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On the topological Helly theorem

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Abstract

The main result of this paper is the following theorem, related to the missing link in the proof of the topological version of the classical result of Helly: Let $\{X_i\}_{i=0}^2$ be any family of simply connected compact subsets of $\mathbb{R}^2$ such that for every $i, j \in \{0, 1, 2\}$ the intersections $X_i \cap X_j$ are path connected and $\bigcap_{i=0}^2 X_i$ is nonempty. Then for every two points in the intersection $\bigcap_{i=0}^2 X_i$ there exists a cell-like compactum connecting these two points, in particular the intersection $\bigcap_{i=0}^2 X_i$ is a connected set.

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1. Introduction

A topological space X is said to be *simply connected* if it is path connected and has a trivial fundamental group, $\pi_1(X) = 1$. It is well known that for every subspace $X \subset \mathbb{R}^2$ of the plane, $\pi_1(X) = 1$ if and only if for every Jordan curve $\mathcal{J} \subset X$ and every point

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$y \in \mathbb{R}^2 \backslash \mathcal{J}$ from the bounded component of $\mathbb{R}^2 \backslash \mathcal{J}$, y lies in X (see, e.g. [13, Chapter 10, §61. II, Theorem 5] or [16, p. 107, Proposition 2.51]) or equivalently, no Jordan curve $\mathcal{J} \subset X$ is a retract of X.

Throughout this paper all singular and Čech (co)homology groups will be assumed to have the integer coefficients $\mathbb{Z}$. A topological space X is called a *singular cell* if all its singular homology groups are trivial, $H_*(X) = H_*(pt)$. Next, X is said to be *acyclic* if all its Čech cohomology groups are trivial, $\check{H}^*(X) = \check{H}^*(pt)$. A planar compactum is acyclic if and only if it is cell-like (see, e.g. [6]). A space X is said to be *cell-like connected* if for every two points a and b there exists cell-like continuum C in X such that $a, b \in C$.

If a subspace $X \subset \mathbb{R}^2$ of the plane is not simply connected then, as it was mentioned above, X contains a Jordan curve $\mathcal{T} \subset X$ which is a retract of X and therefore the group $H_1(X)$ cannot be trivial. If a space X is simply connected then by the Hurewicz Theorem (see, e.g. [15, Theorem VII.5.5]), all homotopy groups of X are naturally isomorphic to the corresponding singular homology groups of X. However, all planar spaces are aspherical (see, [17,3]). Therefore a subspace X of the plane $\mathbb{R}^2$ is a singular cell if and only if X is simply connected.

On the other hand, there exist simply connected spaces which are not acyclic (e.g. the Warsaw circle, see [14, p. 5]). The following classical result is due to Helly (see, e.g. [5,8, 10]):

Theorem 1.1 *(Topological Helly Theorem). Let $\mathcal{K} = \{K_i\}_{i=0}^m$, $m \geqslant n$, be any finite family of closed subsets of the n-dimensional Euclidean space $\mathbb{R}^n$ such that the intersection of every k members of $\mathcal{K}$, is a singular cell, for every $k \leqslant n$, and is nonempty, for $k = n + 1$. Then the intersection $\bigcap_{i=0}^m K_i$ is a singular cell.*

All known proofs of Theorem 1.1. are inductive and the initial step (i.e. when $m = n = 2$) is based on the following assertion:

(∗) *Any family $\{X_i\}_{i=0}^2$ of three simply connected compact subsets of the plane $\mathbb{R}^2$ has a simply connected intersection provided that the intersection $X_i \cap X_j$, $i, j \in \{0, 1, 2\}$ of any two of its members is path connected and the intersection $\bigcap_{i=0}^2 X_i$ of all three members is nonempty.*

Apparently, for several years nobody questioned the validity of assertion (∗). However, Bogatyi [1, p. 399] has recently pointed out that no complete proof of (∗) can be found in the existing literature.

Any intersection $\bigcap_{\lambda \in \Lambda} X_\lambda$ of simply connected subsets $X_\lambda \subset \mathbb{R}^2$, $\lambda \in \Lambda$, has a trivial fundamental group, with respect to any of its points. Indeed, consider any Jordan curve $\mathcal{J} \subset \bigcap_{\lambda \in \Lambda} X_\lambda$. Since by hypothesis, every element X_λ, of the family is simply connected, the bounded region of $\mathbb{R}^2$ determined by $\mathcal{J}$ is a subset of X_λ, therefore it is a subset of the intersection $\bigcap_{\lambda \in \Lambda} X_\lambda$ for every $\lambda \in \Lambda$. Consequently, the fundamental group of the intersection is trivial, $\pi_1(\bigcap_{\lambda \in \Lambda} X_\lambda) = 1$. Hence, in order to prove assertion (∗) it is necessary to verify that the intersection $\bigcap_{i=0}^2 X_i$ of all three sets is *path connected.* In the present paper we provide the first step towards filling this gap—by establishing the cell-like connectedness of $\bigcap_{i=0}^2 X_i$:

Theorem 1.2. *A family $\{X_i\}_{i=0}^2$ of three simply connected compact subsets of $\mathbb{R}^2$ has a cell-like connected intersection $\bigcap_{i=0}^2 X_i$ provided that the intersection $X_i \cap X_j$, $i, j \in \{0, 1, 2\}$, of any two of its members is path connected and the intersection $\bigcap_{i=0}^2 X_i$ of all three is nonempty.*

The corresponding result for acyclic spaces can be found in [4,7,9,12]. The requirement from assertion $(*)$ that the intersection $X_i \cap X_j$ of any two of the sets be *path connected* cannot be weakened to just *connectedness*—as the following result from [11] demonstrates:

Theorem 1.3. *There exist three simply connected compact subsets of $\mathbb{R}^2$ such that intersection of any two of these sets is connected and the intersection of all three of them is a disconnected two-point set.*

2. Preliminaries

Lemma 2.1. *Let A and B be disjoint subcontinua of a compactum X. Then there exists a continuum $C \subset X$ such that $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$ if and only if the inclusion-induced homomorphism $\varphi : \check{H}^0(X) \to \check{H}^0(A \cup B)$ of the Čech cohomology groups is not an epimorphism.*

Proof. ($\Rightarrow$) Suppose that there exists a continuum $C \subset X$ *connecting* A and B, i.e. $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$. Then obviously, $\check{H}^0(A \cup B \cup C) \cong \mathbb{Z}$. Since $A \cap B = \emptyset$ it follows that $\check{H}^0(A \cup B) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the composition of the inclusion-induced homomorphisms $\check{H}^0(X) \to \check{H}^0(A \cup B \cup C) \to \check{H}^0(A \cup B)$ cannot be an epimorphism.

($\Leftarrow$) Conversely, let U be a *clopen* (i.e. open and closed) subset of X which contains A. Such a set always exists, take for example X. Let C be the intersection of all such sets (i.e. C is the *quasi-component* of the set A). Note that the quasi-component of any compact space is always a continuum (see, e.g. [13, Chapter 5, §47. II, Theorem 2]).

Suppose that $B \cap C = \emptyset$. Then there exists a clopen set $U \subset X$ which contains A and does not intersect B. Recall that zero-dimensional Čech cohomology $\check{H}^0(Y)$ is always naturally isomorphic to the group of locally constant functions from Y into the group of integers $\mathbb{Z}$ with the discrete topology. Now, since A and B are connected and U is clopen in X, any locally constant function on A U B in this case can be extended over $A \cup B \cup U$ and hence over X. Therefore φ must be an epimorphism. Contradiction. □

Lemma 2.2. *Let C and D be acyclic subcontinua of the plane $\mathbb{R}^2$. Then each component of connectedness of the intersection $C \cap D$ is an acyclic continuum.*

Proof. Consider the cohomology Mayer–Vietoris exact sequence:

$$\cdots \to \check{H}^1(C) \oplus \check{H}^1(D) \to \check{H}^1(C \cap D) \to \check{H}^2(C \cup D) \to \cdots$$

Since C and D are acyclic spaces and $C \cup D$ is a planar set we have that $\check{H}^1(C) \cong \check{H}^1(D) \cong \check{H}^2(C \cup D) \cong 0$. It follows that $\check{H}^1(C \cap D) \cong 0$. Again by the Mayer–Vietoris exact sequence it follows that for every quasi-component A of $C \cap D$ the first cohomology vanishes, $\check{H}^1(A) \cong 0$. Since in compact spaces every quasi-component is a component and for any planar set M the higher Čech cohomologies are trivial, $\check{H}^n(M) = 0,\ n \geqslant 2$, it follows that every component of $C \cap D$ is an acyclic space. □

Let Δ_n, $n \in \mathbb{N}$, be the standard n-dimensional simplex $[e_0 e_1 \cdots e_n]$ with vertices $e_0, e_1, \ldots, e_n$. Let I^{n+1} be the $(n+1)$-dimensional prism $\Delta_n \times [0,1]$. Let $I_{[i_0 i_1 \cdots i_k]}$, $0 \leqslant k \leqslant n$, be its $(k+1)$-dimensional face $[e_{i_0} e_{i_1} \cdots e_{i_k}] \times [0,1]$, generated by the vertices $e_{i_0}, e_{1_1}, \ldots, e_{i_k}$. Denote by $A = \Delta_n \times \{1\}$ and $B = \Delta_n \times \{0\}$ the top and the bottom faces of the prism, respectively. Let $J_i = A \cup B \cup I_i$, where $I_i = I_{[01 \cdots \hat{i} \cdots n]}$ is the n-dimensional face generated by all vertices $e_0, e_1, \ldots, e_n$, except the vertex e_i.

The following result is of its own interest and its special case for $n = 2$ will play the key role in the proof of our Theorem 1.3:

Proposition 2.3. *Suppose that the prism I^{n+1} is covered by a family $\{F_i\}_{i=0}^n$ of closed sets and that for every i, the face I_i is contained in F_i. Then there exists a continuum $C \subset \bigcap_{i=0}^n F_i$ such that $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$.*

3. Proof of Proposition 2.3: Special case

First, suppose that $J_i \subset F_i$. By Lemma 2.1 it suffices to prove that the inclusion-induced homomorphism $\check{H}^0(\bigcap_{i=0}^n F_i) \to \check{H}^0(A \cup B)$ is not an epimorphism. From the Mayer–Vietoris exact sequence for the pair $(\bigcap_{i=0}^{n-k-1} F_i, \bigcup_{j=n-k}^n F_j)$ and the equalities:

$$\bigcap_{i=0}^{n} F_i = \left(\bigcap_{i=0}^{n-1} F_i\right) \cap F_n,$$

$$\left(\bigcap_{i=0}^{n-k} F_i\right) \cup \left(\bigcup_{j=n-k+1}^{n} F_j\right)$$
$$= \left(\bigcup_{i=0}^{n-k-1} F_i \cup \left(\bigcup_{j=n-k+1}^{n} F_j\right)\right) \cap \left(F_{n-k} \cup \left(\bigcup_{j=n-k+1}^{n} F_j\right)\right),$$

and

$$\left(\bigcap_{i=0}^{n-k-1} F_i \cup \left(\bigcup_{j=n-k+1}^{n} F_j\right)\right) \cup \left(F_{n-k} \cup \left(\bigcup_{j=n-k+1}^{n} F_j\right)\right)$$
$$= \left(\bigcap_{i=0}^{n-k-1} F_i\right) \cup \left(\bigcup_{j=n-k}^{n} F_j\right),$$

we get for $k = 0$ the natural boundary homomorphism:

$$\check{H}^0\left(\bigcap_{i=0}^{n} F_i\right) \to \check{H}^1\left(\left(\bigcap_{i=0}^{n-1} F_i\right) \cup F_n\right),$$

whereas for every $1 \leqslant k \leqslant n$ we obtain the homomorphisms:

$$\check{H}^k\left(\left(\bigcap_{i=0}^{n-k} F_i\right) \cup \left(\bigcup_{j=n-k+1}^{n} F_j\right)\right) \to \check{H}^{k+1}\left(\left(\bigcap_{i=0}^{n-k-1} F_i\right) \cup \left(\bigcup_{j=n-k}^{n} F_j\right)\right).$$

The composition of these homomorphisms for $k = 0, 1, \ldots, (n-1)$ yields the following homomorphism:

$$\check{H}^0\left(\bigcap_{i=0}^{n} F_i\right) \to \check{H}^n\left(\bigcup_{j=0}^{n} F_j\right).$$

By the Mayer–Vietoris exact sequence for the pair $(\bigcap_{i=0}^{n-k-1} J_i, \bigcap_{j=n-k}^{n} J_j)$ we obtain for $k = 0$ the following natural epimorphism:

$$\check{H}^0\left(\bigcap_{i=0}^{n} J_i\right) \to \check{H}^1\left(\left(\bigcap_{i=0}^{n-1} J_i\right) \cup J_n\right) \to 0$$

and for every $k \in \{1, \ldots, n\}$ the following epimorphisms:

$$\check{H}^k\left(\left(\bigcap_{i=0}^{n-k} J_i\right) \cap \left(\bigcup_{j=n-k+1}^{n} J_j\right)\right) \to \check{H}^{k+1}\left(\left(\bigcap_{i=0}^{n-k-1} J_i\right) \cap \left(\bigcup_{j=n-k}^{n} J_j\right)\right) \to 0$$

since the spaces $\bigcap_{i=0}^{n-k-1} J_i$ and $\bigcup_{j=n-k}^{n} J_j$ are contractible for every $k = 0, 1, \ldots, (n-1)$.

Since $\bigcap_{i=0}^{n} J_i = A \cup B$, the composition of these homomorphisms for $k = 0, 1, \ldots, n$ gives an epimorphism $\delta : \check{H}^0(A \cup B) \to \check{H}^n(\bigcup_{j=0}^{n} J_j)$. So we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\check{H}^0(\bigcap_{i=0}^{n} F_i) & \longrightarrow & \check{H}^n(\bigcup_{j=0}^{n} F_j) & & \\
\downarrow \varphi^0 & & \downarrow \varphi^n & & \\
\check{H}^0(A \cup B) & \xrightarrow{\delta} & \check{H}^n(\bigcup_{j=0}^{n} J_j) & \longrightarrow & 0
\end{array}$$

Since $\bigcup_{j=0}^{n} F_j = I^{n+1}$ and I^{n+1} is a contractible space, the group $\check{H}^n(\bigcup_{j=0}^{n} F_j)$ is trivial and so the homomorphism φ^n must also be trivial. However, the epimorphism δ is not trivial since $\bigcup_{j=0}^{n} J_j = \partial(I^{n+1})$ and $H^n(\bigcup_{j=0}^{n} J_j) \cong \mathbb{Z}$. Therefore φ^0 cannot be an epimorphism. Hence by Lemma 2.1 there must exist a continuum $C \subset \bigcap_{i=0}^{n} F_i$ which connects A and B.

4. Proof of Proposition 2.3: General case

Suppose now that $I_i \subset F_i$. Let $G_i = F_i \cup A \cup B$. As we have already proved in Chapter 3, there exists a continuum $C \subset \bigcap_{i=0}^{n} G_i$ which connects A and B. Let $C_0 = C \cap (\bigcap_{i=0}^{n} F_i)$ and let C_x be the component of the point x in the space C_0. Let M be a clopen set in C_0 containing C_x. Then M intersects either A or B. Indeed, if $M \cap A = \emptyset$ and $M \cap B = \emptyset$ then for some open in C set U we would get $M = C_0 \cap U = C_0 \cap (U \setminus (A \cup B)) = U \setminus (A \cup B)$ and M would be clopen in C. However, this is impossible since C is a continuum.

It follows that C_x must intersect $A \cup B$. Consider now the union $\bigcup_{x \in C_0 \cap A} C_x$. This space is closed in C_0. Indeed, consider any limit point x_0 of the set $\bigcup_{x \in C_0 \cap A} C_x$. Let M be a clopen set in C_0 containing x_0. Since sets C_x are connected it follows that either $C_x \subset M$ or $C_x \cap M = \emptyset$. Since x_0 is the limit point there exists x such that $C_x \subset M$. It follows that $M \cap A \neq \emptyset$. Thus $C_{x_0} \cap A \neq \emptyset$ and $\bigcup_{x \in C_0 \cap A} C_x$ is a closed in C_0, hence a compact space.

Similarly, $\bigcup_{x \in C_0 \cap B} C_x$ is a compact space. It follows that

$$C \subset \left(A \cup \left(\bigcup_{x \in C_0 \cap A} C_x\right)\right) \cup \left(B \cup \left(\bigcup_{x \in C_0 \cap B} C_x\right)\right)$$

and since C is connected

$$\left(A \cup \left(\bigcup_{x \in C_0 \cap A} C_x\right)\right) \cap \left(B \cup \left(\bigcup_{x \in C_0 \cap B} C_x\right)\right) \neq \emptyset.$$

Therefore for some x, $C_x \cap A \neq \emptyset$ and $C_x \cap B \neq \emptyset$. So there again exists a continuum $C_x \subset \bigcap_{i=0}^{n} F_i$ which connects A and B.

5. Proof of Theorem 1.2

Consider any two points a and b of the intersection $\bigcap_{i=0}^{2} X_i$. Since $X_i \cap X_{i+1}$ is path connected (indices are considered mod 3) there exists an arc γ_i in $X_i \cap X_{i+1}$, connecting a and b. The union $\gamma_i \cup \gamma_{i+1} \subset X_{i+1}$ is a Peano continuum. Let R_{i+1} be the unbounded complementary domain of $\gamma_i \cup \gamma_{i+1}$ in the plane $\mathbb{R}^2$. Let C_{i+1} be the union of $\gamma_i \cup \gamma_{i+1}$ with all bounded complementary domains. The boundaries of R_i and C_i are the same. It follows by the characterization theorem for planar continua [16, p. 113] that C_i are simply connected Peano continua, for every i. By the Borsuk Theorem [2, Theorem 13.1, Chapter V] we can therefore conclude that all C_i are *AR's*. Since X_i is simply connected, it follows that $C_i \subset X_i$, for every i.

We shall associate to points a and b of the intersection $\bigcap_{i=0}^{2} C_i$, the mapping $f : I^3 \to \bigcup_{i=0}^{2} C_i$ of the prism I^3 in the following manner. Let f^0 map the faces A and B (defined in Chapter 2) to points a and b, respectively. Let f^1 be a mapping $f^1 : A \cup B \cup (\bigcup_{i=0}^{2} I_{[i]}) \to \bigcup_{i=0}^{2} X_i$, which maps $I_{[i]}$, $i \in \{0, 1, 2\}$, bijectively on the corresponding γ_i.

Since the sets C_i are simply connected there exists a mapping $f^2 : \partial(I^3) \to \bigcup_{i=0}^{2} C_i$ which is an extension of f^1. Now, all planar subsets are known to be aspherical ([17], see also [3]), so there exists an extension $f : I^3 \to \bigcup_{i=0}^{2} C_i$ of the mapping f^2 such that $J_i \subset f^{-1}(C_i)$.

By Proposition 2.3 there exists a continuum $C \subset \bigcap_{i=0}^{2} f^{-1}(C_i)$ which connects A and B. Then $f(C) \subset \bigcap_{i=0}^{2} C_i$ and $f(C)$ is a continuum. By Lemma 2.2 the component of connectedness of $\bigcap_{i=0}^{2} C_i$ containing $f(C)$ is acyclic and therefore a cell-like continuum connecting a and b in $\bigcap_{i=0}^{2} X_i$. Since a, b were arbitrary points of the intersection $\bigcap_{i=0}^{2} X_i$ if follows that this intersection is a cell-like connected set.

6. Epilogue

We remark that a special case of assertion (∗), namely for Peano continua, has recently been verified by Bogatyi [1]:

Theorem 6.1. *Any finite family of simply connected Peano continua in $\mathbb{R}^2$ has a nonempty simply connected intersection, provided that intersection of any two of its members is connected and the intersection of any three of its members is nonempty.*

Bogatyi's proof of Theorem 6.1 is based on the following technical lemma [1, p. 395]:

Lemma 6.2. *Suppose that the square $[0,1]\times[0,1]$ is a union of two closed sets B_0 and B_1 such that $\{i\}\times[0,1]\subset B_i$, $i\in\{0,1\}$. Then there exists a continuum $C\subset B_0\cap B_1$ such that $C\cap([0,1]\times\{i\})\neq\emptyset$, $i\in\{0,1\}$.*

We wish to point out that Lemma 6.2 follows from our Proposition 2.3 (for $n=1$). We shall conclude the paper by the following conjecture, a positive answer to which would prove Assertion (∗).

Conjecture 6.3. *Every component of the intersection of any finite family of planar ARs is an AR.*

Note that there exist two topological disks X_1 and X_2 in $\mathbb{R}^3$ such that the intersection $X_1\cap X_2$ is homeomorphic to the Topologist's Sine Curve T and hence is not an *AR*. Indeed, let X_1 be the square $[0,1]\times[0,1]\times\{0\}\subset\mathbb{R}^3$. The set T can be considered as a subspace of X_1. Let X_2 be the square X_1 slightly deformed in such a way that only the points which do not belong to T are moved to the points with the same first and second coordinates and with the positive third coordinate. Obviously, such a deformation always exists and the intersection $X_1\cap X_2$ is clearly homeomorphic to T, as asserted.

There also exist two Peano continua Y_1 and Y_2 in $\mathbb{R}^2$ such that the intersection $Y_1\cap Y_2$ is homeomorphic to the Topologist's Sine Curve T and hence is not an *AR*. Let us demonstrate this: define the following subsets of the plane:

$$A=\{(x,y)\in\mathbb{R}^2\mid x\in[0,1]\text{ and }y=0\text{ or }y=1/n,\ n\in\mathbb{N}\},$$
$$B_{n,m}=\{(m/2^n,y)\in\mathbb{R}^2\mid y\in[0,1/2^{n-1}],\ 0<m<2^n,\ n\in\mathbb{N}\},$$
$$C_{n,m}=\{(m/3^n,y)\in\mathbb{R}^2\mid y\in[0,1/2^{n-1}],\ 0<m<3^n,\ n\in\mathbb{N}\},$$

and

$$D_n=\{(((-1)^n+1)/2,y)\in R^2\mid y\in[1/2^n,1/2^{n-1}],\ n\in\mathbb{N}\}.$$

Define the planar Peano continua Y_1 and Y_2 as follows:

$$Y_1=A\cup\left(\bigcup_{n=1}^{\infty}D_n\right)\cup\left(\bigcup_{n=1}^{\infty}\left(\bigcup_{0<m<2^n}B_{n,m}\right)\right)$$

and

$$Y_2 = A \cup \left(\bigcup_{n=1}^{\infty} D_n\right) \cup \left(\bigcup_{n=1}^{\infty}\left(\bigcup_{0<m<2^n} C_{n,m}\right)\right).$$

Obviously, $Y_1 \cap Y_2 = A \cup (\bigcup_{n=1}^{\infty} D_n) \cong T$, so our assertion follows.

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