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On the union of simply connected planar sets

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Abstract

We prove that the union of any two simply connected compact subspaces of the plane is simply connected if their intersection is path connected and cellular. We also show that there exist two simply connected compact subspaces of the plane with a simply connected intersection and a nonsimply connected union. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

It follows by the Seifert–van Kampen theorem that a topological space is simply connected if it is the union of two simply connected open subspaces, the intersection of which is path connected (see, e.g., [9]). The condition of openness in this statement is necessary since for example Griffiths [4] constructed a nonsimply connected compact subset in $\mathbb{R}^3$ which is the union of two contractible compact spaces with a one-point intersection. In the present paper we investigate unions of two simply connected closed subspaces of $\mathbb{R}^2$. Our main results are:

Theorem 1.1. *Let $X \subset \mathbb{R}^2$ be the union of two simply connected compact subspaces $X_1, X_2 \subset X$ and let the intersection $X_1 \cap X_2$ be path connected and cellular. Then X is simply connected.*

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Theorem 1.2. *There exist two simply connected compact subspaces of $\mathbb{R}^2$ with a simply connected intersection and a nonsimply connected union.*

Any path connected planar continuum is simply connected if and only if it has the fixed-point property [5, Theorem 9.1], so we also obtain some results which are connected with the additivity of the fixed-point property for planar continua.

2. Preliminaries

We shall use the notations and definitions from the [1–3,5,7]. By a *topological interval* we mean a space homeomorphic to the open interval (0,1). Every topological interval in the circle S^1 has two ends if its closure does not equal the circle, so every topological interval can be denoted as (a, b), where $a, b \in S^1$ (on S^1 we have a fixed clockwise orientation).

A space is said to be *simply connected* if it is path connected and its fundamental group is trivial. A subset A of $\mathbb{R}^2$ is called *cellular* if there exists a sequence D_i of 2-disks in $\mathbb{R}^2$ such that $D_{i+1} \subset \operatorname{Int} D_i$ $(i = 1, 2, \ldots)$ and $A = \bigcap_{i=1}^{\infty} D_i$.

A compact subspace of $\mathbb{R}^2$ is cellular if and only if it is acyclic in Čech (co)-homology, or if it is cell-like, or it has trivial shape [2]. A path connected planar continuum X has a trivial fundamental group if and only if every Jordan curve in X bounds a disk in X (see [5]).

3. Proof of Theorem 1.1

Suppose the assertion was false. Since X is obviously path connected, there exists a Jordan curve $f : S^1 \to X_1 \cup X_2 \subset \mathbb{R}^2$ which does not bound any disk in X [5, p. 4525]. Therefore there exists a point $a \in \mathbb{R}^2$ which belongs to the bounded component of $\mathbb{R}^2 \setminus \operatorname{Im} f$ and which does not lie in X. Since $\operatorname{Im} f$ is a retract of $\mathbb{R}^2 \setminus \{a\}$ (see, e.g., [7, Theorem 10, §61.IV], f is a homotopically essential map of S^1 into $\mathbb{R}^2 \setminus \{a\}$.

Since $X_1 \cap X_2$ is cellular in $\mathbb{R}^2$ and $a \notin X_1 \cap X_2$, there exists a disk D such that $X_1 \cap X_2 \subset \operatorname{Int} D \subset X \setminus \{a\}$. Since $X_1 \cap X_2$ is compact, there exists a number $\varepsilon > 0$ such that the ε-neighborhood $N_\varepsilon(X_1 \cap X_2)$ of $X_1 \cap X_2$ lies in D, i.e., $N_\varepsilon(X_1 \cap X_2) \subset D$. Since f is uniformly continuous there exists a $\delta > 0$ such that $\rho(f(x_1), f(x_2)) < \varepsilon$, whenever $\rho(x_1, x_2) < \delta$.

The open set $f^{-1}((X_1 \cup X_2) \setminus (X_1 \cap X_2))$ is a disjoint union of topological intervals in S^1. Only a finite number of them, say

$$(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \quad (i = 1, 2, \ldots, n),$$

can be of diameter at least δ.

Since $f(a_i)$ and $f(b_i)$ lie in $X_1 \cap X_2$ and since $X_1 \cap X_2$ is path connected, there is an arc $f_i' : [a_i, b_i] \to X_1 \cap X_2$ for every i, connecting the points a_i and b_i . Since intervals (a_i, b_i) are connected, $f^{-1}(X \setminus X_1)$ and $f^{-1}(X \setminus X_2)$ are open and disjoint, and since $(a_i, b_i) \subset f^{-1}(X \setminus X_1) \cup f^{-1}(X \setminus X_2)$, it follows that $f([a_i, b_i]) \subset X_1$ or $f([a_i, b_i]) \subset X_2$.

Since X_1 and X_2 are simply connected, the restriction $f|_{[a_i,b_i]}$ is homotopic rel the points a_i, b_i to f_i' in $X_1 \cup X_2$ and hence also in $\mathbb{R}^2 \setminus \{a\}$.

Therefore f is homotopic in $\mathbb{R}^2 \setminus \{a\}$ to the mapping f':

$$f'(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{i=1}^n (a_i, b_i), \\ f_i'(x) & \text{if } x \in (a_i, b_i). \end{cases}$$

However, Im $f' \subset D$, hence f is inessential in $\mathbb{R}^2 \setminus \{a\}$. Contradiction.

4. Proof of Theorem 1.2

Consider in $\mathbb{R}^2$ the "condensed sinusoid":

$$T = \left\{(a,b) \in \mathbb{R}^2 \mid 0 < a \leqslant 1,\ b = \sin\left(\frac{2\pi}{a}\right) \text{ or } a = 0,\ -1 \leqslant b \leqslant 1\right\}.$$

Let A (respectively, B) be the union of the convex hulls of the components of the path connectedness of the intersection

$$T \cap \left\{(a,b) \in \mathbb{R}^2 \mid b \geqslant 0\right\},$$

and

$$T \cap \left\{(a,b) \in \mathbb{R}^2 \mid b \leqslant 0\right\},$$

respectively. Let L be any arc in $\mathbb{R}^2$ with end points in $(0,-1), (1,0)$, which does not intersect $A \cup B$ in any other point. Let $X_1 = A \cup L \cup T$, $X_2 = B \cup L \cup T$ and $X = X_1 \cup X_2$.

Then the compactum X is not simply connected because it contains the topological circle $S^1 = L \cup \{(a,b) \in \mathbb{R}^2 \mid 0 \leqslant a \leqslant 1,\ b = 0\}$ which is its retract. But X_1, X_2 and $X_1 \cap X_2$ are all simply connected, since every Peano subcontinuum of these spaces is obviously contractible (see Fig. 1).

The following related result was communicated to us by K. Omiljanowski, answering our Question 5.5 from [6].

Theorem 4.1. *If a* 1*-dimensional subset $X \subset \mathbb{R}^2$ is a union of two simply connected compact spaces X_1, X_2 such that the intersection $X_1 \cap X_2$ is path connected, then X is simply connected.*

Proof. Indeed, X is path connected since it is the union of two path connected spaces with a path connected intersection. Suppose that X was not simply connected. There is a simple closed curve $S \subset X$ which cannot lie in either X_1 or in X_2 (since X_1 and X_2 are both simply connected and 1-dimensional (see [5, p. 4525])). Let (a,b) be a component of $S \setminus X_2$. The points a, b belong to $X_1 \cap X_2$. Therefore there is an arc $L \subset X_1 \cap X_2$ which joins the points a and b. Then $J = L \cup [a,b]$ is a simple closed curve which is contained in X_1. Since X_1 is 1-dimensional and simply connected, this contradicts [5]. □

5. Connection with the fixed-point theory

By [5, Theorem 9.1], Theorems 1.1, 4.1 and 1.2, respectively, can be reformulated in terms of the fixed-point theory as follows:

Theorem 5.1. *Let $X \subset \mathbb{R}^2$ be the union of two path connected compact subspaces $X_1, X_2 \subset X$ with the fixed-point property and let the intersection $X_1 \cap X_2$ be path connected and cellular. Then X has the fixed-point property.*

Theorem 5.2. *Let a 1-dimensional space $X \subset \mathbb{R}^2$ be the union of two path connected compact subspaces $X_1, X_2 \subset X$ with the fixed-point property and let the intersection $X_1 \cap X_2$ be path connected. Then X has the fixed-point property.*

Theorem 5.3. *There exist two closed subspaces $X_1, X_2 \subset \mathbb{R}^2$ such that*:

(a) *$X_1, X_2, X_1 \cap X_2$ have the fixed-point property*;
(b) *$X_1, X_2, X_1 \cap X_2$ are path connected*; *and*
(c) *$X_1 \cup X_2$ does not have the fixed-point property.*

The planar continuum $X = X_1 \cup X_2$ (see Fig. 1) does not have the fixed-point property (the composition of the retraction of X onto S^1 and the rotation of S^1 is a continuous mapping of X into X without fixed points). But all spaces $X_1, X_2, X_1 \cap X_2$ are simply connected and by [5, Theorem 9.1] they have the fixed-point property. It is interesting to compare this result with results of Štan'ko [10], Yandl [8,11] and Bing [1], stated below:

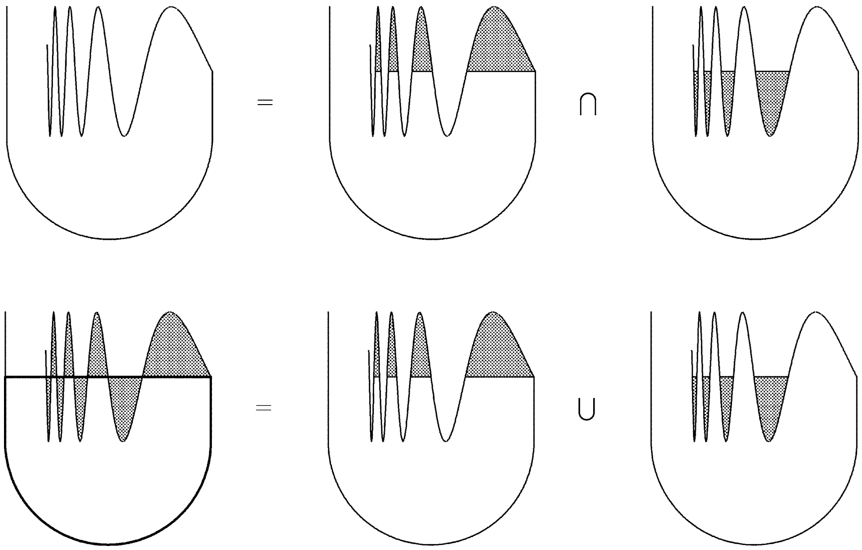

Fig. 1.

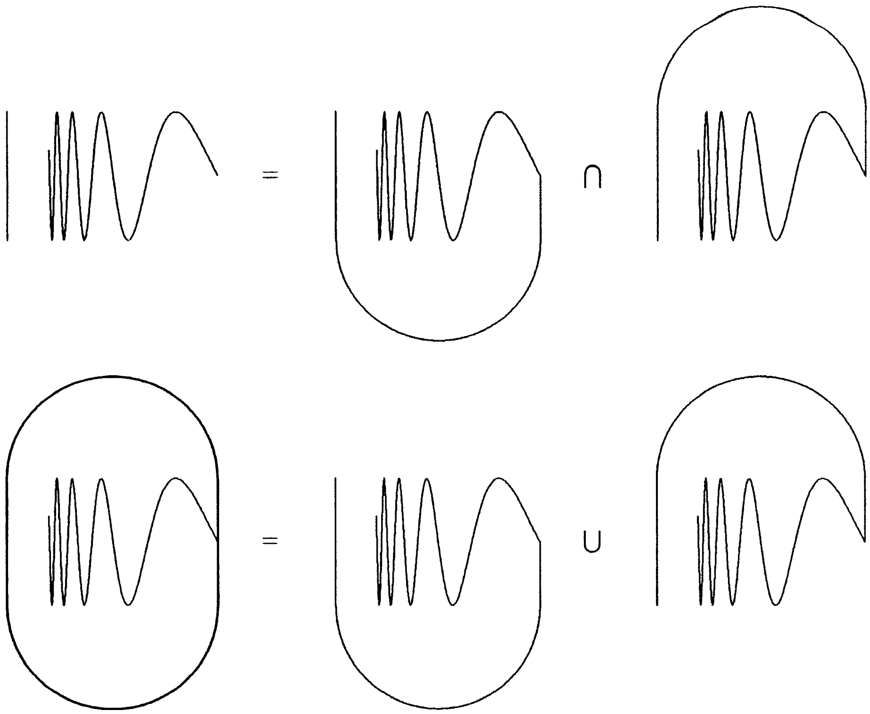

Fig. 2.

Theorem 5.4 (Štan'ko). *If X and Y are* 1*-dimensional continua with the fixed-point property and $X \cap Y$ is an AR, then $X \cup Y$ has the fixed-point property.*

Theorem 5.5 (Yandl). *There exists a* 1*-dimensional planar continuum without the fixed-point property which is the union of continua X and Y, such that X, Y and $X \cap Y$ all have the fixed-point property.*

Remark. Fig. 2 also shows that one cannot omit the condition of path connectedness from Theorem 1.1. Indeed, X and Y are simply connected, the intersection $X \cap Y$ is cellular but not path connected, and the union $X \cup Y$ clearly fails to be simply connected.

Theorem 5.6 (Bing). *There exists a* 1*-dimensional continuum X in $\mathbb{R}^3$ with the fixed-point property, and a disk D such that $D \cap X$ is an arc but $D \cup X$ does not have the fixed-point property.*

Bing [1] originated the following still open problem:

Question 5.7 (Bing). *If C is a planar continuum with the fixed-point property and D is a disk which intersects C in an arc, must then $C \cup D$ have the fixed-point property*?

A partial answer follows from our Theorem 1.1—the answer is affirmative if additionally C is path connected and $C \cup D$ is a subset of $\mathbb{R}^2$.

Question 5.8. *If C is a simply connected planar continuum and D is a disk which intersects C in an arc, must then $C \cup D$ have the fixed-point property*?

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