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An exotic factor of $S^3 \times \mathbb{R}$

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1. *Introduction*

Cannon's *recognition problem* [**10**] asks for a short list of topological properties that is reasonably easy to check and that *characterizes* topological manifolds. In dimensions below three the answer has been known for a long time: see [**6**, **24**]. In dimensions above four it is now known, due to the work of J. W. Cannon [**11**], R. D. Edwards [**14**] (see also [**12**] and [**18**]), and F. S. Quinn [**21**], that topological n-manifolds ($n \geqslant 5$) are precisely *ENR* $\mathbb{Z}$-homology n-manifolds with Cannon's *disjoint disc property* (*DDP*) [**11**] and with a vanishing Quinn's *local surgery obstruction* [**23**]. In dimension four there is a resolution theorem of Quinn [**22**] (with the same obstruction as in dimensions $\geqslant 5$) and a 1-LCC shrinking theorem of M. Bestvina and J. J. Walsh [**5**]. However, it is still an open problem to find an effective analogue of Cannon's DDP for this dimension, one which would yield a shrinking theorem along the lines of that of Edwards [**14**]. For more on the history of the recognition problem see the survey [**24**].

We are interested in the 3-*dimensional* problem where the unresolved status of the Poincaré Conjecture plays a crucial role. So far it has been established by D. Repovš and R. C. Lacher [**25**] that, modulo the Poincaré Conjecture, an *ENR* $\mathbb{Z}$-homology 3-manifold X whose potential singularities are known to be restricted to some 0-dimensional subset of X, is a topological 3-manifold if and only if X possesses either one of the following two general position properties – the *Dehn's lemma property* (*DLP*) or the *map separation property* (*MSP*) [**17**]. The purpose of this paper is to show how strongly a *negative* answer to the Poincaré Conjecture would affect the 3-dimensional recognition problem. We prove the following theorem:

THEOREM 1·1. *If fake* 3-*cells exist then there is a topological space X with the following properties*:

(i) *X is a totally singular $\mathbb{Z}$-homology* 3-*manifold*;
(ii) *X does not admit a resolution*;
(iii) *X is homogeneous*;
(iv) *X is a compact ANR*;
(v) *X has the Dehn's lemma property*;
(vi) *X has the map separation property*; *and*
(vii) *$X \times \mathbb{R}$ is homeomorphic to $S^3 \times \mathbb{R}$.*

The construction of the space X is a slight modification of W. Jakobsche's technique of producing non-manifold homogeneous compact 3-dimensional *ANR*'s

[15]. Recently, F. D. Ancel and L. C. Siebenmann have incorporated this method into their work on the compactification of Davis' non-euclidean universal covering spaces [2]. They have also given the general axiomatic description of this technique and we shall use in it Section 3, where we shall study a class of homogeneous spaces. We remark that a construction, similar to the one in [15] was already used in 1930 by L. S. Pontryagin [20] in his work on dimension theory. For some other applications see R. F. Williams [28].

2. *Preliminaries*

We shall be working in the category of locally compact Hausdorff spaces and continuous maps throughout the paper. Manifolds are assumed to have no boundary unless otherwise specified. Homeomorphism (resp. homotopy, (co)homology) equivalence will be denoted by $\cong$ (resp. $\simeq$, $\sim$). Integer coefficients will be assumed in every (co)homology used in this paper. A *homotopy n-cell* is a compact n-manifold with boundary M such that $M \simeq B^n$, the standard n-cell. The definition of a *homotopy n-sphere* is analogous. A *fake* 3-*cell* (resp. 3-*sphere*) is a homotopy 3-cell (resp. 3-sphere) which fails to be homeomorphic to the standard 3-cell B^3 (resp. 3-sphere S^3). The *Poincaré Conjecture* asserts that fake 3-cells cannot exist or, equivalently, that there are no fake 3-spheres. A space is said to satisfy *Kneser finiteness* if no compact subset of it contains more than finitely many pairwise disjoint fake 3-cells.

A compact subset K of an n-manifold M is *cellular in M* if K is the intersection of a sequence of n-cells B_i^n in M which are *properly nested*, i.e. $B_{i+1}^n \subset \operatorname{int} B_i^n$ for each i. A space X is *cell-like* if there exist a manifold N and an embedding $f: X \to N$ such that $f(X)$ is cellular in N. A closed map is *proper* if its point inverses are compact. A map $f: X \to Y$ is *one-to-one over* $Z \subset Y$ if $f^{-1}(z)$ is a point for every $z \in Z$.

Let G be a decomposition of a space X into continua and let $\pi: X \to X/G$ be the corresponding quotient map. An element $g \in G$ is *non-degenerate* if g is not a point. A set $U \subset X$ is *G-saturated* if $U = \pi^{-1}(\pi(U))$. A decomposition G is *upper semi-continuous* if π is a closed map. A countable family of compacta $\{C_\alpha\}$ is called a *null-sequence* if for every $\epsilon > 0$, all but finitely many among the C's have diameter less than ϵ.

Let $f: X \to Y$ be a map. The *non-degeneracy set* (or *singular set*) of f is defined by $\Sigma(f) = \mathrm{Cl}\{x \in X | f^{-1}f(x) \neq x\}$. For $X \subset Y$ we shall denote by $\mathrm{Int}_Y X$ (resp. $\mathrm{Cl}_Y X$) the *interior* (resp. *closure*) *of X with respect to Y* and we shall write $\mathrm{Fr}_Y X = \mathrm{Cl}_Y X \backslash \mathrm{Int}_Y X$. A space X is *homogeneous* if for every pair of its points $x, y \in X$ there is a homeomorphism $h: X \to X$ such that $h(x) = y$.

A space is a *generalized n-manifold*, where $n \in \mathbb{N}$, if (i) X is a *euclidean neighbourhood retract*, i.e. X is a locally compact, finite-dimensional separable metrizable *ANR*; and (ii) X is a *$\mathbb{Z}$-homology n-manifold*, i.e., for every $x \in X$,

$$H_*(X, X\backslash\{x\}; \mathbb{Z}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}; \mathbb{Z}).$$

The *singular set of X* is defined by

$$S(X) = \{x \in X | x \text{ has no neighbourhood in } X \text{ homeomorphic to } \mathbb{R}^n\}.$$

A generalized manifold X is said to be *totally singular* if $S(X) = X$. A *resolution* of a generalized n-manifold X is a pair (M, f) where M is a topological n-manifold and $f: M \to X$ is a proper, cell-like surjection.

A space X is said to have the *map separation property* (*MSP*) if given any collection of maps $f_1, \ldots, f_k : B^2 \to X$ such that $\Sigma(f_i) \cap \partial B^2 = \varnothing$ for every i and

$$f_i(\partial B^2) \cap f_j(B^2) = \varnothing$$

for $i \neq j$, and given a neighbourhood $U \subset X$ of $\bigcup_{i=1}^k f_i(B^2)$, there exist maps $F_1, \ldots, F_k$: $B^2 \to U$ such that $F_i|\partial B^2 = f_i|\partial B^2$ for every i and $F_i(B^2) \cap F_j(B^2) = \varnothing$ for $i \neq j$. A space X is said to have *Dehn's lemma property* (*DLP*) (cf. [**17**]) if for every map $f: B^2 \to X$ such that $\Sigma(f) \cap \partial B^2 = \varnothing$ and for every neighbourhood $U \subset X$ of $f(\Sigma(f))$ there is an embedding $F: B^2 \to f(B^2) \cup U$ such that $F(\partial B^2) = f(\partial B^2)$.

3. *A class of homogeneous spaces*

Let M^n be a closed topological n-manifold and L^n a compact topological n-manifold with (possibly empty) boundary. To every such pair of M^n and L^n we shall associate a new topological space $X(M^n, L^n)$ which will be defined as the inverse limit of an inverse sequence $\{L_i, \alpha_{i,i+1}\}_{i \in \mathbb{N}}$, where $\alpha_{i,i+1}: L_{i+1} \to L_i$ are the bonding maps, with the following properties for every $i \in \mathbb{N}$:

(i) L_i is a connected sum of L and finitely many (possibly zero) copies of M;

(ii) Ω_i is a finite collection of pairwise disjoint bicollared n-cells in L_i;

(iii) $\alpha_{i,i+1}$ is one-to-one over the complement of the set $\bigcup\{\operatorname{Int} C \mid C \in \Omega_i\}$;

(iv) for every $C \in \Omega_i$, $\alpha_{i,i+1}^{-1}(C)$ is a punctured connected sum of finitely many copies of M;

(v) for every $j > i$, if $C \in \Omega_i$ and $D \in \Omega_j$, then $\partial C \cap \alpha_{i,j}(D) = \varnothing$, where $\alpha_{i,j}: L_j \to L_i$ is the composition $\alpha_{i,j} = \alpha_{i,i+1} \cdots \alpha_{j-1,j}$; and

(vi) the collection $\{\alpha_{i,j}(C) \mid j \geqslant i, C \in \Omega_j\}$ is a dense null-sequence in L_i.

For $n = 3$, $M^3 =$ homotopy 3-sphere $\not\cong S^3$, and $L^3 \cong S^3$, the associated space $X(M^3, L^3)$ is precisely the example from [**15**] of a non-manifold homogeneous generalized 3-manifold. This class of homogeneous spaces has recently attracted renewed interest because F. D. Ancel and L. C. Siebenmann [**2**] have recognized $X(M^n, L^n)$, where $M^n = \mathbb{Z}$-homology n-sphere $\not\cong S^n$, $L^n = S^n$, and $n \geqslant 3$, as a compactification of the Davis non-euclidean universal covering space of a closed $(n+1)$-manifold: see [**13**]. Also, Pontryagin disks, which one obtains from $M^2 =$ 2-torus, $L^2 =$ 2-cell, and $n = 2$, were used in the work of W. J. R. Mitchell, D. Repovš, and E. V. Ščepin [**19**] on the 4-dimensional case of the cell-like mapping problem.

The following properties of spaces $X(M^n, L^n)$ are easily proved using the techniques of [**15**] and some standard properties of inverse limits. They also follow as special cases from a forthcoming paper of W. Jakobsche [**15a**].

PROPOSITION 3·1. *Let M^n and L^n be as in the definition of $X(M^n, L^n)$ above. Then*

(i) *for every two inverse sequences $\{L_i, \alpha_{i,i+1}\}_{i \in \mathbb{N}}$ and $\{K_i, \beta_{i,i+1}\}_{i \in \mathbb{N}}$ satisfying the requirements* (i)–(vi) *above, the associated spaces $X(M^n, L^n)$ are homeomorphic*;

(ii) *every space $X(M^n, L^n)$ embeds in $\mathbb{R}^m$ for some sufficiently large m*;

(iii) *every space $X(M^n, L^n)$ is homogeneous*;

(iv) *if M^n is a $\mathbb{Z}$-homology n-sphere then $X(M^n, L^n)$ is a Čech $\mathbb{Z}$-cohomology n-manifold*;

(v) *if M^n is a homotopy n-sphere then $X(M^n, L^n)$ is an ANR.* ▮

We shall use the following notation: for every $Z \subset L_i$, let $Z' = \alpha_i^{-1}(Z)$, hence $Z' \subset X(M^n, L^n)$. Furthermore, given a family Λ of subsets of L_i, let $\Lambda' = \{Z' | Z \in \Lambda\}$. Next, given a map $f: D \to X(M^n, L^n)$ of a finite disjoint sum of discs and a subset $A \subset X(M^n, L^n)$, let $A^* = f^{-1}(A) \cup A_1$ where A_1 is the union of all components of $D \setminus f^{-1}(A)$ which do not intersect ∂D.

The next lemma follows immediately from Proposition 3·1 (ii):

LEMMA 3·2. *In every space $X(M^n, L^n)$ there is a metric ρ such that*

$$\lim_{i\to\infty}(\sup\{\operatorname{diam}_\rho(C') | C \in \Omega_i\}) = 0. \quad \blacksquare$$

Hereafter we shall assume that $X(M^n, L^n)$ is equipped with such a metric ρ.

Before we state our next lemma we must define a new collection:

$$\Omega_i^E = \{\alpha_{i,j}(C) | j > i, C \in \Omega_j, \text{ and } \alpha_{i,j}(C) \text{ is not contained in } \alpha_{i,k}(D) \text{ for any } k \in \{i+1, \ldots, j-1\} \text{ and any } D \in \Omega_k\} \cup \Omega_i,$$

for $i \in \mathbb{N}$. Clearly the collection Ω_i^E consists (informally) of the 'attaching' n-cells for the nth stage, plus the 'attaching' n-cells for the $(n+1)$st stage disjoint from the ones added previously, plus the 'attaching' n-cells for the $(n+2)$nd stage disjoint from those for the two previous stages, etc. Also note that the conditions (i)–(vi) imply that the canonical projection $\alpha_i: X(M^n, L^n) \to L_i$ is one-to-one over the complement of $\bigcup\{\operatorname{Int} C | C \in \Omega_i^E\}$, for every $i \in \mathbb{N}$.

LEMMA 3·3. *Consider the upper semi-continuous decomposition of the space $X(M^n, L^n)$ whose non-degeneracy set is $(\Omega_i^E)'$, for some $i \in \mathbb{N}$, and let $Q_i = X(M^n, L^n)/(\Omega_i^E)'$ be the corresponding quotient space. Then Q_i is homeomorphic to L_i. In particular, Q_i is a topological n-manifold.*

Proof. Following the remark concerning Ω_i^E, we see that Q_i is homeomorphic to the quotient space L_i/Ω_i^E, since $\alpha_i: X(M^n, L^n) \to L_i$ is one-to-one over the complement of the interiors of the cells Ω_i^E and the upper semi-continuous cellular decomposition of L_i into points and a null-sequence of bicollared 3-cells, determined by Ω_i^E, is clearly shrinkable [12]. The assertion now follows by the Bing shrinking criterion [12]. $\blacksquare$

Note that for $n \geq 5$, every space $X(M^n, L^n)$ satisfies Cannon's *disjoint discs property* [11]: every two maps of a 2-cell into $X(M^n, L^n)$ can be approximated arbitrarily closely by maps with disjoint images. To see this, let $f_1, f_2: B^2 \to X = X(M^n, L^n)$ be any two maps. Consider the canonical projection $q_m: X \to X/(\Omega_m^E)'$, i.e. the decomposition of X whose non-degenerate elements are determined by $(\Omega_m^E)'$. Let $Q_m = X/(\Omega_m^E)'$. By Lemma 3·3, Q_m is a topological n-manifold. Let $f_i' = q_m f_i$ for $i = 1, 2$. Since $n \geq 5$, we have the disjoint discs property in Q_m and hence we can approximate f_i' by a map f_i'', for $i = 1, 2$, so that $f_1''(B^2) \cap f_2''(B^2) = \varnothing$. Let $T = q_m(\bigcup\{C | C \in (\Omega_m^E)'\})$. Then T is a countable dense subset of Q_m so we can apply theorem 7·2 on p. 140 of [4] to approximate f_1'' and f_2'' by maps $g_1', g_2': B^2 \to Q_m$ such that $(g_1'(B^2) \cup g_2'(B_2)) \cap T = \varnothing$. Let $g_i = q_m^{-1} g_i'$ for $i = 1, 2$. Clearly the g_i's are well-defined, and by taking m large enough and each g_i' to be a sufficiently close approximation, we may assume that $g_1(B^2) \cap g_2(B^2) = \varnothing$ and that each g_i is as close to f_i as we wish. $\blacksquare$

Henceforth, we shall deal only with 3-dimensional spaces and for convenience we denote $X(M^3, L^3)$ simply by X^3.

LEMMA 3·4. *For every* $i \in \mathbb{N}$ *and every* $\epsilon_i > 0$ *there exist a null-sequence* $\Gamma_i = \{B_C \mid C \in \Omega_i^E\}$ *of bicollared 3-cells in* L_i *and a null-sequence* $\Sigma_i = \{S_C \mid C \in \Omega_i^E\}$ *of 2-spheres in* L_i *such that*

(i) *for every* $C \in \Omega_i^E$, $C \subset \operatorname{Int} B_C$ *and* $\partial B_C = S_C$;

(ii) *for every* $C \neq F \in \Omega_i^E$, $S_C \cap S_F = \varnothing$;

(iii) $(\bigcup\{S \mid S \in \Sigma_i\}) \cap (\bigcup\{C \mid C \in \Omega_i^E\}) = \varnothing$;

(iv) *for every* $C \in \Omega_i^E$, $\operatorname{diam} B'_C \leqslant \epsilon_i + \operatorname{diam} C'$.

We wish to point out that the bicollared 3-cells B_C need not be pairwise disjoint.

Proof. Let $p_i : L_i \to K_i = L_i/\Omega_i^E$ be the quotient map of the upper semi-continuous decomposition of L_i into points and the (countably many) elements of Ω_i^E. Then it follows by Lemma 3·3 that $K_i \cong L_i$. Furthermore, p_i maps $\bigcup\{C \mid C \in \Omega_i^E\}$ onto a countable dense subset $W_i \subset K_i$. For every $C \in \Omega_i^E$, choose an open neighbourhood $U^C \subset K_i$ of $p_i(C)$ such that $\{p_i^{-1}(U^C) \mid C \in \Omega_i^E\}$ is a null-sequence in L_i. Number the elements of $\Omega_i^E = \{C_1, C_2, \ldots\}$. For every k, we now inductively find a 2-sphere $Z_k \subset U^{C_k}$ bounding a 3-cell D_k in $U^{C_k} \backslash (Z_1 \cup \ldots \cup Z_{k-1})$ such that $Z_k \cap p_i(\Omega_i^E) = \varnothing$ and $p_i(C_k) \subset D_k$. We do this as follows: suppose that we have already found Z_j for $j < k$. Then $U = U^{C_k} \backslash (Z_1 \cup \ldots \cup Z_{k-1})$ is a neighbourhood of $p_i(C_k)$ and by theorem IV·7·2 of [**4**], we can find the desired 2-sphere Z_k inside $U \backslash \bigcup\{p_i(C) \mid C \in \Omega_i^E\}$.

For every $C \in \Omega_i^E$ we now let $S_C = p_i^{-1}(Z_C)$ and $B_C = p_i^{-1}(D_C)$. To satisfy the condition (iv) we only need to choose the sets U^C sufficiently small. ∎

Choose a sequence $\{\epsilon_i > 0\}_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} \epsilon_i = 0$ and let Γ_i and Σ_i be the corresponding null-sequences from Lemma 3·4. It then follows that the corresponding collections Γ'_i and Σ'_i of subsets of X^3 are null-sequences, too, and that for every $B \in \Gamma_i$, we have $\operatorname{Fr} B' = S'$, where $S = \partial B \in \Sigma_i$.

LEMMA 3·5. *Let* $n \in \mathbb{N}$, *let* $U \subset X^3$ *be an open* Γ'_n*-saturated set, and let* $f : D \to X^3$ *be a map of a disjoint union of finitely many 2-cells in* X^3 *such that* $U \cap f(\partial D) = \varnothing$ *and* f *is one-to-one over* U. *Then there exists a map* $g : D \to X^3$ *such that*

(i) $g(f^{-1}(U) \cup (\bigcup\{(B')^* \mid B' \in \Gamma_U\})) \subset U$, *where* $\Gamma_U = \{B' \in \Gamma'_n \mid B' \subset U\}$;

(ii) $g|Y = f|Y$, *where* $Y = D \backslash (f^{-1}(U) \cup (\bigcup\{(B')^* \mid B' \in \Gamma_U\}))$;

(iii) g *is one-to-one over* U; *and*

(iv) $g(D) \cap U \cap (\bigcup\{C' \mid C \in \Omega_n^E\}) = \varnothing$.

Remark. This is our Main Lemma. It will enable us to pass from X^3 to the quotient space $X^3/(\Omega_i^E)' \cong L_i/\Omega_i^E \cong L_i$, a 3-manifold, so we shall be able to apply 3-manifold properties, in particular the DLP and the MSP (see [**25**]). The problem one encounters when transporting Dehn discs from X^3 to $X^3/(\Omega_i^E)'$ is evident: the discs may fail to remain Dehn. We succeed in pushing them off the collection $(\Omega_i^E)'$ inside a prescribed open set U while keeping control over the size of each move, i.e. we taper things off as we get close to $\partial B'_C$, $C \in \Omega_n^E$. The process converges although there are infinitely many C''s because $(\Omega_n^E)'$ is a null-sequence. This, i.e. the convergence, is essentially the assertion of Lemma 3·5.

Proof of Lemma 3·5. For every $B' \in \Gamma_U$, we define a countable family $\Phi(B')$ of continua in $S' = \partial B'$ as follows: let Y be a component of $f(D) \cap S'$ and let $V_Y = \operatorname{Int}_D Y^*$. Define $\Phi(B')$ to be the family of components Y of $f(D) \cap S'$ such that (i) $V_Y \neq \varnothing$ for every Y and (ii) there is no component E of $f(D) \cap S'$ such that for any

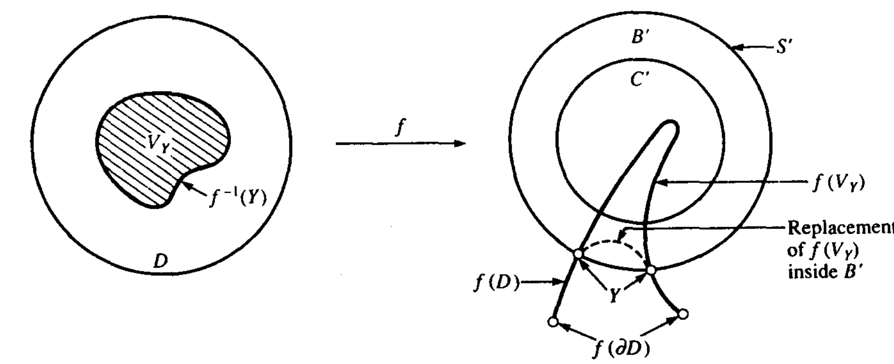


Fig. 1

$Y \in \Phi(B')$, $Y \neq E$, we have $Y \subset f(V_E)$. Condition (i) implies that each V_Y is a non-empty open subset of D, hence $\Phi(B')$ has at most countably many elements. Also, condition (ii) implies that $V_Y \cap V_E = \varnothing$ for every $Y \neq E \in \Phi(B')$.

Let us explain further the nature of the family $\Phi(B')$. This is a family of components of $f(D) \cap S'$, not of arbitrary subcontinua of this intersection. In particular, in the condition (ii), E is any component of $f(D) \cap S'$ whereas $Y \in \Phi(B')$. The motivation for the introduction of the family $\Phi(B')$ was the following: we wish to change $f(D)$ inside each $B'(B = B_C,\ C \in \Omega_n^E)$ so that the new $f(D)$ will miss C', but the change of the map f ought to be restricted (to the interior of V_Y) (see Figure 1). We perform these operations (one B' at a time) for each component $X \in \Phi(B')$. It is not clear that the result is compact. The proof of this is the main task of this lemma. Once we know that, we construct a certain shrinkable upper semi-continuous, cell-like decomposition G of X^3 and transfer the whole problem to the quotient space X^3/G where we are able to find a disc replacement with certain nice properties. Finally, we show how to lift this new disc from X^3/G back to X^3.

Perhaps it is worthwhile to explain why we did not simply take *all* components of $f(D) \cap S'$ (rather than just the countable collection $\Phi(B')$). This is because the situation can be more complicated, e.g. we can have the intersections as shown in Figure 2.

There is no need to consider the components E_1 and E_2 because they will disappear after the modification corresponding to Y. On the other hand, Y is not (by the definition of $\Phi(B')$) contained in $f(V_E)$ for any other $E \in \Phi(B')$, so we have to eliminate it. This explains the condition (ii) in the definition of $\Phi(B')$ above. It is easy to see why we need the condition (i) as well.

By hypothesis, $f|f^{-1}(U): f^{-1}(U) \to X^3$ is an embedding. Also, by definition, $S' \subset U$, so if S' were completely contained in $f(D)$, then $f^{-1}(S')$ would be a 2-sphere embedded in D which is clearly impossible. Therefore the set $S' \backslash f(D)$ is always non-empty. So take an arbitrary $x \in S' \backslash f(D)$ and identify $S' \backslash \{x\}$ with $\mathbb{R}^2$. Consider $\Phi(B')$ as a family of planar continua, i.e. $\bigcup\{Y \mid Y \in \Phi(B')\} \hookrightarrow \mathbb{R}^2$. For every $Y \in \Phi(B')$, let $Z_Y = \mathbb{R}^2 \backslash H_Y$, where $H_Y \subset \mathbb{R}^2$ is the unbounded component of $\mathbb{R}^2 \backslash Y$. Clearly Z_Y is cellular in $\mathbb{R}^2$ and for every $Y \neq E \in \Phi(B')$ we have $Z_Y \subset Z_E$ or $Z_E \subset Z_Y$ or $Z_E \cap Z_Y = \varnothing$.

Let $q_{B'}: B' \to B$ be the quotient map of the upper semi-continuous decomposition $G_{B'}$ of B' whose non-degenerate elements are the cell-like sets from the null-sequence $\Omega_{B'}^E = \{C' \in (\Omega_n^E)' \mid C' \subset B'\}$. By Lemma 3·3, B is a 3-cell. Also $T = q_{B'}(\bigcup\{C' \mid C' \in \Omega_B^E\})$ is

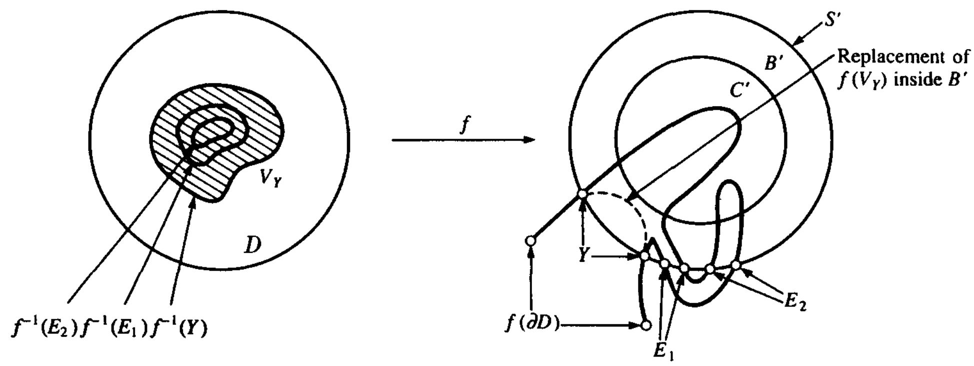


Fig. 2

a countable dense subset of B. We identify the 2-spheres $S = \partial B = q_{B'}(\partial B')$ and $S' = \partial B'$ via the homeomorphism $q_{B'}|S'$. This will allow us to use the same notation for the subsets of S' as for their images under $q_{B'}$ in S. In particular, we shall consider $\{x\}$, $Y \in \Phi(B')$ and Z_Y as subsets of S.

Let $I \subset B$ be an arc such that $I \cap S = \{x\} \subset \partial I$, $I \cap T = \varnothing$, and such that $(B \backslash I, S \backslash \{x\})$ is homeomorphic to (and can thus be identified with) $(\mathbb{R}^2 \times \mathbb{R}_+, \mathbb{R}^2 \times \{0\})$. Then we have for every $Y \in \Phi(B')$ that $Z_Y \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}_+$.

Given any $Y \in \Phi(B')$, let $d_Y : Z_Y \to \mathbb{R}_+$ be the function given by $d_Y(z) = \mathrm{dist}_\rho(z, Y)$, where we are referring to the usual metric ρ on $\mathbb{R}^2$. Let $G_Y \subset \mathbb{R}^2 \times \mathbb{R}_+$ be the graph of d_Y. It is easy to check that each G_Y is cellular in $\mathbb{R}^2 \times \mathbb{R}_+$ and, consequently, in B as well. Also, for every $Y \neq E \in \Phi(B')$, we see that $G_Y \cap G_E = \varnothing$. Due to theorem IV·7·2 of [**4**], we can assume that $T \cap G_Y = \varnothing$ for every $Y \in \Phi(B')$.

For every $Y \in \Phi(B')$, let $P_Y = q_{B'}^{-1}(G_Y)$ and let $\Psi = \bigcup\{\Phi(B') \mid B' \in \Gamma_U\}$. Consider the space

$$\bar{D} = (f(D) \backslash \bigcup\{f(V_Y) \mid Y \in \Psi\}) \cup (\bigcup\{P_Y \mid Y \in \Psi\}).$$

Then $\bar{D} \cap U \cap (\bigcup\{C' \mid C \in \Omega_n^E\}) = \varnothing$. We shall prove that $\bar{D}$ is a closed subspace of X^3. This is the key assertion of the proof and it is highly non-trivial – note that $\bar{D}$ need not be homeomorphic to $f(D)$ because the sets Y^* (which contains $\mathrm{Cl}_D V_Y$) and P_Y are in general not homeomorphic (and even when they are they can be embedded in a different way), so we do not really know much about $\bar{D}$.

Now to show that $\bar{D}$ is closed in X^3 it suffices to verify that for every $B' \in \Gamma_U$ the space $B' \cap \bar{D}$ is compact. Indeed, let $\{a_i\}$ be a sequence of points in $\bar{D}$ such that $a_i \to x \in X^3$. Then we have the following possibilities:

Case 1. There exists a subsequence $\{a_{i_k}\}$ of $\{a_i\}$ such that $a_{i_k} \in B' \cap \bar{D}$ for a fixed B'. Then $x \in B' \cap \bar{D} \subset \bar{D}$ by compactness of $B' \cap \bar{D}$.

Case 2. There exists a subsequence $\{a_{i_k}\}$ of $\{a_i\}$ such that for every k, $a_{i_k} \in \bar{D} \backslash \bigcup\{\mathrm{Int}\,B' \mid B' \in \Gamma_U\}$. Then $x \in \bar{D}$ again since

$$\bar{D} \backslash \bigcup\{\mathrm{Int}\,B' \mid B' \in \Gamma_U\} = f(D) \backslash \bigcup\{f(V_Y) \mid Y \in \Psi\}$$

is a closed subset of $\bar{D} \subset X^3$.

Case 3. Every a_i is contained in some subset of X^3 of the form $\bar{D} \cap \mathrm{Int}\,B'$, $B' \in \Gamma_U$, and every such subset contains at most finitely many points a_i. Then for every point $a_i \in \bar{D} \cap \mathrm{Int}\,B'$ we can find a point $b_i \in \bar{D} \cap \mathrm{Fr}\,B'$ and thus obtain a sequence $\{b_i\}$ such that $b_i \in \bar{D}\backslash\bigcup\{\operatorname{Int} B' \mid B' \in \Gamma_U\}$. It follows that $\lim_{i\to\infty} b_i = \lim_{i\to\infty} a_i = x$ since Γ_U is a null-sequence, and so by Case 2, $x \in \bar{D}$.

Next, we shall prove that for every $B' \in \Gamma_U$, $B' \cap \bar{D}$ is compact. But first we need a sublemma. We shall use Čech cohomology with integer coefficients.

SUBLEMMA 3·6. *Let $Y \in \Phi(B')$ and let $Y_0 \subset Y$ be an arbitrary subcontinuum of Y. Then for every $0 \neq y \in \check{H}^1(Y_0)$, there exists a closed neighbourhood $V \subset \mathbb{R}^2 = S\backslash\{x\} = q_{B'}(S'\backslash\{x\})$ of Y_0 such that there are no $Y_1 \in \Phi(B')$, $Y_1 \neq Y$, satisfying the following condition:*

(*) there exists a cohomology class $y_1 \in \check{H}^1(Y_1 \cap V)$ and $v \in \check{H}^1(V)$ such that $i_1^*(v) = y_1$ and $i_2^*(v) = y$, where $i_1: Y_1 \cap V \to V$ and $i_2: Y_0 \to V$ are inclusions.

Proof. For every non-zero cohomology class $y \in \check{H}^1(Y_0)$, we have $\bar{y} \neq 0$, where $\bar{y} = f^*(y) \in \check{H}^1(f^{-1}(Y_0))$. (Note that f is one-to-one over U.) It then easily follows from the condition (ii) of the definition of the family $\Phi(B')$ that there is a closed neighbourood $\bar{V}$ of $f^{-1}(Y_0)$ in $f^{-1}(U) \subset D$ such that for every $Y_1 \in \Phi(B')$, $Y_1 \neq Y$, we have no $y_1 \in \check{H}^1(f^{-1}(Y_1) \cap \bar{V})$ and $\bar{v} \in \check{H}^1(\bar{V})$ so that $\bar{\imath}_1^*(\bar{v}) = \bar{y}_1$ and $\bar{\imath}_2^*(\bar{v}) = \bar{y}$, where $\bar{\imath}_1: f^{-1}(Y_1) \cap \bar{V} \to \bar{V}$ and $\bar{\imath}_2: f^{-1}(Y_0) \to \bar{V}$ are inclusions.

Since f is one-to-one over U it follows that there is a closed neighbourhood $V_1 \subset U$ of Y_0 which meets the same requirements as $\bar{V}$ with respect to y (instead of $\bar{y}$), Y_0 (instead of $f^{-1}(Y_0)$), Y_1 (instead of $f^{-1}(Y_1)$), and $y_1 \in \check{H}^1(Y_1 \cap V_1)$ (instead of $\bar{y}_1 \in \check{H}^1(f^{-1}(Y_1) \cap V_0)$. Let $V = q_{B'}(V_1) \cap S$. Then V is a neighbourhood of the type which we were looking for. ∎

We now return to the problem of compactness of $B' \cap \bar{D}$. It suffices to show that $q_{B'}(B' \cap \bar{D})$ is closed in $B = q_{B'}(B')$ since the closedness of $q_{B'}(B' \cap \bar{D})$ implies the closedness of $B' \cap \bar{D}$. So suppose this were not the case. Then there would exist a sequence of points $\{a_n\} \subset q_{B'}(B' \cap \bar{D})$ converging to some point $a \in B\backslash q_{B'}(B' \cap \bar{D})$. Since $S \cap q_{B'}(B' \cap \bar{D})$ is compact it follows that $a \notin S$. We identify $(B\backslash I, S\backslash\{x\})$ with $(\mathbb{R}^2 \times \mathbb{R}_+, \mathbb{R}^2 \times \{0\})$ as before and we use the notation introduced above. Moreover $a \notin I$. Indeed, $S \cap q_{B'}(B' \cap \bar{D})$ is compact so it gives rise to a compact set $\bigcup\{Z_Y \mid Y \in \Phi(B')\} \subset \mathbb{R}^2 \times \{0\}$. This does not imply that the set $\bigcup\{G_Y \mid Y \in \Phi(B')\} \subset \mathbb{R}^2 \times \mathbb{R}_+$ is compact but it does imply that it is bounded in the usual metric on $\mathbb{R}^2 \times \mathbb{R}_+$. This, in turn, implies that $a \notin I$. Consequently, if we identify $B\backslash I$ with $\mathbb{R}^2 \times \mathbb{R}_+$, we conclude that $a \in (\mathbb{R}^2 \times (\mathbb{R}_+\backslash\{0\}))\backslash\bigcup\{G_Y \mid Y \in \Phi(B')\}$.

We have an infinite sequence $\{G_i\}$ of sets $G_i = G_{Y_i}$ such that $a_i \in G_i$. Since every G_i is compact we can assume that $G_i \neq G_j$ for all $i \neq j$. Let $Z_i = Z_{Y_i}$. We must consider the following possibilities:

Case 1. There exists an infinite sequence $i_1 < i_2 < i_3 < \ldots$ such that $Z_{i_1} \subset Z_{i_2} \subset Z_{i_3} \subset \ldots$. Then there exists $Y \in \Phi(B')$ such that $Z = \bigcup\{Z_{i_k} \mid k \in \mathbb{N}\}$ is one of the components of $Z_Y\backslash Y$. Indeed, $Y_{i_k} \cap Y_{i_j} = \varnothing$ for $k < j$, so $Z_{i_k} \subset \operatorname{Int} Z_{i_j}$. This implies that Z is an open subset of D, and of course, Z is connected since every Z_{i_k} is connected. On the other hand, $\bar{D} \cap S'$ is compact and so for every sequence of points $\{x_k\}$ such that $x_k \in Y_{i_k}$ and $x_k \to x \in D$, we have that $x \in \bar{D} \cap S'$. This implies that $\operatorname{Fr} Z \subset \bar{D} \cap S'$. By the definition of Z we have $\partial D \cap Z = \varnothing$, so from the connectedness of Z it follows that $\operatorname{Fr} Z$ is contained in one component Y of $\bar{D} \cap S'$ and so Z is one of the components of $Z_Y\backslash Y$.

Let $Y_0 = \operatorname{Fr} Z \subset Y$ and let $y \in \check{H}^1(Y_0)$ be any non-zero cohomology class corresponding (via duality in $\mathbb{R}^2$) to the image of a generator of $H_0(Z)$ in $H_0(\mathbb{R}^2\backslash Y_0)$ (note that Z is an open set). Then by Sublemma 3·6, there exists a closed neighbourhood V of Y_0 in $\mathbb{R}^2$ satisfying all the requirements of (3·6). However, from the definition of the sequence $\{i_k\}$, it follows that $Y_{i_k} \subset \operatorname{Int} V$ for sufficiently large k and by duality, there are $y_{i_k} \in \check{H}^1(Y_{i_k})$ and $v \in \check{H}^1(V)$ such that $i_1^*(v) = y_{i_k}$ and $i_2^*(v) = y$, where $i_1 : Y_{i_k} \to V$ and $i_2 : Y_0 \to V$ are inclusions. This is clearly a contradiction.

Case 2. There exists an infinite sequence $i_1 < i_2 < i_3 < \dots$ such that $Z_{i_1} \supset Z_{i_2} \supset Z_{i_3} \supset \dots$. Let ξ_k be the supremum of the diameters of all balls in Z_{i_k}. Clearly the sequence $\{\xi_k\}$ is monotone. If $\lim_{k\to\infty} \xi_k = 0$ then by the properties of the functions $d_{Y(i,k)}$ and their graphs $G_{Y(i,k)}$, where $Y(i,k) = Y_{i_k}$, we must have $a = \lim_{k\to\infty} a_{i_k} \in \mathbb{R}^2 \times \{0\}$, a contradiction. On the other hand, if $\lim_{k\to\infty} \xi_k = \xi \neq 0$ then it follows by the compactness of $\bar{D} \cap S'$ that there are a component Z of the compactum $\bigcap\{Z_{i_k} \mid k \in \mathbb{N}\}$ and a continuum $Y \in \Phi(B')$ such that $Z = Z_Y$ (note that $\xi_k \to \xi > 0$ implies that $V_Y \neq \emptyset$). Let $Y_0 = \operatorname{Fr}_S Z$ and let $y \in \check{H}^1(Y_0)$ be the non-zero cohomology class which corresponds by duality to a generator of $H_0(S \backslash Z)$. (Note that $Z \subset \mathbb{R}^2 \subset S$ and S is identified with S' by means of the map $q_{B'}$.) Again, by (3·6), one can find a neighbourhood V of Y_0 which contradicts the choice of the sequence $\{i_k\}$.

Case 3. There exists an infinite sequence $i_1 < i_2 < i_3 < \dots$ such that $Z_{i_k} \cap Z_{i_j} = \emptyset$ for all $i \neq j$. Then by the compactness of $S' \cap \bar{D}$ we have $\lim_{k\to\infty} \xi_k = 0$, where $\{\xi_k\}$ is defined as in Case 2. Hence again $a \in S'$, a contradiction.

This completes the verification that $B' \cap \bar{D}$ must be compact. Thus $\bar{D}$ is always compact and we can proceed with our proof of Lemma 3·5.

Let $q_n : X^3 \to Q_n$ be the upper semi-continuous decomposition of X^3 into points and the cell-like continua from the null-sequence $(\Omega_n^E)'$. Since $Q_n \cong L_n/\Omega_n^E$ it follows by Lemma 3·3 that $Q_n \cong L_n$, and hence in particular, Q_n is a 3-manifold. Let $\bar{U} = q_n(U)$ and consider $q_n(\bar{D})$. Clearly

$$q_n(\bar{D}) \cap \bar{U} \cap q_n(\bigcup\{C' \mid C' \in (\Omega_n^E)'\}) = \emptyset.$$

Let $q : Q_n \to \bar{Q}_n$ be the decomposition of Q_n whose only non-degenerate elements are continua from the collection

$$\Delta = \{q_n(P_y) \mid Y \in \Psi\} = \{G_Y \mid Y \in \Psi\}.$$

Since every G_Y is cellular in some $B = q_{B'}(B')$ it is also cellular in Q_n. For every $B' \in \Gamma_n'$, $B' \cap \bar{D}$ is compact and so $q_n(\bar{D}) \cap B'$ too is compact. This implies that $q|B : B \to q(B)$ is an upper semi-continuous decomposition (this is the only place in the proof where we use the fact that $\bar{D}$ is closed in X^3). Therefore q is a proper map since Γ_n' is a null-family.

Assertion. The decomposition $G(q)$ of Q_n determined by the map $q : Q_n \to \bar{Q}_n$ is shrinkable.

Proof. Let $\delta > 0$ and define

$$\mathscr{B}_1 = \{B' \in \Gamma_n' \mid \operatorname{diam} q_n(B') \geqslant \delta\} \quad \text{and} \quad \Delta^1 = \{G(Y) \in \Delta \mid \text{for some } B' \in \mathscr{B}_1,\ Y \in \Phi(B')\}.$$

Let

$$\Delta^2 = \{q_n(B') \mid B' \in \Gamma_n' \backslash \mathscr{B}_1 \text{ and } B' \text{ is not contained in the interior of any } C' \in \Gamma_n' \backslash \mathscr{B}_1\}$$

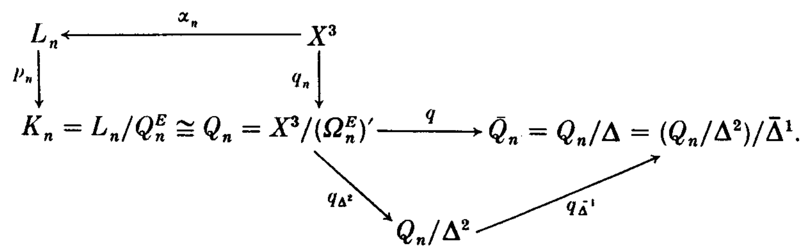
and define an upper semi-continuous cellular decomposition Δ_δ of Q_n whose non-degeneracy set is given by $\Delta_\delta = \Delta^1 \cup \Delta^2$.

Since Δ^2 is a null-sequence of bicollared 3-cells in Q_n, the decomposition of Q_n into points and elements of Δ^2 is shrinkable (see [12]), so by the Bing shrinking criterion the quotient map $q_{\Delta^2}: Q_n \to Q_n/\Delta^2$ is approximable by homeomorphisms.

Clearly the decomposition $\bar{\Delta}^1 = q_{\Delta^2}(\Delta^1)$ of Q_n/Δ^2 is upper semi-continuous, countable, cellular and compact (since $\mathscr{B}_1$ is finite). (Note, however, that $\bar{\Delta}^1$ is *not* a null-sequence.) Therefore by corollary II·7·4·A of [12], $\bar{\Delta}^1$ is (strongly) shrinkable, so by the Bing shrinking criterion, the quotient map $q_{\bar{\Delta}^1}: Q_n/\Delta^2 \to (Q_n/\Delta^2)/\bar{\Delta}^1$ is approximable by homeomorphisms and hence $Q_n \cong Q_n/\Delta^2 \cong (Q_n/\Delta^2)/\bar{\Delta}^1$. Since obviously $Q_n/\Delta_\delta = (Q_n/\Delta^2)/\bar{\Delta}^1$, it follows by the Siebenmann-Armentrout cellular approximation theorem (see [3, 26]) that Δ_δ is shrinkable.

Now, given any $\epsilon > 0$ and any Δ-saturated open covering $\mathscr{U}$ of Q_n, we can find $\delta > 0$ such that there is an open, Δ_δ-saturated refinement $\mathscr{V}$ of $\mathscr{U}$. By the argument above, the decomposition Δ_δ of Q_n is shrinkable, so for our $\epsilon > 0$ there exists a homeomorphism $h: Q_n \to Q_n$ such that h is $\mathscr{V}$-close to the identity id_{Q_n} and for every $g \in \Delta_\delta$, $\operatorname{diam} h(g) < \epsilon$. Clearly h is the desired shrinking of the decomposition Δ, too. ∎

Remark. For a better understanding of the proof above we have included a diagram showing the maps involved:



It follows by the Bing shrinking criterion that $\bar{Q}_n \cong Q_n$ and so $\bar{Q}_n \cong L_n$ by Lemma 3·3.

Consider a map $f_0: D \to \bar{Q}_n$ defined by

$$f_0(t) = \begin{cases} (q \circ q_n \circ f)(t) & \text{if } t \in D \setminus \bigcup\{Y^* \mid Y \in \Psi\} \\ q(G_Y) & \text{if } t \in Y^* \text{ for some } Y \in \Psi. \end{cases}$$

Clearly $f_0(D) = q(q_n(\bar{D}))$. Since the families $(\Omega_n^E)'$ and Δ are countable it follows that the image under $q \circ q_n$ of the non-degeneracy set T_0 of the map $q \circ q_n$ contains only countably many points, some of which may lie in $f_0(D)$. Note that $f_0 | f_0^{-1} q(\bar{U})$ is cellular and that it maps $f^{-1}(U)$ onto $f(f^{-1}(U)) \subset q \circ q_n(\bar{D})$, so we can replace f_0 by a map $f_1: D \to f_0(D)$ such that $f_1 | (D \setminus f_0^{-1} q(\bar{U})) = f_0 | (D \setminus f_0^{-1} q(\bar{U}))$ and such that $f_1 | f_0^{-1} q(\bar{U})$ is one-to-one over U. This implies that $f_1(D) \cap U$ is a 'boundary' set in U, so we can apply theorem IV·7·2 of [4] to replace f_1 by a map $f_2: D \to \bar{Q}_n$ such that $f_2 | (D \setminus f_0^{-1} q(\bar{U})) = f_0 | (D \setminus f_0^{-1} q(\bar{U}))$, $f_2 | f_0^{-1} q(\bar{U})$ is one-to-one over U, and $f_2(D) \cap T_0 \cap q(q_n(U)) = \varnothing$. We now define the map $g: D \to X^3$ promised by (3·5) as follows:

$$g(t) = \begin{cases} f(t) & \text{if } t \in D \setminus f_2^{-1}(U) \\ (q \circ q_n)^{-1} \circ f_2(t) & \text{otherwise.} \end{cases}$$

This completes the proof of Lemma 3·5. ∎

THEOREM 3·7. *Every space $X^3 = X(M^3, L^3)$ has the map separation property.*

Proof. Suppose that $\{f_i: D_i \to X^3 \mid 1 \leqslant i \leqslant k\}$ is a family of Dehn discs such that

$f_i(D_i) \cap f_j(\partial D_j) = \emptyset$ for all $i \neq j$, and choose a neighbourhood $V \subset X^3$ of $\bigcup\{f_i(D_i) \mid 1 \leqslant i \leqslant k\}$. Set $D = \coprod_{i=1}^k D_i$ and $f = \coprod_{i=1}^k f_i : D \to X^3$. Then there is a regular neighbourhood $A = \bigcup_{i=1}^k A_i$ of ∂D in D, where $A_i \subset D_i$ are annuli such that $f^{-1}(f(A)) = A$ and $f|A$ is an embedding. Identify A_i with $S^1 \times [0,1]$, where ∂D_i is identified with $S^1 \times \{1\}$. Then

$$S^1 \times [\tfrac{1}{3}, \tfrac{2}{3}] \subset S^1 \times [0,1] \cong A_i$$

corresponds to an annulus $E_i \subset A_i$ and $S^1 \times [\frac{1}{3}, \frac{1}{2}]$ corresponds to an annulus $G_i \subset E_i$. Set $E = \bigcup_{i=1}^k E_i$ and $G = \bigcup_{i=1}^k G_i$.

Using Lemmas 3·2 and 3·4, we can find Γ'_n-saturated neighbourhoods $V_1, V_2 \subset V$ of $f(E)$ and a number $n \in \mathbb{N}$ so that the following requirements are met: (i) $\operatorname{Cl} V_1 \subset V_2$; (ii) $f((B')^*) \cap (X \backslash V_2) = \emptyset$ for every $B' \in \Gamma'_n$ such that $B' \subset V_1$; (iii) $f^{-1}(V_2) \subset A \backslash \partial A$; and (iv) $V_1 \cap f(D) \subset f(A)$. We now invoke Lemma 3·5 for $U = V_1$, n, and f. We get a map $g : D \to V$ such that $g(E) \subset V_1$, $g|g^{-1}(V_1)$ is an embedding, $g(D) \cap (\bigcup\{C' \mid C \in \Omega_n^E\}) \cap V_1 = \emptyset$, and $g|(D \backslash f^{-1}(V_2)) = f|(D \backslash f^{-1}(V_2))$. (The last condition follows by Lemma 3·5 (i).)

Let $q_n : X^3 \to Q_n \cong L_n$ be the quotient map from Lemma 3·3. Set $h = q_n \circ g$. Then $h : D \to Q_n$ and $W = h(D) \cap q_n(V_1)$ is an embedded surface. If W is not locally flat we can change h slightly by first approximating W by a PL surface (see [7]) and then using theorem IV·7·2 of [4] we can achieve that

$$h(D) \cap V_1 \cap q_n(\bigcup\{C' \mid C \in \Omega_n^E\}) = \emptyset.$$

Clearly, after this operation, W becomes locally flat (in $q_n(V_1)$).

For every $1 \leqslant i \leqslant k$, let $\tilde{D}_i \subset D_i$ be a disc such that $\partial \tilde{D}_i \subset \partial G_i$ and $G_i \subset \tilde{D}_i$. Furthermore, let $Y_i \subset D_i$ be an annulus such that one of the components of ∂Y_i is ∂D_i and such that Y_i forms one of the components of $\operatorname{Cl}_D(A_i \backslash E_i)$. Set $\tilde{D} = \bigcup_{i=1}^k \tilde{D}_i$ and $Y = \bigcup_{i=1}^k Y_i$.

We now consider the non-compact 3-manifold $N = X^3/(\Omega_n^E)' \backslash h(Y)$. By 'cutting N along $h(E \backslash G)$' we obtain a non-compact 3-manifold N' with boundary and a map $p : N' \to N$ with the following properties: (i) $p|R : R \to N \backslash h(E \backslash G)$ is a homeomorphism, where $R = (N' \backslash \partial N') \cup p^{-1}(h(\partial G \backslash \partial E))$; and (ii) $p|Q : Q \to h(\operatorname{Int}_D(E \backslash G))$ is a double covering, where $Q = \partial N' \backslash p^{-1}(h(\partial G \backslash \partial E))$. Note that $p^{-1} \circ h(\partial G \backslash \partial E) \subset \partial N'$. The fact that the points of $\partial N'$ have euclidean neighbourhoods in N' follows from the fact that the surface W is locally flat in $q_n(V_1)$.

Let $F : \tilde{D} \to N'$ be the map defined by $F = p^{-1} \circ h|\tilde{D}$ (note that this is well-defined). The maps $F_i : \tilde{D}_i \to N'$, defined by $F_i = F|\tilde{D}_i$, are then Dehn discs (because $g(E) \subset V_1$ and $g|g^{-1}(V_1)$ is an embedding) such that $F_i(\partial \tilde{D}_i) \subset \partial N'$, so by corollary 2·4 of [25], we can find maps $F'_i : \tilde{D}_i \to p^{-1}q_n(V \backslash h(Y))$ such that $F'_i(\tilde{D}_i) \cap F'_j(\tilde{D}_j) = \emptyset$ for $i \neq j$, and $F'_i|\partial \tilde{D}_i = F_i|\partial \tilde{D}_i$. We can assume that for every i,

$$F'_i(D) \cap p^{-1}(q_n(\bigcup\{C' \mid C \in \Omega_n^E\}) = \emptyset,$$

using the fact that the set $p^{-1}(q_n(\bigcup\{C' \mid C \in \Omega_n^E\}))$ is countable and invoking the standard Baire category argument (see e.g. [12]).

Finally, let $F_i^* : D_i \to X^3$ be the maps given by

$$F_i^*(t) = \begin{cases} (q_n^{-1} \circ h_i)(t) & \text{if } t \in D_i \backslash \tilde{D}_i \\ (q_n^{-1} \circ p \circ F'_i)(t) & \text{otherwise.} \end{cases}$$

Then the maps F_i^* for $1 \leqslant i \leqslant k$ satisfy the requirements of the MSP. ∎

THEOREM 3·8. *Every space* $X^3 = X(M^3, L^3)$ *has the Dehn's lemma property.*

Proof. Suppose that $f: D \to X^3$ is a Dehn disc and let $V \subset X^3$ be a neighbourhood of $f(\Sigma(f))$. Let $A \subset D$ be a regular neighbourhood of ∂D in D such that $A \cap \Sigma(f) = \emptyset$. We identify A with $S^1 \times [0,1]$ so that ∂D is identified with $S^1 \times \{1\}$. Then we denote by E (resp. Y) the annulus in A which is identified with $S^1 \times [0, \frac{1}{2}]$ (resp. $S^1 \times [\frac{1}{2}, 1]$). As in the proof of Theorem 3·7 one can find Γ'_n-saturated neighbourhoods V_1 and V_2 of $f(\Sigma(f))$ in V, and a number $n \in \mathbb{N}$ such that (i) $\mathrm{Cl}\, V_1 \subset V_2$ and $\mathrm{Cl}\, V_2 \subset V$, (ii) $\mathrm{Fr}\, V_i \cap (\bigcup\{B' \mid B' \in \Gamma'_n\}) = \emptyset$, for every i, (iii) $f((B')^*) \cap V_1 = \emptyset$, for every $B' \in \Gamma'_n$ such that $B' \cap \mathrm{Cl}\, V_2 = \emptyset$, and (iv) $f^{-1}(\mathrm{Cl}\, V_2) \cap A = \emptyset$ and no element $B' \in \Gamma'_n$ meets both $f(Y)$ and $f(\mathrm{Cl}_D(D \backslash A))$.

We now invoke Lemma 3·5 for this f and n, and for $U = X^3 \backslash (H \cup \mathrm{Cl}\, V_2)$, where $H = f(Y) \cup H'$ and H' is the sum of all $B' \in \Gamma'_n$ for which $B' \cap f(Y) \neq \emptyset$. Clearly H is compact, and so U is open and Γ'_n-saturated in X^3. By Lemma 3·5, there is a map $g: D \to X^3$ such that $g \mid g^{-1}(X^3 \backslash \mathrm{Cl}\, V_2)$ is an embedding, $g \mid Y = f \mid Y$, $g(\Sigma(g)) = f(\Sigma(f)) \subset V_2$, $f(\mathrm{Cl}_D(g^{-1}(U)) \cap \mathrm{Cl}\, V_1 = \emptyset$ (this condition follows by Lemma 3·5 (i) and from (ii) above), and $g(D) \cap (\bigcup\{C' \mid C \in \Omega_n^E\}) \cap U = \emptyset$.

Now apply Lemma 3·5 again, this time for g, n, and $U_1 = V_2 \backslash \mathrm{Cl}\, V_1 \backslash H_1$, where $H_1 = f(\mathrm{Cl}_D(g^{-1}(U)) \cup H'_1$ and H'_1 is the sum of all $B' \in \Gamma'_n$ such that $B' \cap f(\mathrm{Cl}_D(g^{-1}(U))) \neq \emptyset$. Clearly H_1 is compact, and so U_1 is open and Γ'_n-saturated. From Lemma 3·5 we get a map $h: D \to X^3$ such that $h \mid h^{-1}(X^3 \backslash \mathrm{Cl}\, V_1)$ is an embedding, $h \mid Y = f \mid Y$, $h \mid h^{-1}(U_1)$ is an embedding, $h(\Sigma(h)) = f(\Sigma(f)) \subset V_1$, and

$$h(D) \cap (\bigcup\{C' \mid C \in \Omega_n^E\}) \cap (X^3 \backslash (H \cup \mathrm{Cl}\, V_1)) = \emptyset$$

(this follows since $\mathrm{Fr}\, V_2 \cap (\bigcup\{B' \mid B' \in \Gamma'_n\}) = \emptyset$).

Let $q_n: X^3 \to Q_n$ be the quotient map from Lemma 3·3 and let $F: D \to Q_n$ be the composition $F = q_n \circ h$. By our choice of h, $F \mid (D \backslash (f^{-1}(H) \cup h^{-1}(\mathrm{Cl}\, V_1)))$ is an embedding and $F(D) \cap T \cap q_n(X^3 \backslash (H \cup \mathrm{Cl}\, V_1)) = \emptyset$, where $T = q_n(\bigcup\{C' \mid C \in \Omega_n^E\})$. This implies that $F(\Sigma(F)) \subset q_n(H) \cup q_n(\mathrm{Cl}\, V_1))$.

As in the proof of Theorem 3·7 one can use [**7**] to change F slightly if necessary so that $F(D) \cap q_n(U)$ is locally flat in $q_n(U)$. As in the preceding proof we consider the 3-manifold $N = X^3/(\Omega_n^E)' \backslash q_n(H)$ and 'cut along $F(A) \backslash q_n(H)$' to obtain a non-compact 3-manifold N' with boundary and a map $p: N' \to N$ with the following properties: (i) $p \mid R: R \to N \backslash F(\mathrm{Int}_D E)$ is a homeomorphism, where $R = (N' \backslash \partial N') \cup p^{-1}(F(\partial E \backslash Y))$; and (ii) $p \mid Q: Q \to F(\mathrm{Int}_D E) \backslash q_n(H)$ is a double covering, where $Q = \partial N' \backslash p^{-1}(F(\partial E \backslash \partial Y))$. Note that N' is a 3-manifold because $F(D) \cap q_n(U)$ is locally flat in U.

Let $F': D' \to N'$ be the map $F' = p^{-1} \circ (F \mid D')$, where $D' = \mathrm{Cl}_D(D \backslash A)$. Clearly $F': D' \to N'$ is a Dehn disc with $F'(\partial D') \subset \partial N'$, and

$$F'(\Sigma(F')) \subset q_n(\mathrm{Cl}\, V_1) \subset q_n(V_2 \backslash H_1).$$

Therefore, by corollary 2·2 of [**25**], we can find an embedding $F'': D' \to N'$ such that

$$F''(D') \subset F'(D') \cup q_n(U_1 \cup \mathrm{Cl}\, V_1) = F'(D') \cup q_n(V_2 \backslash H_1).$$

By theorem IV·7·2 of [**4**], we may assume that $p^{-1}(T) \cap F''(D') = \emptyset$. Therefore we can define an embedding $F^*: D \to f(D) \cup V$, required for the DLP, as follows:

$$F^*(t) = \begin{cases} f(t) & \text{if } \; t \in f^{-1}((X^3 \backslash V_2) \cup H_1) = g^{-1}(X^3 \backslash V_2) \\ (q_n^{-1} \circ p \circ F'')(t) & \text{if } \; t \in f^{-1}(V_2 \backslash H_1). \end{cases}$$

This is well-defined because $p^{-1}(T) \cap F''(D') = \varnothing$ so that q_n^{-1} is a function on $(p \circ F'')(D) \cap q_n(V_2 \backslash S_1)$, and by the construction of g, h, and F'', both agree on

$$f^{-1}(\mathrm{Fr}\, V_2 \backslash H_1) \subset f^{-1}(\mathrm{Cl}\, V_2 \backslash H_1) \cap f^{-1}((X^3 \backslash V_2) \cup H_1).$$

Also, F^* is an embedding because it is an embedding on each of the pieces and by the choice of H_1,

$$F^*(f^{-1}(\mathrm{Cl}\, V_2) \backslash H_1) \cap F^*(f^{-1}(X^3 \backslash V_2) \cup H_1) = F^*(\mathrm{Fr}\, V_2 \backslash H_1).$$

4. *Proof of Theorem* 1·1

The construction of the example X is a minor modification of the one in [15] which is equivalent to the one described in [15] which is equivalent to the one described in Section 3 for $n = 3$, for M^3 a homotopy 3-sphere $H^3 \not\cong S^3$, and $L = S^3$. The purpose of the modification is to ensure that the bonding maps are cell-like.

So let (H^3, F^3) be a polyhedral pair, where $F^3 \subset H^3$ is a fake 3-cell and $H^3 \backslash \mathrm{Int}\, F^3$ is a real 3-cell. Let (T, T_0) be a triangulation of the pair (H^3, F^3) such that the 2-skeleton $(T_0)^{(2)}$ contains a *spine* of F^3, i.e. a compact 2-dimensional subpolyhedron $K^2 \subset \mathrm{Int}\, F^3$ such that $F^3/K^2 \cong B^3$, the standard 3-cell. Note that K^2 is always cell-like (and never cellular in F^3): see [16]. Let $f\colon (F^3, \partial F^3) \to (B^3, \partial B^3)$ be the corresponding PL spine map; hence f is a proper, cell-like surjection whose only non-degenerate point-inverse is the spine K^2. We may assume that $f|\partial B^3$ is the identity. With this modification the construction in [15] yields a space $X = X(H^3, S^3)$ as the inverse limit of the inverse sequence $\{L_i, \alpha_{i,i+1}\}_{i \in \mathbb{N}}$, and all the bonding maps $\alpha_{i,i+1}\colon L_{i+1} \to L_i$ as well as the canonical projections $\alpha_i\colon X \to L_i$ are proper, cell-like surjections.

By [15], X is a 3-dimensional, homogeneous compact ANR and it is not a manifold. By our Theorems 3·1, 3·7 and 3·8, X is also a $\mathbb{Z}$-homology 3-manifold, hence a generalized 3-manifold and (because of homogeneity) totally singular (i.e. $S(X) = X$), and X possesses both the Dehn's lemma property and the map separation property. This satisfies the assertions (i) and (iii)–(vi) of Theorem 1·1 and so it remains to establish (ii) and (vii) in order to complete the proof.

First we shall prove assertion (vii). Consider the following inverse sequence:

$$L_1 \times S^1 \xleftarrow{\alpha_{1,2} \times \mathrm{id}} L_2 \times S^1 \xleftarrow{\alpha_{2,3} \times \mathrm{id}} L_3 \times S^1 \xleftarrow{\alpha_{3,4} \times \mathrm{id}} \cdots.$$

which is obtained from the inverse sequence $(L_i, \alpha_{i,i+1}\}_{i \in \mathbb{N}}$ by 'crossing with S^1 and with the identity map'. Clearly the bonding maps $\alpha_{i,i+1}\colon L_{i+1} \times S^1 \to L_i \times S^1$ are proper, cell-like and onto (see [16]) and since they are maps between topological 4-manifolds we can invoke F. S. Quinn's 4-dimensional cell-like approximation theorem [22] to conclude that they are *near-homeomorphisms*, i.e. approximable by homeomorphisms. Finally, we use M. Brown's theorem for inverse sequences of near-homeomorphisms (see [8]; for a very nice short proof see also [1]) to conclude that $X \times S^1 \cong L_1 \times S^1$. Recall that in [15], $L_1 = S^3$ and pass to the universal covering spaces to conclude that $X \times \mathbb{R} \cong S^3 \times \mathbb{R}$. This proves (vii).

Remark. In [15] it was established that X embeds in $\mathbb{R}^7$. It follows from our argument above and Theorem 3·1 (i) that X embeds already in $\mathbb{R}^4$.

It remains to show that X does not admit a resolution (assertion (ii)). To see this we shall argue with *Kneser finiteness*: the idea is to construct a generalized 3-manifold

Y and a proper, cell-like surjection $g: X \to Y$ such that Y will contain a null-sequence of pairwise disjoint fake 3-cells. (Note that Y will therefore not be, like X, totally singular – its singular set will be precisely the complement of that null-sequence of fake 3-cells.) The argument will then be as follows: if X had a resolution, say $h: M \to X$, with h some proper cell-like surjection and M some topological 3-manifold, then the composition $gh: M \to Y$ would be a resolution of Y: see [**16**]. However, Y cannot resolve, by a finiteness theorem of Bryant and Lacher [**9**] because it fails to satisfy Kneser finiteness. This contradiction will therefore establish assertion (ii).

We shall construct Y as the inverse limit of an inverse sequence $\{M_i, \beta_{i,i+1}\}_{i\in\mathbb{N}}$ which, in turn, will be built inductively. The map $g: X \to Y$ will be defined as the limit of the maps $g_i: L_i \to M_i$, $i \in \mathbb{N}$, which will make the following diagram commutative:

$$\begin{array}{ccccccccc}
L_1 & \xleftarrow{\alpha_{1,2}} & L_2 & \xleftarrow{\alpha_{2,3}} \cdots \longleftarrow & L_i & \xleftarrow{\alpha_{i,i+1}} & L_{i+1} & \longleftarrow \cdots & X \\
\downarrow g_1 & & \downarrow g_2 & & \downarrow g_i & & \downarrow g_{i+1} & & \downarrow g \\
M_1 & \xleftarrow{\beta_{1,2}} & M_2 & \xleftarrow{\beta_{2,3}} \cdots \longleftarrow & M_i & \xleftarrow{\beta_{i,i+1}} & M_{i+1} & \longleftarrow \cdots & Y.
\end{array} \qquad (*)$$

Before we begin our construction we need to review the construction of $X = \varprojlim \{L_i, \alpha_{i,i+1}\}_{i\in\mathbb{N}}$ from [**15**]. (Note, however, that we are using different notation.) Let $L_1 = S^3$, with some fixed triangulation τ_1 and some fixed orientation. Choose an orientation also for F^3 and assume hereafter that the restriction $f|\partial F^3$ of the map $f: (F^3, \partial F^3) \to (B^3, \partial B^3)$, introduced earlier in this section, is an orientation-reversing homeomorphism.

To obtain L_2 do the following: take the second barycentric subdivision τ_1'' of the triangulation τ_1. For every 3-simplex $\sigma \in \tau_1$, take the star $\sigma^* = \mathrm{st}(\hat{\sigma}, \tau_1'')$ of the barycentre $\hat{\sigma}$ of σ (with respect to τ_1'') and replace σ^* by a copy F_σ of the fake 3-cell F^3, using as the glueing map the identity

$$f_\sigma = f|\partial F^3 : \partial F^3 \to \partial\sigma^* \cong \partial B^3.$$

Fix once and for all a subdivision T_* of T_0 which makes f_σ PL. Define

$$L_2 = (L_1 \backslash \bigcup\{\mathrm{Int}\,\sigma^* | \sigma \in \tau_1\}) \bigcup_{\{f_\sigma\}} (\coprod\{F_\sigma | \sigma \in \tau_1\})$$

with the triangulation τ_2 which is induced by T_* on the F_σ's and by τ_1 on $L_1 \backslash \bigcup\{\mathrm{Int}\,\sigma^* | \sigma \in \tau_1\}$. Let $\alpha_{1,2}: L_2 \to L_1$ be the PL map which equals

$$f: (F^3, \partial F^3) \to (\sigma^*, \partial s^*) = (B^3, \partial B^3)$$

over every σ^* and is the identity over the rest of L_1.

To obtain L_3, take the second barycentric subdivision τ_2'' of τ_2 and for every 3-simplex $\sigma \in \tau_2$, replace the star $\sigma^* = \mathrm{st}(\hat{\sigma}, \tau_2)$ by a copy F_σ of F^3, etc.

Assertion. Suppose that for every i, $K_i \subset L_i$ is a subcomplex of L_i such that $\alpha_{i,i+1}^{-1}(K_i) \subset K_{i+1}$. Then there exists an inverse system $\{M_i, \beta_{i,i+1}\}_{i\in\mathbb{N}}$ with subcomplexes $N_i \subset M_i$ such that for every i,

(1) $M_i \backslash N_i = L_i \backslash K_i$;

(2) $\beta_{i,i+1} | (M_{i+1} \backslash \beta_{i,i+1}^{-1}(N_i)) = \alpha_{i,i+1}^{-1} | (L_{i+1} \backslash \alpha_{i,i+1}^{-1}(K_i))$;

(3) $\beta_{i,i+1} | \beta_{i,i+1}^{-1}(N_i) : \beta_{i,i+1}^{-1}(N_i) \to N_i$ is a homeomorphism;

(4) $N_{i+1} \backslash \beta_{i,i+1}^{-1}(N_i) = K_{i+1} \backslash \alpha_{i,i+1}^{-1}(K_i)$; and

(5) $N_{i+1} \supset \beta_{i,i+1}^{-1}(N_i)$;

and there exist maps $\{g_i: L_1 \to M_i\}_{i\in\mathbb{N}}$ such that the diagram (*) above commutes.

Remark. As a result, we obtain maps $g: X \to Y$, where $X = \varprojlim \{L_i, \alpha_{i,i+1}\}$ and $Y = \varprojlim \{M_i, \beta_{i,i+1}\}$. It follows by (3) and (5) that $\beta_i | \beta_i^{-1}(N_i): \beta_i^{-1}(N_i) \to N_i$ is a homeomorphism.

Proof. We argue by induction on i. To begin take $g_1 = \mathrm{id}$, $M_1 = L_1$ and $N_1 = K_1$. Suppose now that we have constructed M_j, N_j and $\beta_{j,j+1}$ for all $j \leqslant i$ and construct M_{i+1} from M_i as follows: whenever (in the above notation) any $\sigma^* \subset L_{i+1}$ lies in $\alpha_{i,i+1}^{-1}(K_i)$, we replace it by a copy of $B^3 \cong \sigma^*$ (i.e. itself) rather than F_σ. We then take N_{i+1} to be the subcomplex corresponding (in the evident sense) to K_{i+1}. The verification of properties (1), (4) and (5) is straightforward: we define $\beta_{i,i+1}: M_{i+1} \to M_i$ to be $\alpha_{i,i+1}$ modified to the identity (instead of f) over the σ^*'s which were replaced 'by themselves'. Finally, $g_{i+1}: L_{i+1} \to M_{i+1}$ is defined as the identity everywhere except on the F_σ's in L_{i+1} replaced by σ^*'s in M_{i+1}, where it is defined as $f: F_\sigma \to \sigma^* \cong B^3$. Commutativity of the diagram (*) as well the properties (2) and (3) now follow easily. ∎

Our example of a non-resolvable Y is now constructed using the assertion above as follows: take $K_1 = \varnothing$ and $K_2 = \alpha_{1,2}^{-1}(\sigma_1)$, where σ_1 is a fixed 3-simplex of L (with respect to τ_1). Next, let $K_3 = \alpha_{2,3}^{-1}(K_2 \cup \sigma_2)$, where σ_2 is a fixed 3-simplex of L_2 (with respect to τ_2) such that $\operatorname{Int} \sigma_2 \cap K_2 = \varnothing$, etc. It is then clear that $N_1 = \varnothing$, $N_2 = F$, $N_3 =$ two copies of F (possibly identified along a boundary edge), etc. and that Y contains the requisite sequence of copies of fake cubes F. ∎

Note that by taking $K_i = L_i$ for all i, we produce a proper cell-like surjection $g: X^3 \to S^3$. Moreover it is clear that g is homogeneous in the sense that a PL homeomorphism $h: S^3 \to S^3$ can be 'lifted' to $H: X^3 \to X^3$ such that $g \circ H = h \circ g$.

Using a resolution theorem of Bryant and Lacher [**9**], one can further strengthen the assertion of Theorem 1·1: it follows that X does not admit even an (almost) $\mathbb{Z}_2$-acyclic resolution. Namely, one can show that for every proper, monotone surjection $f: M \to X$ defined on a topological 3-manifold M, the dimension of the set $\{x \in X \mid \check{H}^1(f^{-1}(x); \mathbb{Z}_2) \not\cong 0\}$ is at least 1.

Question 4·1. Suppose fake 3-cells exist. Is there then a generalized 3-manifold X with $\dim S(X) = 0$ (in particular, $S(X) \neq \varnothing$) satisfying Kneser finiteness and possessing (at least one of) the properties DLP and MSP?

Remarks. (1) If such an example exists, then $S(X)$ cannot have any isolated points and, moreover, $S(K)$ is not 1-LCC embedded in X.

(2) The condition concerning Kneser finiteness is important: without it, it is easy to find such an example (see [**24**]).

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