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ON EMBEDDABILITY OF $X \times I$ INTO EUCLIDEAN SPACE

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ABSTRACT. Our main result is the following Theorem: Suppose that X is an acyclic polyhedron, such that $X \times I$ is embeddable into $\mathbb{R}^{n+1}$ (or that $\mathbb{R}^{n+1}$ even contains an uncountable collection of pairwise disjoint copies of X). Then $\tilde{X}^* = \{(x,y) \in X \times X \mid x \neq y\}$ admits an equivariant inessential map to S^n.

One consequence is that if X is an acyclic polyhedron such that $X \times I$ embeds into $\mathbb{R}^{n+1}$ and $\dim X \leq \frac{2n}{3} - 1$, then X embeds into $\mathbb{R}^n$. We prove this independently for collapsible polyhedron X (without dimension restrictions) and for any homologically $(2\dim X - n - 1)$-connected manifold X. We also prove that for each n-dimensional polyhedron X, $X \times I$ embeds into $\mathbb{R}^{2n+1}$.

1. Introduction.

It is well-known that multiplying with an interval may improve properties of topological spaces ([7]). In this paper we study these improvements regarding the theory of embeddings (see also [5], [15]). Our results can be also considered as a generalization of the theorem that the plane does not

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contain uncountably many pairwise disjoint triods [11]. For other generalizations see [1], [2], [4], [13], [20].

Let us first fix some notations. By X we shall denote a compactum. Let $\tilde{X}^* = \{(x,y) \in X^2 \mid x \neq y\}$ be the deleted product of X. We consider the involution $t : \tilde{X}^* \to \tilde{X}^*$, given by $t(x,y) = (y,x)$ for every $x, y \in \tilde{X}^*$ and the antipodal involution on S^n. Denote by $\chi : (\tilde{\mathbb{R}}^{n+1})^* \to S^n$ the map $\chi(x,y) = \frac{x-y}{\|x-y\|}$. $C = c_0 \cup \bigcup_{m=1}^{\infty} c_m$ is a convergent sequence, $c_0 = \lim_{m\to\infty} c_m$. A compactum X is called *acyclic* if $\tilde{H}^n(X) = 0$ for each $n \geq 0$.

Theorem 1.1. *Let X by an acyclic polyhedron such that either:*

(1) *$X \times I$ embeds into $\mathbb{R}^{n+1}$; or*
(2) *$X \times I$ quasi-embeds into $\mathbb{R}^{n+1}$; or*
(3) *$\mathbb{R}^{n+1}$ contains an uncountable collection of pairwise disjoint copies of X; or*
(4) *$X \times C$ embeds into $\mathbb{R}^{n+1}$.*

Then there is an inessential equivariant mapping $\tilde{X}^* \to S^n$.

The idea of the proof is clearest if we use (1). Suppose that $X \times I \subset \mathbb{R}^{n+1}$. Define a map $\chi' : X^2 \to S^n$ by $\chi'(x,y) = \chi((x,0),(y,1))$. Since X is acyclic, X^2 is acyclic and so χ' is inessential. Also $F : \tilde{X}^* \times I \to S^n$, $F((x,y),t) = \chi((x,0),(y,t))$ is a homotopy between $\chi\,|_{\tilde{X}^*}$ and $\chi'\,|_{\tilde{X}^*}$. So, $\chi\,|_{\tilde{X}^*}$ is inessential, too. Also $\chi\,|_{\tilde{X}^*}$ is equivariant, and the theorem follows.

This proof obviously can be refined to obtain an inessential equivariant mapping $\tilde{X}^*_p \to (\tilde{\mathbb{R}}^{n+1})^*_p \sim S^{(n+1)(p-1)-1}$ where $\tilde{X}^*_p$ is the p-fold deleted product of X. This proof is also valid for an arbitrary space X. However, in general neither (2) nor (3) nor (4) imply the conclusion of Theorem (1.1). For example, if X is the pseudo-arc, then each of (2), (3), (4) is fullfilled for $n = 1$. Since X has no separating points, $\tilde{X}^*$ is connected. But if $\tilde{X}^*$ admits an inessential equivariant map to S^1, then by [13, (2,1)], $\tilde{X}^*$ is not connected, which is a contradiction. Nevertheless, either one of (2), (3), (4) implies that for each $\epsilon > 0$ there exists an inessential equivariant mapping on $\tilde{X}^*_\epsilon = \{(x,y) \in X^2 \mid \operatorname{dist}(x,y) \geq \epsilon\}$. This suffices to obtain:

Corollary 1.2. *If there exists a map $f : S^n \to X$ onto an acyclic compactum, which does not identify antipodes, then none of the conditions (1)-(4) above can be fullfilled.*

A theorem on n-dimensional triods [20] follows easily from this corollary.

Theorem 1.3. *Suppose that $X \times I$ PL-embeds into $\mathbb{R}^{n+1}$, where X is either (a) an acyclic polyhedron and* $\dim X \leq \frac{2n}{3} - 1$*; or*

(b) *any homologically* $(2\dim X - n - 1)$*-connected manifold and* $\dim X \leq \frac{2n}{3} - 1$*; or*

(c) *a collapsible polyhedron.*

Then X PL-embeds into $\mathbb{R}^n$.

In Theorem 1.3a the condition of PL-embeddability of $X \times I$ into R^{n+1} can be weakened to either one of (2), (3), (4). Using this fact we construct an example which is interesting in connection with Borsuk's problem on embeddability of n-dimensional AR's into $\mathbb{R}^{2n}$ [3].

Example 1.4. For every n there exists an n-dimensional contractible space which does not embed into $\mathbb{R}^{2n}$.

Theorem 1.3 does not hold for general polyhedra X. Indeed, take an n-dimensional polyhedron X which is not embeddable into $\mathbb{R}^{2n}$. However, $\mathbb{R}^{2n+1}$ contains $X \times I$. This follows from the next improvement of the Nöbeling-Pontryagin theorem (for products of graphs it follows from [9]):

Theorem 1.5. *If X is an n-dimensional compact polyhedron then $X \times I$ embeds into $\mathbb{R}^{2n+1}$.*

2. Proofs.

Proof of Theorem 1.1. The implication (3) $\Rightarrow$ (4) was actually proved in [17] (for a direct proof see [13]). From either embeddability of $X \times C$ or quasi-embeddability of $X \times I$ into $\mathbb{R}^{n+1}$ it follows that for each $\epsilon > 0$ there are ϵ-close ϵ-maps $f, g : X \to \mathbb{R}^{n+1}$ with disjoint images. Let us derive from this the existence of inessential equivariant mapping $\tilde{X}^*_\epsilon \to S^n$. Since f is an ϵ-map it follows that a map $\chi_1 : \tilde{X}^*_\epsilon \to S^n$, $\chi_1(x,y) = \chi(f(x), f(y))$ is well-defined. Take a map $\chi_2 : X^2 \to S^n$ defined by $\chi_2(x,y) = \chi(g(x), f(y))$. Since X is acyclic, X^2 is acyclic and so χ_2 is inessential. Since $\mathrm{dist}(g(x), f(x)) < \epsilon$ for each $x \in X$, for any $(x,y) \in \tilde{X}^*_\epsilon, \chi_1(x,y)$ and $\chi_2(x,y)$ are not antipodal points of S^n. Therefore $\chi_1 |_{\tilde{X}^*_\epsilon}$ is homotopic to $\chi_2 |_{\tilde{X}^*_\epsilon}$ and is also inessential.

Remark 2.1. If ind and coind_L are the Smith index and cohomological index over a ring L (see [6], [19]), then $\text{ind}\tilde{X}^*_\epsilon \leq n-1$ and $\text{coind}_L\tilde{X}^*_\epsilon \leq n-1$.

Proof of Corollary 1.2. Since f does not identify antipodes there exists $\epsilon > 0$ such that $\text{dist}(f(y),(f(-y)) > \epsilon$, for each $y \in S^{n-1}$. Suppose on the contrary, that either one (1), (2), (3), (4) is fullfilled. Then there is an inessential equivariant map $g : \tilde{X}^*_\epsilon \to S^{n-1}$. Let $\tilde{f} : S^{n-1} \to X^2$ be a map, defined by $\tilde{f}(x) = (f(x), f(-x))$. By the choice of $\epsilon, \tilde{f}(S^{n-1}) \subset \tilde{X}^*_\epsilon$. Since g is inessential, it follows that $g\tilde{f} : S^{n-1} \to S^{n-1}$ is inessential. Since $\tilde{f}$ does not identify antipodes and g is equivariant, it follows that $g\tilde{f}$: $S^{n-1} \to S^{n-1}$ does not identify antipodes. This contradicts the Borsuk-Ulam theorem. □

Proof of Theorem 1.3a. By the remark after Theorem 1.1, for each $\epsilon > 0$ there exists an inessential equivariant map $\tilde{X}^*_\epsilon \to S^n$. By [6, 3.12], it follows that for each $\epsilon > 0$, there exists an equivariant map $\tilde{X}^*_\epsilon \to S^{n-1}$. Since X is a polyhedron, for some $\epsilon > 0$ there is an equivariant retraction $\tilde{X}^* \to \tilde{X}^*_\epsilon$ [10]. So, there exists an equivariant mapping $\tilde{X}^* \to S^{n-1}$. By [21] X is embeddable into $\mathbb{R}^n$, since $\dim X \leq \frac{2n}{3} - 1$. □

Proof of Theorem 1.3b. If $X \times I \subset \mathbb{R}^{n+1}$, then the normal bundle of X in $\mathbb{R}^{n+1}$ has a cross-section and so $\bar{\omega}_{n-\dim X}(X) = 0$, [10, (12.1)]. Therefore by [21], X is embeddable in $\mathbb{R}^n$. □

Question 2.2. *Can the restriction* $\dim X \leq \frac{2n}{3} - 1$ *be weakened in Theorem (1.3a,b)? (From the result of [16] it follows that the restriction* $\dim X \leq \frac{2n}{3} - 1$ *cannot be weakened in Weber's theorem [21].)*

Proof of Theorem 1.3c. Suppose $X \times I$ is PL-embedded in $\mathbb{R}^{n+1}$. Take a regular neighbourhood U of $X \times 0$ for a sufficiently small triangulation of $\mathbb{R}^{n+1}$. Since X is collapsible, $\partial U \cong S^n$. We also have that $X \times I \cap \partial U \cong X$. Therefore X is embeddable in S^n. Since $X \neq S^n$, X is PL-embeddable in $\mathbb{R}^n$. □

Construction of Example 1.4. Let F^{n-1} be Flores' $(n-1)$-dimensional polyhedron, not embeddable in $\mathbb{R}^{2n-2}$. Let X be the union of con F^{n-1} and an arc, one of whose ends is the vertex of con F^{n-1} and the other end is denoted by a. By [15], [18], [22], X is not embeddable in $\mathbb{R}^{2n-1}$. Since

$n \leq \frac{2 \cdot 2n}{3} - 1$, by the remark after Theorem (1.3), $(X \times C)/(a \times C)$ is not embeddable in $\mathbb{R}^{2n}$. Therefore $(X \times C)/(a \times C)$ is the space with the desired property.

Proof of Theorem 1.5. Suppose that $f : X \to \mathbb{R}^{2n}$ is a general position mapping. Then $S(f) = \{x \in X \mid |f^{-1}(f(x))| \geq 2\}$ is finite. Triangulate X so that points from $S(f)$ lie in disjoint closed simplices of the triangulation. For each pair of simplexes σ_0, σ_1 of the triangulation, for which $f(\sigma_0) \cap f(\sigma_1) = \{a\}$, take $\epsilon > 0$, such that $\mathcal{O}_\epsilon(a) \cap f(X) = \mathcal{O}_\epsilon(a) \cap f(\sigma_0 \cup \sigma_1)$. For $x \in \sigma_i \cap f^{-1}\mathcal{O}_\epsilon(a)$, let $g(x) = (1 - \frac{\operatorname{dist}(a, f(x))}{\epsilon})\frac{i}{2} \in I$, $g(x) = 0$ for all other x. Then $f_1 : X \times [0, \frac{1}{4}] \to \mathbb{R}^{2n} \times I$, $f_1(x,t) = (f(x), g(x) + t)$ is the desired embedding. □

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