = \sigma(b)$. Since the left projection in the diagram is an isomorphism, there exists c' such that $\sigma(\varphi_3(c')) = \sigma(b)$. It follows that $b - \varphi_3(c') = \varphi_1(d)$ for some element d. Then $\varphi_4(b - \varphi_3(c')) = \varphi_4\varphi_1(d)$, but $\varphi_4(\varphi_3(c')) = 0$ and $\varphi_4(b) = a$, therefore $a = \varphi_2(d)$ and the right column is an exact sequence.

By Proposition 2.4 we have that $\operatorname{Ker}\sigma \cong \prod_{i\in\mathbb{N}}\mathbb{Z}/\sum_{i\in\mathbb{N}}\mathbb{Z}$. By Proposition 2.3, the right column splits since the group $\prod_{i\in\mathbb{N}}\mathbb{Z}/\sum_{i\in\mathbb{N}}\mathbb{Z}$ is algebraically compact and torsion-free [7, Corollary 42.2]. Therefore we have the following exact sequence which splits:

$$0 \to \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right) \to H_1(HA) \xrightarrow{p} \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right) \to 0$$

and hence

$$H_1(HA) \cong \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right) \oplus \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right).$$

However, obviously,

$$\left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right) \oplus \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right) \cong \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right),$$

therefore

$$H_1(\mathcal{HA}) \cong H_1(HA) \cong \left(\prod_{i=1}^{\infty}\mathbb{Z}\right)\Big/\left(\sum_{i=1}^{\infty}\mathbb{Z}\right).$$

3.2. Proof of Theorem 1.3

Let $U_1 = U$ (where U was defined in the proof of Theorem 1.2) and U_i at $i > 1$ be the following open subspaces of HA:

$$U_i = HA \cap \left\{(x,y,z) \in \mathbb{R}^3 \,:\, y > \frac{2i+1}{2i(i+1)}\right\} \cup U.$$

Obviously, U_i is homotopy equivalent to the Modified Hawaiian Earring and therefore to the Hawaiian Earring. Since the Hawaiian Earring can be presented as the inverse limit of bouquets of finite numbers of cycles:

$$S_1^1 \xleftarrow{\pi_1} S_1^1 \vee S_2^1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_{n-1}} \bigvee_{j=1}^{n} S_j^1 \xleftarrow{\pi_n} \bigvee_{j=1}^{n+1} S_j^1 \longleftarrow \dots,$$

where the projections π_n map the corresponding circles S_{n+1}^1 to the base point of the bouquets and map all other circles identically, it follows that the 1-dimensional Čech cohomology of the Hawaiian Earring is isomorphic to $\sum_{j=1}^{\infty}\mathbb{Z}$. Therefore $\check{H}^1(U_i) \cong \sum_{j=1}^{\infty}\mathbb{Z}$. By Proposition 2.5, we have the following exact sequences:

$$0 \to \varprojlim{}^{(1)}\check{H}^1(U_i) \to \check{H}^2(HA) \to \varprojlim \check{H}^2(U_i) \to 0.$$

The embedding $U_i \subset U_{i+1}$ generates the monomorphism $\check{H}^1(U_i) \leftarrow \check{H}^1(U_{i+1})$ which we can identify with $\sum_{j=1}^{\infty} \mathbb{Z} \xleftarrow{p} \sum_{j=1}^{\infty} \mathbb{Z}$, where p acts by the rule $p(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots)$, therefore

$$\varprojlim{}^{(1)} \check{H}^1(U_i) \;\cong\; \varprojlim{}^{(1)} \left(\sum_{j=1}^{\infty} \mathbb{Z} \xleftarrow{p} \sum_{j=1}^{\infty} \mathbb{Z} \right).$$

We have following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \sum_{i=1}^{\infty} \mathbb{Z} \arrow[r, "q"] & \sum_{i=1}^{\infty} \mathbb{Z} \arrow[r] & 0 & \\
0 \arrow[r] & \sum_{i=1}^{\infty} \mathbb{Z} \arrow[r, "p"] \arrow[u, "p"] & \sum_{i=1}^{\infty} \mathbb{Z} \arrow[r] \arrow[u, "q"] & \mathbb{Z} \arrow[r] \arrow[u, "l_1"] & 0 \\
0 \arrow[r] & \sum_{i=1}^{\infty} \mathbb{Z} \arrow[r, "p^2"] \arrow[u, "p"] & \sum_{i=1}^{\infty} \mathbb{Z} \arrow[r] \arrow[u, "q"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[u, "l_2"] & 0 \\
 & \cdots \arrow[u, "p"] & \cdots \arrow[u, "q"] & \cdots \arrow[u, "l_3"] &
\end{tikzcd}$$

where q is the identity mapping and $l_n \colon \sum_{i=1}^{n} \mathbb{Z} \to \sum_{i=1}^{n-1} \mathbb{Z}$ is the projection defined by

$$l_n(a_1, a_2, \dots, a_{n-1}, a_n) = (a_1, a_2, \dots, a_{n-1}).$$

For the inverse limit functor and its first derived functor we have following exact sequence [9, Property 5]:

$$\begin{aligned}
0 \to \varprojlim \left(\sum_{i=1}^{\infty} \mathbb{Z} \xleftarrow{p} \sum_{i=1}^{\infty} \mathbb{Z} \right) \to \varprojlim \left(\sum_{i=1}^{\infty} \mathbb{Z} \xleftarrow{q} \sum_{i=1}^{\infty} \mathbb{Z} \right) \to \varprojlim \left(\sum_{i=1}^{n-1} \mathbb{Z} \xleftarrow{l_n} \sum_{i=1}^{n} \mathbb{Z} \right) \to \\
\to \varprojlim{}^{(1)} \left(\sum_{i=1}^{\infty} \mathbb{Z} \xleftarrow{p} \sum_{i=1}^{\infty} \mathbb{Z} \right) \to \varprojlim{}^{(1)} \left(\sum_{i=1}^{\infty} \mathbb{Z} \xleftarrow{q} \sum_{i=1}^{\infty} \mathbb{Z} \right) \to \varprojlim{}^{(1)} \left(\sum_{i=1}^{n-1} \mathbb{Z} \xleftarrow{l_n} \sum_{i=1}^{n} \mathbb{Z} \right) \to 0.
\end{aligned}$$

Obviously,

$$\begin{aligned}
&\varprojlim \left(\sum_{i=1}^{\infty} \mathbb{Z} \xleftarrow{p} \sum_{i=1}^{\infty} \mathbb{Z} \right) \cong 0, &\qquad &\varprojlim \left(\sum_{i=1}^{\infty} \mathbb{Z} \xleftarrow{q} \sum_{i=1}^{\infty} \mathbb{Z} \right) \cong \sum_{i=1}^{\infty} \mathbb{Z}, \\
&\varprojlim \left(\sum_{i=1}^{n-1} \mathbb{Z} \xleftarrow{l_n} \sum_{i=1}^{n} \mathbb{Z} \right) \cong \prod_{i=1}^{\infty} \mathbb{Z}, &\qquad &\varprojlim{}^{(1)} \left(\sum_{i=1}^{\infty} \mathbb{Z} \xleftarrow{q} \sum_{i=1}^{\infty} \mathbb{Z} \right) \cong 0,
\end{aligned}$$

since q is the identity mapping. It follows from this diagram that

$$\varprojlim{}^{(1)} \left(\sum_{i=1}^{\infty} \mathbb{Z} \xleftarrow{p} \sum_{i=1}^{\infty} \mathbb{Z} \right) \cong \left(\prod_{i=1}^{\infty} \mathbb{Z} \right) \Big/ \left(\sum_{i=1}^{\infty} \mathbb{Z} \right),$$

therefore

$$\varprojlim{}^{(1)} \check{H}^1(U_i) \cong \left(\prod_{i=1}^{\infty} \mathbb{Z} \right) \Big/ \left(\sum_{i=1}^{\infty} \mathbb{Z} \right).$$

Since U_i are homotopy equivalent to 1-dimensional space (Hawaiian Earring), it follows that $\varprojlim \check{H}^2(U_i) \cong 0$ and by Proposition 2.5 we have

$$\check{H}^2(HA) \cong \left(\prod_{i=1}^{\infty} \mathbb{Z}\right) \Big/ \left(\sum_{i=1}^{\infty} \mathbb{Z}\right).$$

Since, as it was mentioned in the proof of Theorem 1.3, $\varprojlim \check{H}^1(U_i) \cong 0$, it follows from the exact sequence

$$0 \to \varprojlim{}^{(1)} \check{H}^0(U_i) \to \check{H}^1(HA) \to \varprojlim \check{H}^1(U_i) \to 0$$

that $\check{H}^1(HA) \cong 0$. Since $\dim HA = 2$ it follows that $\check{H}^n(HA) \cong 0$ for all $n > 2$.

Remark 3.1.
From the homotopical point of view, the spaces $\mathcal{G}$ and $\mathcal{HA}$ are very close to each other – it is possible to show that $\mathcal{G}$ and $\mathcal{HA}$ are homotopy equivalent to $\mathcal{MG}$ and $\mathcal{MHA}$, respectively. However, by definition of $\mathcal{MHA}$, it follows that

$$\mathcal{MHA} = \mathcal{MG} \setminus \{a, b\},$$

for some pair of points a, b (see Figures 2 and 3).

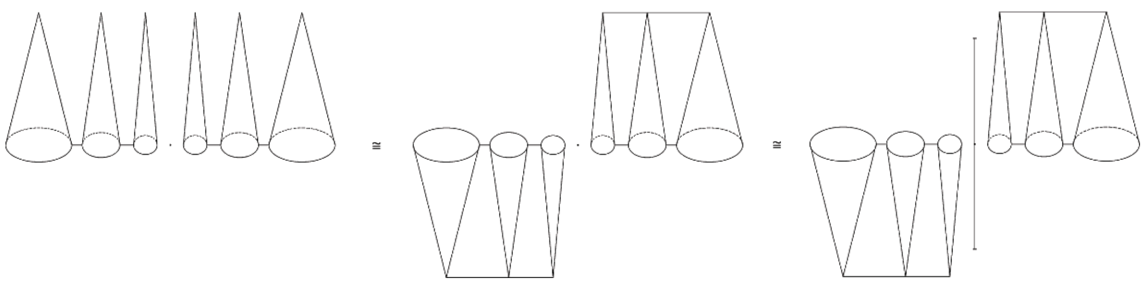

Figure 3. Homotopy representatives of the Harmonic Archipelago $\mathcal{HA}$.

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