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ON SMOOTHNESS OF COMPACTA

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1. Introduction

The original purpose of this paper was to find a simpler proof of the following homogeneity theorem due to Skopenkov and the authors [11] (see also [1, 10, 12]), which would avoid the use of the classical Rademacher theorem on Lipschitz functions [3]:

Theorem 1.1 (homogeneity theorem [11]). *For every $k \in \{1, 2, \dots, n\}$, any ambiently C^k-smoothly homogeneous subset of $\mathbb{R}^n$ is a C^k-smooth submanifold of $\mathbb{R}^n$.*

A subset $M \subset \mathbb{R}^n$ is said to be *ambiently C^k-smoothly homogeneous* if, for every pair of points $x, y \in M$, there exist neighborhoods $O(x), O(y) \subset \mathbb{R}^n$ such that the triples $(O(x), O(x) \cap M, \{x\})$ and $(O(y), O(y) \cap M, \{y\})$ are C^k-diffeomorphic.

As a result of our efforts, this theorem splits into two. The first one describes the typical behavior of the tangent directions in compacta (*tangent regular* points; see the existence Theorem 2.1). The second one characterizes smooth submanifolds as the sets with only typical (i.e., regular) points (see the characterization Theorem 2.2). These two theorems together provide a very short proof of the homogeneity Theorem 1.1, which is contained in Secs. 3 and 7.

The rest of the paper (Secs 4–6) replaces Rademacher's theorem, consisting of the fact that *every* Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ is almost everywhere differentiable. To avoid the use of this theorem, we develop an approach to differentiability via *tangent directions.*

In conclusion, we wish to point out that the initial investigations of smooth homogeneity [1, 10, 11, 12] have produced not only a simple characterization of smooth submanifolds of $\mathbb{R}^n$, but have also inspired a new approach to the classical Hilbert–Smith conjecture (cf. [5, 6, 9, 13]). We believe that the present paper generates a promising new approach to the theory of smooth structures on compact subsets of $\mathbb{R}^n$.

2. Main Results

Points of the unit $(n-1)$-sphere $S^{n-1} \subset \mathbb{R}^n$ will be called *directions.* For a pair of different points $x, y \in \mathbb{R}^n$, the *direction* from x to y is defined as $\frac{y-x}{\|y-x\|}$ and denoted by $\overrightarrow{xy}$. We say that a sequence of ordered pairs $\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ *represents* a direction $\vec{d} \in S^n$ at a point $x \in \mathbb{R}^n$ if

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = x \quad \text{and} \quad \lim_{k\to\infty} \overrightarrow{x_k y_k} = \vec{d}.$$

Let M be an arbitrary locally compact subset of $\mathbb{R}^n$. A direction is said to be a *tangent direction* to M at $x \in M$ if it can be represented at x by a sequence $\{(x_k, y_k)\}_{k \in \mathbb{N}}$ of points of M converging to (x, x). The set of all tangent directions of M at $x \in M$ will be denoted by $T_M(x)$ and called the *tangent set.* Tangent directions that have a special representation by sequences of the form $\{(x_k, y_k)\}_{k \in \mathbb{N}}$, where $x_k = x$ for every $k \in \mathbb{N}$, are called *proper tangent directions,* and the *proper tangent set* is denoted by $t_M(x)$. We are now ready to formulate a key concept of the paper.

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Definition 2.1. A point $x \in M \subset \mathbb{R}^n$ is called *tangent regular* (or simply *regular*) if all of its tangent directions are proper, i.e., $T_M(x) = t_M(x)$.

As usual, points that are not regular are called *singular*. Our first main result states the existence of tangent regular points in any compactum and shows that regular points are typical.

Theorem 2.1 (existence theorem). *For every locally compact subset M of $\mathbb{R}^n$, the set of all singular points of M is of the first Baire category (i.e., is the union of a sequence of nowhere dense subsets).*

We define the *tangent plane* for $M \subset \mathbb{R}^n$ at x as the linear space spanned by $T_M(x)$ and denote it by $L_M(x)$, where $L_M(x) \subset \mathbb{R}^n$. Our main theorem can now be stated as follows:

Theorem 2.2 (characterization theorem). *For every $m \in \{0, 1, \dots, n\}$ and every locally compact subset $M \subset \mathbb{R}^n$, the following statements are equivalent:*

(1) *M is a smooth submanifold of $\mathbb{R}^n$ of class C^1 and dimension m;*

(2) *M is tangent regular and has tangent planes of constant dimension m at every point $x \in M$.*

The condition that the tangent planes are of constant dimension is not sufficient if this dimension is not related to the *geometric* dimension of M (as is the case here). The graph of a continuous nowhere differentiable function of $\mathbb{R}$ has tangent planes all of dimension 2. So, it would be a submanifold of dimension 2, but this does not mean that there are no tangent singular points (despite of the fact that most of its points are regular). Hence the condition of tangent regularity itself is very restrictive. Compacta that satisfy it are similar to *stratified* manifolds.

These two theorems immediately yield a C^1-version of the homogeneity Theorem 1.1 (which, in turn, implies a C^k-version, as is shown in [11] and in Sec. 8). Indeed, ambiently smooth homogeneity implies that all points of the sets considered have the same regularity type and have constant dimension of tangent planes. The existence Theorem 2.1 guarantees that this type is regular. Hence the conditions of the characterization Theorem 2.2 are satisfied and our sets are indeed smooth submanifolds.

The main concept of our *tangent-direction theory* is the notion of a tangentiable mapping.

Definition 2.2. A proper (closed) mapping $f : M_1 \to M_2$ between subsets of the Euclidean spaces is said to be *tangentiable* if, for every sequence $\{(x'_k, x''_k)\}_{k\in\mathbb{N}}$ representing a direction at $x \in M_1$, the sequence $\{(f(x'_k), f(x''_k))\}_{k\in\mathbb{N}}$ represents a direction at $f(x) \in M_2$.

The main result of this new theory is the following:

Theorem 2.3 (on smooth images). *Tangentiable mappings map smooth submanifolds into smooth submanifolds.*

For examples of tangentiable mappings that are not differentiable (and for a discussion of this concept), see Sec. 8.

3. Existence of Regular Points

In this section, we will deal with a fixed locally compact set $M \subset \mathbb{R}^n$ and we will omit the index M in the notation of tangent sets. Let us introduce *sets of ε-tangent directions*

$$T^\varepsilon(x) = [\{\overrightarrow{yz} \mid y, z \in O_\varepsilon(x)\}] \quad \text{and} \quad t^\varepsilon(x) = [\{\overrightarrow{xy} \mid y \in O_\varepsilon(x)\}]\,.$$

Here, the brackets [ ] denote the closure, and $O_\varepsilon(x) = \{y \mid \operatorname{dist}(x, y) < \varepsilon\}$ denotes the open ε-neighborhood of x in M. The following lemma is an immediate consequence of the definitions.

Lemma 3.1. *For every sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of positive numbers converging to 0, we have*

$$\bigcap_{k=1}^{\infty} T^{\varepsilon_k}(x) = T(x) \quad \text{and} \quad \bigcap_{k=1}^{\infty} t^{\varepsilon_k}(x) = t(x)\,.$$

Lemma 3.2. *If a set $A \subset M$ has an interior point in its closure $x \in O_\varepsilon(x) \subseteq [A]$, then*

$$\Big[\bigcup_{a \in A} t^{2\varepsilon}(a)\Big] \supseteq T^\varepsilon(x) \supseteq T(x).$$

Proof. If $y, z \in O_\varepsilon(x)$, then $\operatorname{dist}(y, z) < 2\varepsilon$. Since $y \in O_\varepsilon(x) \subseteq [A]$, there exists a sequence $\{a_k\}_{k \in \mathbb{N}} \in A$, converging to y and such that

$$\operatorname{dist}(a_k, z) < 2\varepsilon \quad \text{for all} \quad k \in \mathbb{N}.$$

Thus $\overrightarrow{a_k z} \in t^{2\varepsilon}(a_k)$ for all $k \in \mathbb{N}$ and $\lim\limits_{k \to \infty} \overrightarrow{a_k z} \in [\bigcup\limits_{a \in A} t^{2\varepsilon}(a)]$. On the other hand, $\lim\limits_{k \to \infty} \overrightarrow{a_k z} = \overrightarrow{yz}$; hence $\overrightarrow{yz} \in [\bigcup\limits_{a \in A} t^{2\varepsilon}(a)]$. However, $\overrightarrow{yz}$ represents an arbitrary element of $T^\varepsilon(x)$.

Lemma 3.3. *Let $W \subset S^{n-1}$ be an open set. Consider the following two sets:*

$$T_W = \{x \in M \mid T(x) \cap W \neq \varnothing\}$$

and

$$t_{-W} = \{x \in M \mid t(x) \cap [W] = \varnothing\}.$$

Then the intersection $T_W \cap t_{-W}$ is of the first Baire category.

Proof. Since $t(x) = \bigcap t^\varepsilon(x)$, one concludes that

$$t_{-W} = \bigcap_{\varepsilon > 0} t^\varepsilon_{-W} = \bigcap_{k=1}^{\infty} t^{1/k}_{-W},$$

where

$$t^\varepsilon_{-W} = \{x \in M \mid t^\varepsilon(x) \cap [W] = \varnothing\}.$$

Thus, it suffices to prove that $T_W \cap t^\varepsilon_{-W}$ is nowhere dense. But if t^ε_{-W} are dense in some V, then

$$[\bigcup_{x \int^\varepsilon_{-W}} t^\varepsilon(x)] \cap W = \varnothing,$$

and by Lemma 3.1, $T(x) \cap W = \varnothing$ for all $x \in V$.

Hence the closure of t^ε_{-W} cannot contain points of T_W in its interior. This means that the intersection $t^\varepsilon_{-W} \cap T_W$ is nowhere dense and our lemma is thus proved.

Proof of the existence Theorem 2.1. Let us consider a countable open base $\{W_k\}_{k \in \mathbb{N}}$ of S^{k-1} (for example all rational balls). The union $\bigcup\limits_{k=1}^{\infty} (T_{W_k} \cap t_{-W_k})$, being of the first Baire category by virtue of Lemma 3.3, obviously contains all singular points of F. Indeed, if $t(x) \neq T(x)$, one can find W_{k_0} such that $W_{k_0} \cap T(x) \neq \varnothing$ and $[W_{k_0}] \cap t(x) = \varnothing$.

4. Directionally Differentiable Functions

A function $f : M \to \mathbb{R}$ ($M \subset \mathbb{R}^n$ a locally compact set) is said to be *differentiable at the point $x \in M$ along the direction $\vec{d} \in T_M(x)$* if, for every sequence $\{(x'_k, x''_k)\}_{k \in \mathbb{N}}$ representing $\vec{d}$, there exists a finite limit

$$\lim_{k \to \infty} \frac{f(x''_k) - f(x'_k)}{\|x''_k - x'_k\|} = \frac{\partial f}{\partial \vec{d}}(x).$$

A function that is differentiable along all directions from $T_M(x)$ is called *directionally differentiable* at x. A function directionally differentiable at any point $x \in M$ is said to be *directionally smooth*. The goal of this section is to reduce the directional differentiability to the continuous differentiability for smooth manifolds.

Lemma 4.1. *A function $f : \mathbb{R} \to \mathbb{R}$ that is directionally differentiable at x is differentiable at x in the usual sense.*

Proof. $T_{\mathbb{R}}(x)$ consists of two elements (positive and negative directions). The directional differentiability immediately implies the existence of left and right derivatives

$$\lim_{\Delta x \to +0} \frac{\Delta f}{\Delta x} \quad \text{and} \quad \lim_{\Delta x \to -0} \frac{\Delta f}{\Delta x}.$$

To see that these limits coincide, let us consider the sequences $\{(x'_k, x''_k)\}_{k\in\mathbb{N}}$, where $x'_{2k} = x = x''_{2k+1}$, $x'_{2k+1} = x + \frac{1}{2k+1}$ and $x''_{2k} = x - \frac{1}{2k}$. For these sequences, one has $x''_k - x'_k = \frac{1}{k}$ for all $k \in \mathbb{N}$; hence $\{(x'_k, x''_k)\}_{k\in\mathbb{N}}$ represents a positive direction, and the directional differentiability of f implies the convergence of the sequence $\left\{\frac{f(x''_k)-f(x'_k)}{x''_k - x'_k}\right\}_{k\in\mathbb{N}}$. But the even members of this sequence converge to the right derivative and the odd members converge to the left derivative. Therefore, the left and right derivatives coincide.

Lemma 4.2. *Consider a C^1-function $f : \mathbb{R}^n \to \mathbb{R}$. Let $x \in \mathbb{R}^n$ be an arbitrary point. Then, for every sequence $\{(x'_k, x''_k)\}_{k\in\mathbb{N}}$ representing a direction $\vec{d}$ at x, there exists a limit*

$$\lim_{k\to\infty} \frac{f(x''_k) - f(x'_k)}{||x''_k - x'_k||},$$

which coincides with the partial derivative of f in the direction $\vec{d}$.

Proof. Without loss of generality, we can assume that $\vec{d} = \vec{e}_1 = (1, 0, \dots, 0)$. The condition $\lim\limits_{k\to\infty} x'_k x''_k = \vec{e}_1$ implies that

$$x''_k - x'_k = \lambda_k \vec{e}_1 + O(||x''_k - x'_k||)\,. \tag{4.1}$$

The continuous differentiability of f implies

$$f(x'_k + \lambda_k \vec{e}_1) - f(x'_k) = \frac{\partial f}{\partial \vec{e}_1}(x'_k) \cdot \lambda_k + O(\lambda_k), \tag{4.2}$$

$$\frac{\partial f}{\partial \vec{e}_1}(x'_k) = \frac{\partial f}{\partial \vec{e}_1}(x) + O(||x'_k - x||)\,. \tag{4.3}$$

We note that condition (4.1) implies

$$||x'_k - x''_k|| = ||\lambda_k \vec{e}_1 + O(x''_k - x'_k)|| = \lambda_k + O(||x''_k - x'_k||)\,. \tag{4.4}$$

Using conditions (4.1)–(4.4), one obtains

$$f(x''_k) - f(x'_k) = f(x'_k + \lambda_k \vec{e}_1) - f(x'_k) + O(||x''_k - x'_k||)$$

$$= \frac{\partial f}{\partial \vec{e}_1}(x'_k) \cdot \lambda_k + O(||x''_k - x'_k||)$$

$$= \frac{\partial f}{\partial \vec{e}_1}(x) \cdot \lambda_k + O(||x''_k - x'_k||) =$$

$$\frac{\partial f}{\partial \vec{e}_1}(x)||x''_k - x'_k|| + O(||x''_k - x'_k||)\,.$$

Lemma 4.3. *Assume that $x_k \to x$ and $\vec{d}_k \to \vec{d}$, where $\vec{d}_k \in T_M(x_k)$. If the function $f : M \to \mathbb{R}$ is differentiable along $\vec{d}$ and all $\vec{d}_k$ then*

$$\lim_{k\to\infty} \frac{\partial f}{\partial \vec{d}_k}(x_k) = \frac{\partial f}{\partial \vec{d}}(x).$$

Proof. Let the sequences $\{(y_k^l, z_k^l)\}_{k\in\mathbb{N}}$ represent $\vec{d}_k \in T_M(x_k)$ for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, choose $l(k)$ such that

(1) $\operatorname{dist}(\overrightarrow{y_k^{l(k)} z_k^{l(k)}}, \vec{d}_k) < \frac{1}{k}$;
(2) $\operatorname{dist}(y_k^{l(k)}, x_k) < \frac{1}{k}$;
(3) $\operatorname{dist}(z_k^{l(k)}, x_k) < \frac{1}{k}$;
(4) $\left| \frac{f(z_k^{l(k)}) - f(y_k^{l(k)})}{\|z_k^{l(k)} - y_k^{l(k)}\|} - \frac{\partial f}{\partial \vec{d}_k}(x_k) \right| < \frac{1}{k}$.

In this case, the "diagonal" sequence $\{(y_k^{l(k)}, z_k^{l(k)})\}_{k\in\mathbb{N}}$ represents the direction $\vec{d}$ because of conditions (1)–(3) and the conditions of the lemma, $\lim_{k\to\infty} \vec{d}_k = \vec{d}$ and $\lim_{k\to\infty} x_k = x$. Therefore, the derivatives $\frac{\partial f}{\partial \vec{d}}$ can be calculated by using this sequence. On the other hand, condition (4) implies

$$\lim_{k\to\infty} \frac{f(z_k^{l(k)}) - f(y_k^{l(k)})}{\|z_k^{l(k)} - y_k^{l(k)}\|} = \lim_{k\to\infty} \frac{\partial f}{\partial \vec{d}_k}(x_k).$$

Theorem 4.1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable if and only if it is directionally differentiable.*

Proof. If f is a C^1-function and $\vec{d} \in T_{\mathbb{R}^n}(x)$ is any direction, then one can choose a coordinate system such that $\vec{d}$ becomes $\vec{e}_1$ and apply Lemma 4.2 to prove the existence of $\frac{\partial f}{\partial \vec{d}}(x)$. If f is directionally smooth, it has partial derivatives by Lemma 4.2 and these derivatives are continuous by Lemma 4.3; this implies that our function is continuously differentiable.

5. Tangent-Direction Functor

Definition 5.1. The *tangent space* of a subset $M \subset \mathbb{R}^n$ is the subspace of $M \times S^{n-1}$ given by

$$T(M) = \{(x, \vec{d}) \mid \vec{d} \in T_M(x)\}.$$

Proposition 5.1 (closedness theorem). *For every locally compact subset M of $\mathbb{R}^n$, $T(M)$ is a closed subset of $M \times S^{n-1}$ and the projection $\pi : T(M) \to M$ is a closed mapping.*

Proof. We will prove that if $\vec{d}_k \in T_M(x_k)$ converges to $\vec{d}$ and $x_k \to x$, then $\vec{d} \in T_M(x)$. Choose sequences $\{(y_k^l, z_k^l)\}_{k\in\mathbb{N}}$ representing $\vec{d}_k$. The "diagonal" sequence $\{(y_k^{l(k)}, z_k^{l(k)})\}$, constructed as in Lemma 4.3 (satisfying conditions (1)–(3)), represents $\vec{d} \in T_M(x)$ and has the same limit as $\{\vec{d}_k\}_{k\in\mathbb{N}}$.

This proves the closedness of $T(M)$. If, in the above proof, the assumption that $\vec{d}_k \to \vec{d}$ is replaced by the requirement that (d_k, x_k) belongs to a fixed closed set $C \subset T(M)$, then the same "diagonal" sequence converges to any partial limit of $\{\vec{d}_k\}_{k\in\mathbb{N}}$ (all of which belong to C) and the images converge to x. This proves that $x \in \pi(C)$, i.e., the closedness of $\pi(C)$, and, hence, that of π.

Corollary 5.1. *Let $F \subset S^{n-1}$ be any closed set of directions. Then the set $T_{-F} = \{x \in M \mid T_M(x) \cap F = \varnothing\}$ is open.*

Proof. Assume that $x \in T_{-F}$. Denote by $\psi : M \times S^n \to S^n$ the projection onto the second factor. Then the inverse image $\psi^{-1}(F)$ is closed by the continuity of ψ. By Proposition 5.1, $\pi(\psi^{-1}(F))$ is closed. Hence x has a neighborhood $O_x \subset M$ such that $O_x \cap \pi(\psi^{-1}(F)) = \varnothing$. It remains to observe that $T_M(y) \cap F = \varnothing$ for every $y \in O_x$; hence, $O_x \subset T_{-F}$.

A mapping $f : M_1 \to M_2$ is said to be *correct* in direction $\vec{d} \in T_{M_1}(x)$ if, for every sequence $\{(x'_k, x''_k)\}_{k\in\mathbb{N}}$ representing $\vec{d}$, the sequence $\{(f(x'_k), f(x''_k))\}_{k\in\mathbb{N}}$ represents a direction, which is denoted by $T_f(\vec{d})$. The mapping $f : M_1 \to M_2$ is said to be *tangentiable at a point* $x \in M_1$ if it is correct in all directions of $T_{M_1}(x)$. A mapping is said to be *tangentiable* if it is tangentiable at every point. Every tangentiable map φ generates the *tangent mapping* $T_\varphi : T(M_1) \to T(M_2)$. A mapping $f : X \to Y$ is called *locally homeomorphic* if every point $x \in X$ has an open neighborhood $O(x)$ such that $f(O(x))$ is an open subset in Y and $f|_{O(x)} : O(x) \to f(O(x))$ is a homeomorphism.

Lemma 5.1. *If $f : M_1 \to M_2$ is tangentiable at $x \in M_1$, then it maps M_1 locally homeomorphically onto its image $f(M_1)$.*

Proof. Since every continuous injective mapping of compacta is a homeomorphism, we have that for locally compact spaces, the local injectivity of a continuous mapping implies that it is a local homeomorphism. The negation of the local injectivity implies the existence of a sequence $\{(x'_k, x''_k)\}_{k\in\mathbb{N}}$ such that $x'_k \neq x''_k$, $f(x'_k) = f(x''_k)$, and $\lim\limits_{k\to\infty}(x'_k) = \lim\limits_{k\to\infty}(x''_k) = x$. In this case, f is incorrect with respect to all partial limits of $\{(x'_k, x''_k)\}_{k\in\mathbb{N}}$.

The following proposition summarizes obvious properties of tangentiable mappings.

Proposition 5.2. *Let $f : M_1 \to M_2$ be a surjective tangentiable mapping between locally compact subsets $M_1, M_2 \subset \mathbb{R}^n$. Then the following assertions hold:*

(1) *$T_f : T(M_1) \to T(M_2)$ is continuous;*

(2) *T_f is surjective, and, moreover, $T_f(T_{M_1})(x) = T_{M_2}(f(x))$ for every x;*

(3) *T_f maps regular points to regular points.*

Proof. (1) Here the same trick with the "diagonal" sequence works as in Lemma 4.3.

(2) Consider a neighborhood $O(x)$ such that $f|_{O(x)}$ is a homeomorphism between $O(x)$ and $f(O(x))$ (Lemma 5.1). Since $f(O(x))$ is a neighborhood of $f(x)$, every direction $\vec{d} \in T_{M_1}(f(x))$ has a representation $\{(y'_k, y''_k)\}_{k\in\mathbb{N}}$ by elements of $f(O(x))$. Let $\varphi : f(O(x)) \to O(x)$ be the inverse to f. Being continuous, φ generates directions $\{(\varphi(y'_k), \varphi(y''_k))\}_{k\in\mathbb{N}}$ at x by choosing convergent subsequences. Such directions transform f into d; this proves $\vec{d} \in T_f(T_{M_1}(x))$.

(3) Obviously, f transforms proper directions into proper ones. Thus, if $t_{M_1}(x) = T_{M_1}(x)$, then $T_f(t_{M_1}(x)) \subset t_{M_2}(f(x))$ and $T_f(t_{M_1}(x)) = T_{M_2}(f(x'))$ by the previous property. Hence $t_{M_2}(f(x)) = T_{M_2}(f(x))$.

A tangentiable mapping $f : M_1 \to M_2$ is said to be *tangently injective* if its tangent mapping T_f is injective.

Lemma 5.2. *If a mapping $\varphi : M_1 \to M_2$ has an inverse $\varphi^{-1} : M_2 \to M_1$ and φ is tangently injective at $x \in M_1$, then φ^{-1} is tangentiable at $\varphi(x)$.*

Proof. Let us consider a sequence $\{(x'_k, x''_k)\}_{k\in\mathbb{N}}$ such that $\{(\varphi(x'_k), \varphi(x''_k))\}_{k\in\mathbb{N}}$ represents some direction $\vec{d} \in T_{M_2}\varphi(x)$. We have to prove that (x'_k, x''_k) represents a direction in $T_{M_1}(x)$. Assume the contrary. Then the sequence $\overrightarrow{x'_k x''_k}$ is not convergent and it has at least two different partial limits $\vec{d_1}$ and $\vec{d_2}$. In this case, T_φ transforms both of them into $\vec{d}$. This contradicts the tangent injectivity of φ.

Lemma 5.3. *A linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is correct in direction $\vec{d} \in T_{\mathbb{R}^n}(x)$ if and only if $\vec{d}$ does not belong to its kernel, i.e., $\vec{d} \notin L^{-1}(0)$.*

The proof is obvious.

Lemma 5.4. *Let $\pi : \mathbb{R}^n \to L_n(x)$ be the orthogonal projection onto the tangent plane $L_n(x)$ at the point $x \in M$. Let $f = \pi|_M$. Then f is tangently injective at x, i.e., the mapping T_f is injective in some neighborhood of x.*

Proof. Let us consider T_f at the point x:

$$T_f : T_M(x) \to T_{L_M(x)}(x) = S \cap L_M(x)\,,$$

where S is the unit sphere. Thus, T_f is an inclusion, due to the linearity and orthogonality of π. Hence x is a point of tangent injectivity of $f = \pi|_M$. Corollary 5.1 and the closedness of the kernel $\pi^{-1}(x)$ now imply that T_f is injective in some neighborhood of x.

6. Tangentiable Mappings

A direction $\vec{d} \in T_M(x)$ is called *continuous* if there exists a continuum $C \subset M$ (i.e., a compact connected set) such that $x \in C$ and $t_C(x)$ consists of a unique element $\vec{d}$.

Lemma 6.1. *If directions $\vec{d_1}, \vec{d_2} \in T_M(x)$ are continuous and x is a regular point of M, then all positive linear combinations $-\alpha_1\vec{d_1} + \alpha_2\vec{d_2}$ belong to $T_M(x)$.*

Proof. Let C_1 and C_2 be continua representing $\vec{d_1}$ and $\vec{d_2}$. For sufficiently small $\varepsilon > 0$, one finds points $x(\varepsilon) \in C_1$ and $y(\varepsilon) \in C_2$ such that $\operatorname{dist}(x(\varepsilon), x) = \alpha_1\varepsilon$ and $\operatorname{dist}(y(\varepsilon), y) = \alpha_2\varepsilon$. Then $\lim\limits_{\varepsilon\to 0} x(\varepsilon)y(\varepsilon)$ represents the direction $-\alpha_1\vec{d_1} + \alpha_2\vec{d_2}$.

Lemma 6.2. *If all directions in $T_M(x)$ are continuous and x is regular, then $L_M(x) = \{\alpha\vec{d} \mid \alpha \geq 0,\ \vec{d} \in T_M(x)\}$.*

Proof. The regularity of x means that $t_M(x) = T_M(x)$. But $T_M(x)$ is symmetric with respect to the origin, since one can replace a sequence of pairs $\{(x_k, y_k)\}_{k\in\mathbb{N}}$ by the sequence $\{(y_k, x_k)\}_{k\in\mathbb{N}}$. Thus, if $\vec{d}$ is continuous then $-\vec{d}$ is also a continuous direction. Thus, for continuous directions $\vec{d_1}$ and $\vec{d_2}$ and for positive α_1 and α_2, we have that the directions along the vector $\alpha_1\vec{d_1} + \alpha_2\vec{d_2}$ belong to $T_M(x)$.

In fact, it suffices to consider Lemma 6.1 with

$$\alpha_1\vec{d_1} + \alpha_2\vec{d_2} = (-\alpha_1)(-\vec{d_1}) + \alpha_2\vec{d_1}\,.$$

One can prove by induction that for a basis $\vec{d_1}, \dots, \vec{d_l}$ of $L_M(x)$, every direction in $L_M(x)$ belongs to $T_M(x)$.

Given a mapping $f : X \to Y$, its *graph* is defined as follows:

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}\,,$$

and the *graph embedding* is the natural homeomorphism of X onto Γ_f denoted by $\Gamma(f) : X \to \Gamma_f \subset X \times Y$.

Lemma 6.3. *A function $f : M \to \mathbb{R}$ is directionally differentiable at x if and only if its graph embedding $\Gamma(f) : M \hookrightarrow M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$ is tangentiable at x and $T_{\Gamma_f}(x, f(x))$ does not contain the "vertical direction" $(0, 0, \dots, 0, 1)$.*

Proof. If a sequence $\{(x'_k, x''_k)\}_{k\in\mathbb{N}}$ represents a direction in $M \subset \mathbb{R}^n$, then it is easy to see that the existence of a finite limit of $\frac{f(x''_k)-f(x'_k)}{\|x''_k - x'_k\|}$ is equivalent to existence of a nonvertical limit for the sequence of directions $\{(f(x''_k), f(x'_k))\}_{k\in\mathbb{N}}$.

Lemma 6.4. *If a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ has tangentiable graph embeddings $\Gamma(\varphi)$ and its graph G has no "vertical" directions, i.e., $T_{\Gamma_\varphi}(x) \cap \{\bar{O} \times \mathbb{R}^n\} = \varnothing$, then φ is continuously differentiable.*

Proof. If $\pi : \mathbb{R}^n \to \mathbb{R}$ is the first coordinate projection of the product, then

$$\mathrm{id} \times \pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$$

projects the graph of φ onto the graph of $\pi \circ \varphi$. The tangentiability of $\pi \times \mathrm{id}|_{\Gamma_\varphi}$ is provided by the condition that Γ_φ has no vertical directions. (This condition means that no element of $T(M)$ belongs to the kernel of $\pi \times \mathrm{id}|_{\Gamma_\varphi}$; see Lemma 5.3.) Since φ has a tangentiable graph, it follows that φ is tangentiable; hence, so is $\pi \circ \varphi$, being a composition of tangentiable maps. Therefore, $\pi \circ \varphi$ also satisfies the "verticality condition." Hence Lemma 6.3 and Sec. 4 produce the continuous differentiability of $\pi \circ \varphi$, and, thus, that of φ.

Theorem 6.1. *The image of every tangentiable mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^{n+m}$ is a C^1-submanifold of $\mathbb{R}^{n+m}$.*

Proof. The case $m = 0$ follows from Lemma 5.1 and the Brouwer theorem on the invariance of domains. Now let $m \geq 1$. First, let us remark that the tangent plane $L_{\varphi(\mathbb{R}^n)}(\varphi(x)) = L^n_x$ is an n-dimensional linear subspace of $\mathbb{R}^{n+m}$. Indeed, note first that for every point $x \in \mathbb{R}^n$, its image $\varphi(x)$ is regular (see Proposition 5.2 (3), applied to $f = \varphi$, $M_1 = \mathbb{R}^n$, and $M_2 = \varphi(\mathbb{R}^n)$). Next, note that the image of every continuous direction is continuous. Finally, by Proposition 5.2 (2), the tangent map is surjective.

Therefore, all tangent directions in $\varphi(\mathbb{R}^n)$ are continuous. It remains to observe that Lemma 6.2 implies the inequality

$$\dim L^n_x \leq n\,,$$

whereas Lemma 7.1 below yields the inequality

$$\dim L^n_x = \dim L_{\varphi(\mathbb{R}^n)}(\varphi(x)) \geq \dim_{\varphi(x)} \varphi(\mathbb{R}^n) = n\,.$$

Denote now by L^m_x the orthogonal complement to L^n_x and by $\pi_n : \mathbb{R}^{m+n} \to L^n_x$ and $\pi_m : \mathbb{R}^{m+n} \to L^m_x$ the canonical orthogonal projections. The composition $\pi_n \circ \varphi : \mathbb{R}^n \to L^n_x$, being locally invertible (Lemma 5.1) and locally tangentially injective (Lemma 5.4), has a local inverse $\psi : V \to O(x)$, which is tangentiable (Lemma 5.2). The graph of $\pi_m(\psi) : V \to L^m_x$ in $L^m_x \oplus L^m_x$ transforms to $\varphi(O(x))$ under the linear homeomorphism $\pi_n \oplus \pi_m : \mathbb{R}^{n+m} \to L^n_x \oplus L^m_x$. Also, the "vertical" direction of $L^n_n \oplus L^n_m$ transforms to the kernel of π_n, which does not intersect $T_{\varphi(M)}(y)$ for $y \in O(x)$ ($O(x)$ is so chosen). Thus, the image of $\varphi(O(x))$ coincides with the graph of a tangential mapping ψ that takes the "vertical" directions. Now, by Lemma 6.4, ψ, being C^1, produces C^1-submanifolds as its image.

Proof of Theorem 2.3 on smooth images. It is implied by Theorem 6.1.

7. Characterization of Submanifolds

We denote by $\dim_x M$ the *topological dimension* of M at x. There are several possibilites for introducing this concept (see, e.g., [2] or [4]). But we need only the following property of such a dimension which is common to all possible approaches to the local dimension.

Proposition 7.1. *For a locally compact set $M \subset \mathbb{R}^n$, one has $\dim_x M = n$ if and only if x is an interior point of M.*

Lemma 7.1. *For every locally compact set $M \subset \mathbb{R}^n$ and every point $x \in M$, one has $\dim_x M \leq \dim L_M(x)$.*

Proof. The orthogonal projection $\pi|_M : M \to L_M(x)$ is a local homeomorphism by Lemmas 5.1 and 5.4. Hence, $\dim_x M \leq \dim O(x) \leq \dim L_M(x)$.

Lemma 7.2. *If $\dim_x M = \dim L_M(x) = k$, then x has a neighborhood $O(x)$ that is a k-dimensional topological submanifold of $\mathbb{R}^n$.*

Proof. Consider the projection $\pi : \mathbb{R}^n \to L_M(x)$ as above. Proposition 7.1 implies that the image of an open neighborhood $O(x)$, where $\pi|_{O(x)}$ is injective, is a neighborhood of $\pi(x)$. Hence, $O(x)$ is a k-dimensional topological manifold and it is a topological submanifold, since its intersection with $\{\pi^{-1}(y)\}_{y\in\pi(O(x))}$ consists of a single point.

Lemma 7.3. *If* $\dim_x M = \dim_M L(x)$ *for all* $x \in M$, *then* M *is a* C^1*-submanifold of* $\mathbb{R}^n$.

Proof. Let the projection $\pi : \mathbb{R}^n \to L_M(x)$ be as above, and let $O(x)$ be as in Lemma 7.2; in addition, we require that $\pi|_{O(x)}$ be tangentially injective (Lemma 5.4). Now, one concludes by Lemma 5.2 that the inverse is tangential, and Theorem 6.1 allows us to prove that the image of $O(x)$ (which is homeomorphic to $\mathbb{R}^n$) is a C^1-submanifold.

A round ball B is said to be *tangent to the set* M *at the point* $x \in M$ if $x \in \partial B$ and Int $B \cap M = \varnothing$.

Lemma 7.4. *If* x *is a regular point of* $M \subset \mathbb{R}^n$ *and it has a tangent ball, then*

$$\dim L_M(x) < n\,.$$

Proof. Let us place x into the origin in such a way that the tangent ball takes a position with coordinate of the center equal to $(r, 0, 0, \ldots, 0)$, where r is its radius. Let us denote by S^{n-1}_+ and S^{n-1}_- the upper and lower hemisphere of S^{n-1}. Hence $S^{n-1}_+ = \{\vec{d} = (d_1, d_2, \ldots, d_n) \mid d_1 > 0\}$.

In this case, it is easy to see that $t(x) \cap S^{n-1}_+ = \varnothing$. However, x is regular and $T(x)$ is always symmetric (as immediately follows from its definition). Hence, $T(x) \cap S^{n-1}_+ = T(x) \cap S^{n-1}_- = \varnothing$, and this means that $T(x)$ is contained in the hyperplane $x_1 = 0$, and so is $L(x)$.

Proof of the characterization Theorem 2.2. According to Lemma 7.3, it suffices to prove that $\dim_M L(x) = \dim_x M$. In the opposite case, there is a point $x \in M$ with $\dim_x M < \dim L(x)$.

Consider the orthogonal projection $\pi : \mathbb{R}^n \to L(x)$. Its restriction is an embedding which does not contain any neighborhood of x. Hence there exists a point $y \in L(x) \setminus \pi(O(x))$ that is arbitrarily close to x. The nearest point of $\pi(O(x))$, which exists for points y sufficiently close to x, since the image $\pi(O(x))$ is locally compact, will be the point z with the tangent ball. But z should be regular as the image of a regular point (Proposition 5.2). Thus, by Lemma 7.4, one concludes that $\dim L_M(z) < \dim L_M(\pi^{-1}(z))$. But π should be tangent injective by Lemma 7.4.

8. Epilogue

Tangent regular points have two fundamental properties: symmetry and continuity. The first one means that for every proper tangent direction, its opposite is also proper. Possibly, the most interesting aspect of our existence Theorem 2.1 is the discovery of the fact that typical points in an arbitrary compactum are tangent symmetric.

The continuity means that the tangent set at a point contains the upper limit of tangent sets converging to the point. The sets $T(x)$ of all tangent directions always have the property of continuity. The concept of improper direction automatically leads to this continuity.

However, the concept of improper tangent direction gives more than the continuity itself. Let us consider a bouquet of two smooth tangent circumferences (see Fig. 1).

Its proper tangent sets (consisting of either two or four points) are symmetric. The proper tangent sets are continuous and the proper tangent plane (linear space spanned by $t_M(x)$) is of constant dimension. However, these are not manifolds.

This example shows the role of improper directions in the characterization Theorem 2.2. The touching point is not tangent regular. All directions of the plane are improper tangent directions to this point.

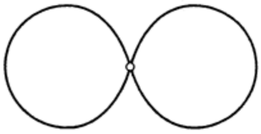

Fig. 1

Our considerations give an approach to introducing *differential structure* via tangent directions. A differential structure is defined when one knows which functions are differentiable. In particular, the differential structure via functions is inherited from smooth manifolds to all of its subsets. Functions on subsets are called *differentiable* if they are restrictions of differentiable functions on an ambient manifold.

Our approach, arising from tangent directions, is as follows. If tangent directions are known, one can introduce the differentiability of functions as the directional differentiability. This approach produces the same result for smooth submanifolds as the one via functions (cf. the main result of Sec. 4). However, for compacta they give different results.

The smooth structure via directions can be defined without embeddings into manifolds. Every continuous $\mathbb{Z}_2$-equivariant mapping $\mathcal{D} : \tilde{X}^2 \to S^n$ of the deleted square $\tilde{X}^2 = \{(x, y) \mid x \neq y\}$ into the n-sphere (the $\mathbb{Z}_2$-action of $\tilde{X}^2$ is the diagonal symmetry $(x, y) \mapsto (y, x)$) generates the directional structure. A sequence $\{(x'_k, x''_k)\}_{k \in \mathbb{N}}$ represents a direction $\vec{d} \in S^n$ if $\lim\limits_{k\to\infty} \mathcal{D}(x'_k, x''_k) = \vec{d}$.

For such a general direction structure, one introduces the concept of proper directions and regular points without any changes. The proof of the existence Theorem 2.1 also proceeds in this general setting without changes. Compacta with all tangent regular points could perhaps play the role of manifolds in such a theory.

In Theorem 6.1 on smooth images, it was proved that every tangentiable mapping $f : \mathbb{R}^n \to \mathbb{R}^{n+m}$ locally factors through a tangentiable homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ and a C^1-smooth embedding $\varphi : \mathbb{R}^n \to \mathbb{R}^{n+m}$. All homeomorphisms of the real line are obviously tangentiable. But for $n > 1$, tangentiable homeomorphisms are very close to diffeomorphisms. In particular, we can prove that the dimension of the set of points of nondifferentiability of such homeomorphism is at most 0-dimensional.

We conclude our paper with the following question.

Question 8.1. *Does there exist a tangentiable homeomorphism of the plane that is not differentiable?*

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REFERENCES

1. D. Dimovski, D. Repovš, and E. V. Shchepin, "C^∞-homogeneous curves on orientable closed surfaces," in: *Geometry and Topology*, G. M. Rassias and G. M. Stratopoulos (Eds.), World Sci. Publ. Co., Singapore (1989), pp. 100–104.

2. R. Engelking, *Dimension Theory*, PWN, Warsaw (1978).

3. H. Federer, *Geometric Measure Theory*, Grund. Math. Wisch. **153**, Springer-Verlag, Berlin (1969).

4. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Univ. Press, Princeton (1941).

5. J. Malešič, "Hilbert–Smith conjecture for Hölder actions," *Usp. Mat. Nauk,* **52**, No. 2, 173–174 (1997).

6. J. Malešič and D. Repovš, "On characterization of Lipschitz manifolds. New developments," in: *Differ. Geom., Budapest,* 1996 (J. Szenthe, Ed.), Kluwer, Dordrecht (1998), pp. 265–278.

7. D. Repovš and E. V. Shchepin, " On smoothness of compacta," *Preprint*, Inst. Math. Phys. Mech. Univ. Ljubljana, **33**, No. 487 (1995).

8. D. Repovš and E. V. Shchepin, "On smoothness of compacta," *Abstr. Amer. Math. Soc.*, **17**, 262, Abstr. No. 96T-57-28 (1996).

9. D. Repovš and E. V. Shchepin, "A proof of the Hilbert–Smith conjecture for actions by Lipschitz maps," *Math. Ann.*, **308**, 361–364 (1997).

10. D. Repovš, A. B. Skopenkov, and E. V. Shchepin, "A characterization of C^1-homogeneous subsets of the plane," *Boll. Unione Mat. Ital. Ser.* A. 7, 437–444 (1993).

11. D. Repovš, A. B. Skopenkov and E. V. Shchepin, "C^1-homogeneous compacta in $\mathbb{R}^n$ are C^1-submanifolds of $\mathbb{R}^n$," *Proc. Amer. Math. Soc.*, **124**, 1219–1226 (1996).

12. D. Repovš, A. B. Skopenkov, and E. V. Shchepin, "Group actions on manifolds and smooth ambient homogeneity," *J. Math. Sci.*, **83**, 546–549 (1997).

13. E. V. Shchepin, "Hausdorff dimension and dynamics of diffeomorphisms," *Mat. Zametki* (in press).