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On embeddability of contractible k-dimensional compacta into $\mathbb{R}^{2k}$

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Abstract

We present, for every integer $k \in \mathbb{N}$, an elementary construction of a contractible k-dimensional compactum which does not embed into $\mathbb{R}^{2k}$.

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1. Introduction

The classical Pontryagin–Nöbeling embedding theorem [2] asserts that every k-dimensional compactum can be embedded in $\mathbb{R}^{2k+1}$. On the other hand, there exists for every k, a compact k-dimensional polyhedron which does not allow any embedding into $\mathbb{R}^{2k}$ (cf. [3,11]).

It is well known that every acyclic compact k-dimensional polyhedron embeds into $\mathbb{R}^{2k}$ (cf. [4–7,10]). The following related question about 2-dimensional compacta has recently been asked in [1]: *Is every acyclic* (*aspherical, cell-like*) 2*-dimensional compactum embeddable in* $\mathbb{R}^4$?

It turns out that the answer is *negative*. Namely, applying [13], the following result was established in [8]:

Theorem 1.1. *For every integer $k \in \mathbb{N}$, there exists a contractible k-dimensional compactum which does not embed into $\mathbb{R}^{2k}$.*

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However, the question remained whether there exists a simple direct proof of Theorem 1.1, based only on elementary constructions. We present such a proof below.

2. Preliminaries

In the complex plane $\mathbb{C}$ consider the following line segment

$$D^1 = \left\{z \in \mathbb{C} \mid z = r\mathrm{e}^{\mathrm{i}\phi},\ 0 \leqslant r \leqslant 1,\ \phi \in \{0, \pi\}\right\},$$

the disk

$$D^2 = \left\{z \in \mathbb{C} \mid z = r\mathrm{e}^{\mathrm{i}\phi},\ 0 \leqslant r \leqslant 1,\ 0 \leqslant \phi \leqslant 2\pi\right\},$$

three sectors (for $n \in \{0, 1, 2\}$)

$$T_n = \left\{z \in \mathbb{C} \mid z = r\mathrm{e}^{\mathrm{i}\phi},\ 0 \leqslant r \leqslant 1,\ \tfrac{2}{3}\pi n \leqslant \phi \leqslant \tfrac{2}{3}\pi(n+1)\right\},$$

the triod

$$T = \left\{z \in \mathbb{C} \mid z = r\mathrm{e}^{\mathrm{i}\phi},\ 0 \leqslant r \leqslant 1,\ \phi = \tfrac{2}{3}\pi n,\ n \in \{0, 1, 2\}\right\},$$

and the following closed countable set with the limit point:

$$S = \left\{z \in \mathbb{C} \mid z = 0 \text{ or } z = \pm\frac{1}{n},\ n \in \mathbb{N}\right\}.$$

Let $p_n : T_n \to T_n \cap T, n \in \{0, 1, 2\}$, be the canonical projections in the respective directions $\mathrm{e}^{2\pi(n+2)\mathrm{i}/3}$. Then projections p_n naturally define the projection $p : D^2 \to T$.

It is easy to see that for any point $a \in D^2$, $d(p(a), p(-a)) = |a|$. We consider on $\mathbb{C}$ the standard metric d, whereas in the product $D^1 \times \prod_1^k D^2$ we take the metric d_m, defined as follows:

$$d_m(\bar{a}, \bar{b}) = \max\bigl(d(a_i, b_i)_{i\in\{0,1,\ldots,k\}}\bigr),$$
$$\bar{a} = (a_0, a_1, \ldots, a_k), \qquad \bar{b} = (b_0, b_1, \ldots, b_k).$$

3. Elementary proof of Theorem 1.1

We claim that the subspace

$$X = \left(S \times \prod_1^k T\right) \cup \left(D^1 \times \prod_1^k \{0\}\right) \subset D^1 \times \prod_1^k T$$

satisfies all properties asserted by Theorem 1.1.

Contractibility and k-dimensionality of the space X are obvious. We shall prove that X does not embed into $\mathbb{R}^{2k}$ by demonstrating that already its subset $S \times \prod_1^k T$ does not embed into $\mathbb{R}^{2k}$.

Suppose to the contrary that there was such an embedding:

$$i : S \times \prod_1^k T \to \mathbb{R}^{2k}.$$

Since $\mathbb{R}^{2k}$ is an AR, there would then exist an extension

$$f : D^1 \times \prod_1^k T \to \mathbb{R}^{2k}$$

of the map i over $D^1 \times \prod_1^k T$.

Consider now the following sequences of segments $I_n = [-1/n, 1/n], n \in \mathbb{N}$, natural embeddings

$$i_n : I_n \times \prod_1^k T \to D^1 \times \prod_1^k T, \quad n \in \mathbb{N},$$

and mappings

$$\mathcal{P}_n : D^1 \times \prod_1^k D^2 \to I_n \times \prod_1^k T, \quad n \in \mathbb{N},$$

and

$$\mathcal{P}_0 : D^1 \times \prod_1^k D^2 \to D^1 \times \prod_1^k T,$$

defined by

$$\mathcal{P}_n\big((a_0, a_1, \ldots, a_k)\big) = \big(\tfrac{1}{n}a_0, p(a_1), p(a_2), \ldots, p(a_k)\big), \quad n \in \mathbb{N},$$

and

$$\mathcal{P}_0\big((a_0, a_1, \ldots, a_k)\big) = \big(0, p(a_1), p(a_2), \ldots, p(a_k)\big).$$

Let

$$\phi : \partial\left(D^1 \times \prod_1^k D^2\right) \to S^{2k}$$

be the homeomorphism onto the $2k$-dimensional sphere, defined by $\phi(\bar{a}) = \bar{a}/\|\bar{a}\|$.

By the classical Borsuk–Ulam theorem on antipodes, applied to the mapping

$$f \circ i_n \circ \mathcal{P}_n \circ \phi^{-1} : S^{2k} \to \mathbb{R}^{2k}, \quad n \in \mathbb{N},$$

there exists a pair of points

$$\bar{b}^n, -\bar{b}^n \in S^{2k}$$

such that

$$(f \circ i_n \circ \mathcal{P}_n \circ \phi^{-1})(\bar{b}^n) = (f \circ i_n \circ \mathcal{P}_n \circ \phi^{-1})(-\bar{b}^n), \quad n \in \mathbb{N}.$$

The points $\bar{a}^n = \phi^{-1}(\bar{b}^n)$ cannot lie in $(\partial D^1) \times \prod_1^k D^2$, for any $n \in \mathbb{N}$, because the restrictions

$$f \circ i_n|_{\{-1/n, 1/n\} \times \prod_1^k T} : \left\{-\frac{1}{n}, \frac{1}{n}\right\} \times \prod_1^k T \to \mathbb{R}^{2k}, \quad n \in \mathbb{N},$$

are injective mappings. Therefore

$$\bar{a}^n \in D^1 \times \partial\left(\prod_1^k D^2\right), \quad n \in \mathbb{N}.$$

However, for all such points and every $n \in \mathbb{N}$, the following holds:

$$\begin{aligned} & d_m\big(\mathcal{P}_n(\bar{a}^n), \mathcal{P}_n(-\bar{a}^n)\big) \\ & \quad = \max_{i\in\{1,2,\ldots,k\}} \left\{ \frac{1}{n}|2a_0^n|,\ d\big(p(a_i^n), p(-a_i^n)\big) \right\} \geqslant 1. \end{aligned}$$

Since the space $D^1 \times \prod_1^k D^2$ is compact, there exists a subsequence $\{\bar{a}^{n_k}\}_{k\in\mathbb{N}}$ which converges to some point $\bar{a}^* = (a_0^*, a_1^*, \ldots, a_k^*)$.

Mappings

$$i_n \circ \mathcal{P}_n : D^1 \times \prod_1^k D^2 \to D^1 \times \prod_1^k T, \quad n \in \mathbb{N},$$

converge to

$$\mathcal{P}_0 : D^1 \times \prod_1^k D^2 \to D^1 \times \prod_1^k T,$$

so we can conclude the following:

$$\begin{aligned} & (f \circ \mathcal{P}_0)(\bar{a}^*) = (f \circ \mathcal{P}_0)(-\bar{a}^*), \\ & d_m\big(\mathcal{P}_0(\bar{a}^*), \mathcal{P}_0(-\bar{a}^*)\big) = 1, \\ & \mathcal{P}_0(\bar{a}^*) \in S \times \prod_1^k T, \\ & \mathcal{P}_0(-\bar{a}^*) \in S \times \prod_1^k T. \end{aligned}$$

This contradicts the injectivity of the mapping i and hence there cannot be any embedding of $S \times \prod_1^k T$ into $\mathbb{R}^{2k}$.

4. Epilogue

The question stated below remains open. It is related to an old, still unsolved, Borsuk problem on whether every k-dimensional compact AR embeds into $\mathbb{R}^{2k}$ (cf. Problem 707 in [12]).

Question 4.1. Does there exist for some integer $k \in \mathbb{N}$, a homologically locally connected contractible k-dimensional compactum which does not embed into $\mathbb{R}^{2k}$?

Note added in proof

It was pointed out by R.J. Daverman that our contractible k-dimensional compactum which does not embed into $\mathbb{R}^{2k}$, actually does not embed into any $2k$-dimensional topological manifold. Indeed, suppose there were an embedding ϕ of X into some $2k$-manifold M^{2k}. Consider the point $\theta = \prod_1^{k+1}\{0\}$ of X. Since ϕ is uniformly continuous there exists a small closed neighborhood of the point θ such that its image lies in an open neighborhood of $\phi(\theta)$ in M^{2k} homeomorphic to $\mathbb{R}^{2k}$. Since the restriction of an embedding is again an embedding and since for the point θ there exist arbitrary small closed neighborhoods which are homeomorphic to X, it would follow that there exists an embedding of X into $\mathbb{R}^{2k}$. Contradiction.

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