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Connected sums of 4-manifolds

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Abstract

We study the following problem for closed connected oriented manifolds M of dimension 4. Let $\Lambda = \mathbb{Z}[\pi_1(M)]$ be the integral group ring of the fundamental group $\pi_1(M)$. Suppose $G \subset H_2(M;\Lambda)$ is a free Λ-submodule. When do there exist closed connected 4-manifolds P and M' such that M is homotopy equivalent to the connected sum $P \# M'$, where $\pi_1(P) \cong \pi_1(M)$, $\pi_1(M') \cong 0$, and $H_2(M';\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda \cong G$. An answer is given in terms of $\pi_1(M)$ and the intersection forms on $H_2(M;\Lambda)$ and $H_2(M;\mathbb{Z})$.

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1. Introduction

We study the problem of splitting a closed topological manifold M into a nontrivial connected sum according to some algebraic data. In dimension 3 the Kneser conjecture gives the answer if $\pi_1(M) = G_1 * G_2$. In dimension 4 a splitting may be given according to a free product of $\pi_1(M)$ or a direct sum of $\pi_2(M)$, or of both (see, for example, [8,10, 12]). In the present paper we study splittings of closed 4-manifolds M^4 up to homotopy

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equivalence according to a direct sum decomposition $\pi_2(M) = H_2(M;\Lambda) = H \oplus G$ (as Λ-modules), where $\Lambda = \mathbb{Z}[\pi_1(M)]$ is the integral group ring of $\pi_1(M)$. Previous results were proved in [2–4]. Our results are built on those obtained by Hambleton and Kreck in [9]. If $D \to B\pi_1(M)$ is the second Postnikov decomposition of M^4, i.e., $\pi_q(D) = 0$ for every $q \geqslant 3$ and there is a map $M \to D$ which induces isomorphisms on π_1 and π_2, Hambleton and Kreck defined $\mathcal{S}_4^{\mathrm{PD}}(D)$ to be the set of homotopy equivalence classes of polarized oriented 4-dimensional Poincaré complexes. We recall that an element of $\mathcal{S}_4^{\mathrm{PD}}(D)$ is represented by a 3-equivalence $f : X \to D$, where X is a Poincaré 4-complex. Let $[X] \in H_4(X;\mathbb{Z})$ be the fundamental class of X. Then the map

$$\mathcal{S}_4^{\mathrm{PD}}(D) \to H_4(D;\mathbb{Z})$$

sending (X, f) to $f_*([X])$ is well-defined. It was shown in [9] that this map is injective if $\pi_1(M)$ is infinite and $H_2(D;\mathbb{Q}) \neq 0$. If $\pi_1(M)$ is finite of order m, then there is an exact sequence

$$0 \to \mathrm{Tor}\big(\Gamma_2\big(\pi_2(D)\big) \otimes_\Lambda \mathbb{Z}\big) \to \mathcal{S}_4^{\mathrm{PD}}(D) \to \mathbb{Z}_m \times H_4(D;\mathbb{Z})$$

where $\Gamma(\cdot)$ denotes the Whitehead functor (see [9, Theorem 1.1]). To state our results we introduce the $\mathbb{Z}$- and Λ-intersection forms

$$\lambda^C : H_2(M;C) \times H_2(M;C) \to C$$

where C is $\mathbb{Z}$ or Λ. If $G \subset H_2(M;C)$ is a submodule, let λ_G^C be the restriction of λ^C to $G \times G$. We denote the adjoint morphism by

$$\hat{\lambda}_G^C : G \to \mathrm{Hom}_C(G, C) = G^*.$$

Then we prove

Theorem A. *Let M^4 be a closed connected oriented topological 4-manifold with infinite fundamental group. Let $G \subset H_2(M;\Lambda) = \pi_2(M)$ be a Λ-submodule such that*

(1) *G is Λ-free and $\hat{\lambda}_G^\Lambda : G \to G^*$ is an isomorphism*;
(2) *Either $H^2(B\pi_1(M);\Lambda) \cong 0$ or $H_2(M;\Lambda)/G$ is trivial as Λ-module* (*that is, the fundamental group $\pi_1(M)$ acts trivially on it*);
(3) *λ_G^Λ is extended from $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda\mathbb{Z}}$.*

Then there exists a homotopy equivalence $\psi : M \to M_1 = P \# M'$, where P is a Poincaré 4-complex with $\pi_1(P) \cong \pi_1(M)$, M' is a simply connected closed 4-manifold, and $G = H_2(M';\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda$.

Moreover, if $\pi_1(M)$ is “good” (see [7] or [6] for slightly different conditions) and $w_2(G \otimes_\Lambda \mathbb{Z}_2) = 0$, then P can be realized as a manifold.

Remark. The connected sum $M_1 = P \# M'$ can be performed by using the top cell of P. The hypotheses imply $G \otimes_\Lambda \mathbb{Z} \subset H_2(M;\mathbb{Z})$. The first part of the theorem holds for any Poincaré 4-complex M.

To prove Theorem A we have to construct P and M' and a polarization $M_1 = P \# M' \to D$ (see Sections 2 and 3). This can be done for any fundamental group π_1. More precisely, we prove the following result:

Theorem B. *Let M^4 be a Poincaré 4-complex with an arbitrary fundamental group. Let $G \subset H_2(M; \Lambda)$ be a free Λ-submodule such that $\hat{\lambda}_G^{\Lambda} : G \to G^*$ is an isomorphism. Then there is a homotopy equivalence $\psi : M^{(3)} \to (P \# M')^{(3)}$ between 3-skeleta, where P is a Poincaré 4-complex and M' is a closed simply connected topological 4-manifold.*

In order to prove Theorem A we have to show that the images of $[M]$ and $[P \# M']$ under $\mathcal{S}_4^{\mathrm{PD}}(D) \to H_4(D; \mathbb{Z})$ coincide. This will be analyzed in Section 4. If $\pi_1(M)$ is finite, one can extend the homotopy equivalence $M^{(3)} \to (P \# M')^{(3)}$ to a map $M \to P \# M'$. But there is no control over the degree of the map. This defines a component in $\mathbb{Z}_m$. On the other hand if $\pi_1(M)$ is infinite, then the degree is shown to be one. Finally, we recall that there are many important results on connected sum decompositions of 4-manifolds: let us just mention the papers [8,13,14,17], and the book [7] (see [5] for corrections). Further results for 4-manifolds with special fundamental groups were proved in [2–4,12,15,18].

2. Preliminary constructions

Let M^4 be (as in Section 1) a closed connected topological 4-manifold with an orientation and a CW-structure with only one 4-cell. We need this special CW-structure only for homotopy constructions, hence it suffices to have a (simple) homotopy equivalence to a 4-dimensional CW-complex with only one 4-cell. By a theorem of Wall (see [19, Lemma 2.9]) this can be assumed if M is smooth or PL. Let $G \subset H_2(M; \Lambda) \cong \pi_2(M)$ be a Λ-free submodule of rank r such that $\hat{\lambda}_G^{\Lambda} : G \to G^*$ is a Λ-isomorphism. We choose a Λ-basis $e_1, \ldots, e_r$ of G and form the CW-complex P obtained from M by attaching 3-cells along $e_1, \ldots, e_r$. We note that $H_p(P, M; \Lambda)$ (respectively $H^p(P, M; \Lambda)$) is trivial for $p \neq 3$, and isomorphic to G (respectively G^*) for $p = 3$. Furthermore, $H_p(P, M; \mathbb{Z})$ (respectively $H^p(P, M; \mathbb{Z})$) is trivial for $p \neq 3$, and isomorphic to $G \otimes_\Lambda \mathbb{Z}$ (respectively $G^* \otimes_\Lambda \mathbb{Z}$) for $p = 3$. We will denote by $f : M \to P$ the canonical inclusion map. It follows that

$$0 \to H_3(P, M; C) \to H_2(M; C) \xrightarrow{f_*} H_2(P; C) \to 0$$

is exact for $C = \Lambda$ or $\mathbb{Z}$. In particular, the inclusion induced homomorphism $f_* : H_4(M; \mathbb{Z}) \to H_4(P; \mathbb{Z})$ is bijective, and we set $[P] = f_*([M])$, where $[M]$ is the fundamental class of M. Since $\hat{\lambda}_G^{\Lambda}$ is an isomorphism, we get the following diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(P; \Lambda) & \xrightarrow{f^*} & H^2(M; \Lambda) & \longrightarrow & H^3(P, M; \Lambda) = G^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bigcap[P]} & & {\scriptstyle \cong}\downarrow{\scriptstyle \bigcap[M]} & & {\scriptstyle \cong}\uparrow{\scriptstyle \hat{\lambda}_G^{\Lambda}} & & \\
0 & \longleftarrow & H_2(P; \Lambda) & \xleftarrow[f_*]{} & H_2(M; \Lambda) & \longleftarrow & H_3(P, M; \Lambda) = G & \longleftarrow & 0
\end{array}$$

From this we conclude that

$$f^*: H^3(P;\Lambda) \to H^3(M;\Lambda), \qquad f_*: H_3(M;\Lambda) \to H_3(P;\Lambda),$$

and

$$\bigcap[P]: H^2(P;\Lambda) \to H_2(P;\Lambda)$$

are isomorphisms. From the diagrams

$$\begin{array}{ccc} H^1(P;\Lambda) & \xrightarrow[\cong]{f^*} & H^1(M;\Lambda) \\ {\scriptstyle\bigcap[P]}\downarrow & & \cong\downarrow{\scriptstyle\bigcap[M]} \\ H_3(P;\Lambda) & \xleftarrow[\cong]{f_*} & H_3(M;\Lambda) \end{array}$$

and

$$\begin{array}{ccc} H^3(P;\Lambda) & \xrightarrow[\cong]{f^*} & H^3(M;\Lambda) \\ {\scriptstyle\bigcap[P]}\downarrow & & \cong\downarrow{\scriptstyle\bigcap[M]} \\ H_1(P;\Lambda) & \xleftarrow[\cong]{f_*} & H_1(M;\Lambda) \cong 0 \end{array}$$

we obtain isomorphisms

$$\bigcap[P]: H^q(P;\Lambda) \to H_{4-q}(P;\Lambda)$$

for any $q = 1, 3$; similarly, for $q = 0, 4$. Hence we have proved the first part of the following lemma:

Lemma 2.1. *The* CW-*complex P is a Poincaré duality complex of formal dimension* 4*, and $f: M \to P$ is of degree* 1*. If the second Stiefel–Whitney class $w_2: H_2(M;\mathbb{Z}) \to \mathbb{Z}_2$ vanishes on $G \otimes_\Lambda \mathbb{Z}$, then the Spivak normal spherical fibration of P reduces to a* TOP-*fibration.*

Proof. Let $\nu_M: M \to \mathrm{BSTOP}$ be the classifying map for the stable normal bundle of M. Since $w_2(e_i) = 0$, we obtain trivializations of $e_i^*(\nu_M)$ which extend over the attached 3-cells, for any $i = 1, \ldots, r$. Therefore, ν_M extends over P. Then the extension must be a reduction of the Spivak normal spherical fibration of P. □

Lemma 2.2. *The kernel of the homomorphism*

$$H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z} \to H_2(P;\Lambda) \otimes_\Lambda \mathbb{Z}$$

is isomorphic to the kernel of $H_2(M;\mathbb{Z}) \to H_2(P;\mathbb{Z})$. This isomorphism coincides with

$$H_3(P, M;\Lambda) \otimes_\Lambda \mathbb{Z} \xrightarrow[\cong]{} H_3(P, M;\mathbb{Z}).$$

Regarding $H_3(P, M;\mathbb{Z}) \subset H_2(M;\mathbb{Z})$, the restriction of $\lambda_M^{\mathbb{Z}}$ to $H_3(P, M;\mathbb{Z}) \times H_3(P, M;\mathbb{Z})$ is obtained by tensoring λ_M^{Λ} over Λ with $\mathbb{Z}$ and restricting to $(H_3(P, M;\Lambda) \otimes_\Lambda \mathbb{Z}) \times (H_3(P, M;\Lambda) \otimes_\Lambda \mathbb{Z})$.

Proof. For $X = M$ or P we have the following well-known sequence (see [1]):

$$H_3(X; C) \to H_3(B\pi_1; C) \to H_2(X; \Lambda) \otimes_\Lambda C \to H_2(X; C) \to H_2(B\pi_1; C) \to 0.$$

Here C is a Λ-module. We will apply it for $C = \mathbb{Z}$. Since

$$H_2(M; \Lambda) \cong H_2(P; \Lambda) \oplus G,$$

we have the isomorphism

$$\mathrm{Tor}_1^\Lambda\big(H_2(M; \Lambda), \mathbb{Z}\big) \underset{\cong}{\longrightarrow} \mathrm{Tor}_1^\Lambda\big(H_2(P; \Lambda), \mathbb{Z}\big),$$

hence the sequence

$$0 \to H_3(P, M; \Lambda) \otimes_\Lambda \mathbb{Z} \to H_2(M; \Lambda) \otimes_\Lambda \mathbb{Z} \to H_2(P; \Lambda) \otimes_\Lambda \mathbb{Z} \to 0$$

is exact. Note also that $f_* : H_3(M; \mathbb{Z}) \to H_3(P; \mathbb{Z})$ is an isomorphism. This gives the following commutative diagram of exact rows and columns:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & H_3(P, M; \Lambda) \otimes_\Lambda \mathbb{Z} & \to & H_3(P, M; \mathbb{Z}) & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
H_3(M; \mathbb{Z}) & \to & H_3(B\pi_1; \mathbb{Z}) & \to & H_2(M; \Lambda) \otimes_\Lambda \mathbb{Z} & \to & H_2(M; \mathbb{Z}) & \to & H_2(B\pi_1; \mathbb{Z}) & \to & 0 \\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \downarrow & & \downarrow\cong & & \\
H_3(P; \mathbb{Z}) & \to & H_3(B\pi_1; \mathbb{Z}) & \to & H_2(P; \Lambda) \otimes_\Lambda \mathbb{Z} & \to & H_2(P; \mathbb{Z}) & \to & H_2(B\pi_1; \mathbb{Z}) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

Now the claim follows from this diagram. □

Let M' be a closed simply-connected topological 4-manifold which realizes the nonsingular symmetric form $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda\mathbb{Z}}$. We can form in an obvious way the connected sum $M_1 = P \# M'$. The manifold M' has the homotopy type of a wedge of r 2-spheres with a top cell attached, i.e., $M' \simeq (\bigvee_1^r \mathbb{S}^2) \cup_\theta D^4$, where $[\theta] \in \pi_3(\bigvee_1^r \mathbb{S}^2)$ corresponds to $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda\mathbb{Z}}$ under the identification $\pi_3(\bigvee_1^r \mathbb{S}^2) = \Gamma(G \otimes_\Lambda \mathbb{Z})$. Here $\Gamma(A)$ denotes Whitehead's quadratic functor of the Abelian group A (see [20]). The 3-skeleton of M_1 is, up to homotopy, $M_1^{(3)} = P^{(3)} \bigvee (M')^{(2)} = P^{(3)} \vee (\bigvee_1^r \mathbb{S}^2)$. Now we will construct a map

$g : M \to M'$ of degree 1. Let $\beta = \bigvee_1^r e_i : (M')^{(3)} = \bigvee_1^r \mathbb{S}^2 \to M$ be the above given basis. The degree 1 property of f defines a splitting of f^* as follows:

$$\begin{array}{ccccccccc}
 & & H^2((M')^{(3)};\mathbb{Z}) & & & & & & \\
 & & \| & & & & & & \\
0 & \longleftarrow & G^* \otimes_\Lambda \mathbb{Z} & \longleftarrow & H^2(M;\mathbb{Z}) & \xleftarrow{f^*} & H^2(P;\mathbb{Z}) & \longleftarrow & 0 \\
 & & & & \cong \downarrow \bigcap[M] & & \cong \downarrow \bigcap[P] & & \\
0 & \longrightarrow & G \otimes_\Lambda \mathbb{Z} & \longrightarrow & H_2(M;\mathbb{Z}) & \xrightarrow{f_*} & H_2(P;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \| & & \beta^* \uparrow & & & & \\
 & & H_2((M')^{(3)};\mathbb{Z}) & = & H_2((M')^{(3)};\mathbb{Z}) & & & &
\end{array}$$

So there are well-defined elements $u_1, \ldots, u_r \in H^2(M;\mathbb{Z})$ satisfying $u_i \cap e_j = \delta_{ij}$, and $(\bigcap[P])^{-1} f_*(u_i \cap [M]) = 0$ (or equivalently, $f_*(u_i \cap [M]) = 0$). The product

$$u_1 \times \cdots \times u_r : M \to \prod_1^r \mathbb{C}P^\infty$$

restricts to a map $g : M^{(3)} \to \bigvee_1^r \mathbb{S}^2 = (\prod_1^r \mathbb{C}P^\infty)^{(2)}$.

Let $M^* = (\bigvee_1^r \mathbb{S}^2) \cup_{\alpha^*} D^4$, where $\alpha^* : \mathbb{S}^3 \to \bigvee_1^r \mathbb{S}^2$ is the restriction of g to the boundary sphere of $M^{(3)}$. Then g extends to a map $M \to M^*$, also denoted by g. It is obvious that $H_4(M^*;\mathbb{Z}) \cong \mathbb{Z}$, hence we put $[M^*] = g_*([M])$. We identify $(M')^{(3)} = (M^*)^{(3)}$. Furthermore, we denote by $e_1^*, \ldots, e_r^* \in H_2(M^*;\mathbb{Z})$ the canonically given basis and by $u_1^*, \ldots, u_r^*$ its dual in $H^2(M^*;\mathbb{Z})$. By construction, $g^*(u_i^*) = u_i$, and $\beta_*(e_j^*) = e_j$, for any $i, j = 1, \ldots, r$. So we have

$$\left(u_i^* \cup u_j^*\right) \cap \left[M^*\right] = \left(g^* u_i^* \cup g^* u_j^*\right) \cap [M] = (u_i \cup u_j) \cap [M]$$

by identifying $H_0(M^*;\mathbb{Z}) = H_0(M;\mathbb{Z}) = \mathbb{Z}$. Therefore, M^* is a Poincaré complex with the same intersection matrix as M', i.e., M^* is homotopy equivalent to M'.

Lemma 2.3. *There is a degree* 1 *map* $g : M \to M'$ *such that*

$$\bigvee_1^r \mathbb{S}^2 = \left(M'\right)^{(2)} = \left(M'\right)^{(3)} \xrightarrow{\beta} M \xrightarrow{g} M'$$

is homotopic to the inclusion, and

$$\left(M'\right)^{(3)} \xrightarrow{\beta} M \xrightarrow{f} P$$

is homotopic to the constant map.

Proof. Using the above notation we have

$$u_i^* \cap g_* \beta_*\left(e_j^*\right) = g^*\left(u_i^*\right) \cap e_j = u_i \cap e_j = \delta_{ij},$$

hence $\{u_i^*\colon\ i = 1, \ldots, r\}$ is the Hom-dual basis of $\{g_* \beta_*(e_j^*)\colon\ j = 1, \ldots, r\}$. So we have $g_* \beta_*(e_j^*) = e_j^*$, for any $j = 1, \ldots, r$. Therefore, the composition map $g \circ \beta : (M')^{(3)} \to$

$(M')^{(3)}$ is a homotopy equivalence. Since $f_*\beta_*(e_i^*) = f_*(e_i) = 0$, the composition map $f \circ \beta$ is homotopic to the constant map. □

3. The homotopy type of $M^{(3)}$

Let $G \subset H_2(M;\Lambda)$ be, as before, a Λ-free submodule such that $\hat{\lambda}_G^\Lambda : G \to G^*$ is an isomorphism. Thus we have a Poincaré complex P of dimension 4, and a degree 1 map $f : M \to P$ with $f_* : \pi_1(M) \xrightarrow[\cong]{} \pi_1(P)$ and $\mathrm{Ker}(f_* : \pi_2(M) \to \pi_2(P)) \cong G$.

Remark. Instead of the above hypothesis one could start with a degree 1 map $f : M \to P$ such that $f_* : \pi_1(M) \xrightarrow[\cong]{} \pi_1(P)$. The difference with the above assumption is that $\mathrm{Ker}(f_* : \pi_2(M) \to \pi_2(P))$ is only stably Λ-free. The proofs go through under this weaker assumption.

For the following it is convenient to recall the natural exact sequence of Whitehead for a CW-complex X (see [20]):

$$H_4(X;\Lambda) \to \Gamma(\Pi_2(X)) \xrightarrow{\rho} \Pi_3(X) \to H_3(X;\Lambda) \to 0.$$

Recall that $\Gamma(A)$ is the quadratic functor defined on Abelian groups A. If A is a Λ-module, then $\Gamma(A)$ inherits from A a Λ-module structure. So $\Gamma(\pi_2(X))$ is a Λ-module. It is well known that there is a natural identification

$$\Gamma\big(\pi_2(X)\big) = \mathrm{Im}\big(\pi_3\big(X^{(2)}\big) \to \pi_3\big(X^{(3)}\big)\big).$$

The homomorphism ρ is induced from $\pi_3(X^{(3)}) \to \pi_3(X)$, and $\pi_3(X) \to H_3(X;\Lambda)$ is the Hurewicz homomorphism.

Lemma 3.1. *The induced homomorphisms of the map $f : M \to P$ satisfy the following properties*:

(a) $f_* : \pi_2(M^{(3)}) \to \pi_2(P^{(3)})$ *is split surjective; and*
(b) $f_* : \pi_3(M^{(3)}) \to \pi_3(P^{(3)})$ *is surjective.*

Proof. (a) follows from the degree 1 property of the map f. Recall from Section 2 that $f_* : H_3(M;\Lambda) \to H_3(P;\Lambda)$ is an isomorphism. From the diagram

$$\begin{array}{ccccccccc}
0 \to & H_4(M;\Lambda) & \to & H_4(M, M^{(3)};\Lambda) & \to & H_3(M^{(3)};\Lambda) & \to & H_3(M;\Lambda) & \to 0 \\
 & \cong\downarrow f_* & & \cong\downarrow & & \downarrow f_* & & \downarrow f_* & \\
0 \to & H_4(P;\Lambda) & \to & H_4(P, P^{(3)};\Lambda) & \to & H_3(P^{(3)};\Lambda) & \to & H_3(P;\Lambda) & \to 0
\end{array}$$

we get that $f_*: H_3(M^{(3)};\Lambda) \to H_3(P^{(3)};\Lambda)$ is an isomorphism. Then property (b) follows from the following diagram of Whitehead's sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\pi_2(M^{(3)})) & \longrightarrow & \pi_3(M^{(3)}) & \longrightarrow & H_3(M^{(3)};\Lambda) & \longrightarrow & 0 \\ & & \downarrow f_{**} & & \downarrow f_* & & \downarrow f_* & & \\ 0 & \longrightarrow & \Gamma(\pi_2(P^{(3)})) & \longrightarrow & \pi_3(P^{(3)}) & \longrightarrow & H_3(P^{(3)};\Lambda) & \longrightarrow & 0 \end{array}$$

since f_{**} is induced from the split-surjective homomorphism

$$f_*: \pi_2\bigl(M^{(3)}\bigr) \to \pi_2\bigl(P^{(3)}\bigr).$$

Note that Γ satisfies $\Gamma(A \oplus B) \cong \Gamma(A) \oplus \Gamma(B) \oplus (A \otimes B)$. □

Corollary 3.2.

(a) $f_*: \pi_2(M) \to \pi_2(P)$ *is split surjective*; *and*
(b) $f_*: \pi_3(M) \to \pi_3(P)$ *is surjective.*

Since $f_*: \pi_1(M) \to \pi_1(P)$ is an isomorphism, there is a map $\alpha: P^{(2)} \to M^{(2)}$ such that

$$(f \circ \alpha)_* = i_*: \pi_1\bigl(P^{(2)}\bigr) \xrightarrow[\cong]{} \pi_1(P),$$

where $i: P^{(2)} \to P$ is the inclusion.

Lemma 3.3. *The map $\alpha: P^{(2)} \to M^{(2)}$ extends to a map over the* 3*-skeleton* (*still denoted by* α) *such that*

$$f_* \circ \alpha_* = i_*: \pi_2\bigl(P^{(3)}\bigr) \to \pi_2(P),$$

where $i: P^{(3)} \to P$ is the inclusion.

Proof. The difference cochain construction defines a bijection of the set of homotopy classes of extensions of $\alpha|_{P^{(1)}}$ with $C^2(\widetilde{P}, \pi_2(M)) = \mathrm{Hom}_\Lambda(C_2(\widetilde{P}), \pi_2(M))$. Here $\widetilde{X}$ denotes the universal covering space of X as usual. Let $d = d(f \circ \alpha, \text{inclusion}) \in C^2(\widetilde{P}, \pi_2(P))$ be the difference cochain between the composition $f \circ \alpha$ and the inclusion map $i: P^{(2)} \to P$. Since $f_*: \pi_2(M) \to \pi_2(P)$ is surjective and $C_2(\widetilde{P})$ is Λ-free, the induced homomorphism $C^2(\widetilde{P}, \pi_2(M)) \to C^2(\widetilde{P}, \pi_2(P))$ is surjective. Therefore, we can lift d to an element $\tilde{d} \in C^2(\widetilde{P}, \pi_2(M))$. Changing α by $\tilde{d}$ defines a map $\alpha': P^{(2)} \to M$ such that $f \circ \alpha': P^{(2)} \to P$ is homotopic to the inclusion. We are going to denote α' by α. Now, let $\omega \in H^3(P; \pi_2(M))$ be the obstruction to extending α over the 3-skeleta. The natural homomorphism

$$H^3\bigl(P; \pi_2(M)\bigr) \to H^3\bigl(P; \pi_2(P)\bigr)$$

maps ω to the obstruction to extending $f \circ \alpha \simeq i: P^{(2)} \to P$ over $P^{(3)}$, so it is zero. But we have isomorphisms $\pi_2(M) \cong \pi_2(P) \oplus G$ and $G \cong \bigoplus_1^r \Lambda$, hence $H^3(P; \pi_2(M)) \xrightarrow[\cong]{} H^3(P; \pi_2(P))$ because $H^3(P; G) \cong H_1(P; G) \cong 0$. Therefore, $\omega = 0$

and α extends over $P^{(3)}$. Now again, since $f_*:\pi_3(M)\to\pi_3(P)$ is surjective, the difference cochain construction applies to give the desired map

$$\alpha:P^{(3)}\to M. \qquad \square$$

Addendum to Lemma 3.3. The map $f\circ\alpha:P^{(3)}\to P$ is homotopic to the inclusion i, hence it extends to a map $\Theta:P\to P$ of degree 1, i.e., $\Theta|_{P^{(3)}}=f\circ\alpha$. So we have the following diagrams:

$$\begin{array}{ccc} H_4(P,P^{(3)};\Lambda) & \xrightarrow{\Theta_*=\mathrm{id}} & H_4(P,P^{(3)};\Lambda) \\ \downarrow{\scriptstyle\partial_*} & & \downarrow{\scriptstyle\partial_*} \\ H_3(P^{(3)};\Lambda) & \xrightarrow[f_*\circ\alpha_*]{} & H_3(P^{(3)};\Lambda) \end{array}$$

and

$$\begin{array}{ccc} \pi_4(P,P^{(3)}) & \xrightarrow{\theta_*=\mathrm{id}} & \pi_4(P,P^{(3)}) \\ \downarrow{\scriptstyle\partial_*} & & \downarrow{\scriptstyle\partial_*} \\ \pi_3(P^{(3)}) & \xrightarrow[f_*\circ\alpha_*]{} & \pi_3(P^{(3)}) \end{array}$$

The maps $f:M\to P$ and $g:M\to M'$ give rise to a map

$$\psi=(f\times g)|_{M^{(2)}}:M^{(2)}\to\left(P\times M'\right)^{(2)}=P^{(2)}\vee\left(M'\right)^{(2)}=M_1^{(2)}.$$

We will extend ψ over the 3-skeleton to a map, also denoted by ψ, and show that

$$\alpha\vee\beta:P^{(3)}\vee\left(M'\right)^{(3)}=M_1^{(3)}\to M^{(3)}$$

is a homotopy inverse.

First we note that the compositions

$$M^{(2)}\xrightarrow{\psi}M_1^{(2)}\xrightarrow{c}P^{(2)}\xrightarrow{i}P,$$
$$M^{(2)}\xrightarrow{\psi}M_1^{(2)}\xrightarrow{c'}(M')^{(2)}\xrightarrow{i'}M',$$

and

$$\left(M'\right)^{(2)}\xrightarrow{\beta}M^{(2)}\xrightarrow{\psi}M_1^{(2)}\xrightarrow{c'}(M')^{(2)}$$

are equal to $f|_{M^{(2)}}$, $g|_{M^{(2)}}$, and $\mathrm{Id}_{(M')^{(2)}}$, respectively.

Here $c:M_1^{(2)}=P^{(2)}\vee(M')^{(2)}\to P^{(2)}$ and $c':M_1^{(2)}\to(M')^{(2)}$ are the projections, and i and i' are the canonical inclusions.

Lemma 3.4. *The map $\psi:M^{(2)}\to M_1^{(2)}$ extends to a map (still denoted by ψ) $\psi:M^{(3)}\to M_1^{(3)}$ such that the composition*

$$c\circ\psi:M^{(3)}\xrightarrow{\psi}M_1^{(3)}\xrightarrow{c}P^{(3)}$$

is homotopic to $f|_{M^{(3)}}:M^{(3)}\to P^{(3)}$.

Proof. Since $\pi_2(M) \cong \pi_2(P) \oplus G$ and $G \cong \bigoplus_1^r \Lambda$, the induced homomorphism $H^3(M;\pi_2(M_1)) \to H^3(M;\pi_2(P))$ is an isomorphism. The obstruction for extending ψ maps to the obstruction for extending $i \circ c \circ \psi \simeq f|_{M^{(2)}}$, under this isomorphism. So it is zero, and ψ extends over $M^{(3)}$. The extensions are classified by equivariant chain maps

$$C_3(\widetilde{M}^{(3)}) \to \pi_3(M_1^{(3)}),$$

i.e., by elements of $\mathrm{Hom}_\Lambda(C_3(\widetilde{M}^{(3)}), \pi_3(M_1^{(3)}))$. Let $d \in \mathrm{Hom}_\Lambda(C_3(\widetilde{M}^{(3)}), \pi_3(P^{(3)}))$ be the difference cochain of $f|_{M^{(3)}}$ and $c \circ \psi$. Since $c_*:\pi_3(M_1^{(3)}) \to \pi_3(P^{(3)})$ is surjective (same proof as for Lemma 3.1(b)), we can lift d to an element $\tilde{d} \in \mathrm{Hom}_\Lambda(C_3(\widetilde{M}^{(3)}), \pi_3(M_1^{(3)}))$. Changing ψ by $\tilde{d}$ gives the desired extension. □

We note that the composition

$$(M')^{(2)} = (M')^{(3)} \xrightarrow{\beta} M^{(3)} \xrightarrow{\psi} M_1^{(3)} \xrightarrow{c'} (M')^{(3)} = (M')^{(2)} \qquad (*)$$

is still homotopic to $\mathrm{Id}|_{(M')^{(3)}}$.

Lemma 3.5. *The induced homomorphism $\psi_*:\pi_2(M^{(3)}) \to \pi_2(M_1^{(3)})$ is surjective.*

Proof. The composition

$$\pi_2(M_1^{(3)}) \xrightarrow{(\alpha \bigvee \beta)_*} \pi_2(M^{(3)}) \xrightarrow{\psi_*} \pi_2(M_1^{(3)})$$

defines a homomorphism

$$\pi_2(P^{(3)}) \oplus (\pi_2((M')^{(2)}) \otimes_{\mathbb{Z}} \Lambda) \to \pi_2(P^{(3)}) \oplus (\pi_2((M')^{(2)}) \otimes_{\mathbb{Z}} \Lambda).$$

Note that all maps are Λ-homomorphisms. Since

$$(M')^{(2)} \xrightarrow{\beta} M^{(3)} \xrightarrow{f} P^{(3)}$$

is homotopic to zero (see Lemma 2.3), it follows from $(*)$ that an element $(0,b) \in \pi_2(P^{(3)}) \oplus (\pi_2((M')^{(2)}) \otimes_{\mathbb{Z}} \Lambda)$ maps to $(0,b)$. An element

$$(a,0) \in \pi_2(P^{(3)}) \oplus (\pi_2((M')^{(2)}) \otimes_{\mathbb{Z}} \Lambda)$$

goes to the element $(a,\chi(a))$ by Lemmas 3.3 and 3.4, where χ is the composite homomorphism

$$\pi_2(P^{(3)}) \xrightarrow{\alpha_*} \pi_2(M^{(3)}) \xrightarrow{\psi_*} \pi_2(M_1^{(3)}) \xrightarrow{\mathrm{proj}} \pi_2((M')^{(2)}) \otimes_{\mathbb{Z}} \Lambda.$$

Therefore, $(\alpha \vee \beta)_* \circ \psi_*$ is surjective; in fact, it is an isomorphism. Hence

$$\psi_*:\pi_2(M^{(3)}) \to \pi_2(M_1^{(3)})$$

is surjective. □

Lemma 3.6. *The induced homomorphism*

$$\psi_*:\pi_2(M^{(3)}) \to \pi_2(M_1^{(3)})$$

is an isomorphism.

Proof. Lemma 3.4 gives the following diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & K_2(f,\Lambda) & \longrightarrow & H_2(M^{(3)};\Lambda)=\pi_2(M^{(3)}) & \xrightarrow{f_*} & H_2(P^{(3)};\Lambda)=H_2(P;\Lambda) & \longrightarrow 0 \\
& \downarrow & & \downarrow \psi_* & & \| & \\
0 \longrightarrow & K_2(c,\Lambda) & \longrightarrow & H_2(M_1^{(3)};\Lambda)=\pi_2(M_1^{(3)}) & \xrightarrow[c_*]{} & H_2(P^{(3)};\Lambda)=H_2(P;\Lambda) & \longrightarrow 0
\end{array}$$

where $K_2(f,\Lambda)$ and $K_2(c,\Lambda)$ denote the kernels of f_* and c_*, respectively. Note that they are Λ-free. Therefore, the surjective homomorphism

$$\psi_*: H_2\big(M^{(3)};\Lambda\big) \to H_2\big(M_1^{(3)};\Lambda\big)$$

induces a surjective homomorphism

$$\psi_*|_{K_2(f,\Lambda)}: K_2(f,\Lambda) \to K_2(c,\Lambda)$$

and

$$K_2(f,\Lambda) \cong K_2(c,\Lambda) \oplus \operatorname{Ker}(\psi_*|_{K_2(f,\Lambda)}).$$

But we have isomorphisms

$$K_2(f,\Lambda)\otimes_\Lambda \mathbb{Z} \cong \bigoplus_1^r \mathbb{Z} \cong K_2(c,\Lambda)\otimes_\Lambda \mathbb{Z},$$

hence

$$\operatorname{Ker}(\psi_*|_{K_2(f,\Lambda)}) \cong 0.$$

Now the claim follows from the above diagram. □

We can now state the main result of this section.

Theorem 3.7. *Let M be a closed connected topological 4-manifold with a CW-structure so that $M = M^{(3)} \cup_\varphi D^4$. Suppose that $G \subset H_2(M;\Lambda)$ is a Λ-free submodule of rank r such that $\hat{\lambda}_G^\Lambda: G \to G^*$ is an isomorphism. Then there are a Poincaré complex P, a degree 1 map $f: M \to P$ with $f_*: \pi_1(M) \xrightarrow[\cong]{} \pi_1(P)$ and $K_2(f,\Lambda) = G$, a closed simply-connected topological 4-manifold M' with $H_2(M';\mathbb{Z}) = G \otimes_\Lambda \mathbb{Z}$, and a homotopy equivalence $\psi: M^{(3)} \to P^{(3)} \vee (M')^{(3)}$.*

Proof. It remains to prove that ψ is a homotopy equivalence. By Lemma 3.6 this follows once we have proved that $\psi_*: H_3(M^{(3)};\Lambda) \to H_3(M_1^{(3)};\Lambda)$ is an isomorphism. Since $f: M \to P$ and $c: M_1 = P \# M' \to P$ (the "projection" onto P) are of degree 1 and $c_*: \pi_1(M_1) \to \pi_1(P)$ is an isomorphism, we obtain isomorphisms $f_*: H_3(M;\Lambda) \to$

$H_3(P; \Lambda)$ and $c_* : H_3(M_1; \Lambda) \to H_3(P; \Lambda)$ (see Section 2). Now the claim follows from the diagram

$$\begin{array}{ccccccccc}
H_4(M;\Lambda) & \longrightarrow & H_4(M, M^{(3)};\Lambda) & \longrightarrow & H_3(M^{(3)};\Lambda) & \longrightarrow & H_3(M;\Lambda) & \longrightarrow & 0 \\
\cong \downarrow f_* & & \cong \downarrow f_* & & \downarrow f_* & & \cong \downarrow f_* & & \\
H_4(P;\Lambda) & \longrightarrow & H_4(P, P^{(3)};\Lambda) & \longrightarrow & H_3(P^{(3)};\Lambda) & \longrightarrow & H_3(P;\Lambda) & \longrightarrow & 0 \\
\cong \uparrow c_* & & \cong \uparrow c_* & & \uparrow c_* & & \cong \uparrow c_* & & \\
H_4(M_1;\Lambda) & \longrightarrow & H_4(M_1, M_1^{(3)};\Lambda) & \longrightarrow & H_3(M_1^{(3)};\Lambda) & \longrightarrow & H_3(M_1;\Lambda) & \longrightarrow & 0
\end{array}$$

and $c_* \circ \psi_* = f_* : H_3(M^{(3)}; \Lambda) \to H_3(P^{(3)}; \Lambda)$ (by Lemma 3.4). Therefore M and $P \# M'$ have the same 3-type (see [16]). □

4. Extending $\psi : M^{(3)} \to M_1^{(3)}$

In this section we will show that the obstruction to extending ψ to a homotopy equivalence (still denoted by ψ), $\psi : M \to M_1$, is detected by the intersection form $\lambda_M^\Lambda : H_2(M; \Lambda) \times H_2(M; \Lambda) \to \Lambda$. Let us first recall it. If X is a 4-dimensional Poincaré complex, then the cup product defines a map

$$H^2(X; \Lambda) \otimes H^2(X; \Lambda) \to H^4(X; \Lambda \otimes_{\mathbb{Z}} \Lambda) \xrightarrow{\cap [X]} H_0(X; \Lambda \otimes_{\mathbb{Z}} \Lambda) \cong \Lambda.$$

Choosing the Λ-module structures as in [19], it is Λ-linear in the first component and anti-Λ-linear in the second one (by using the canonical anti-involution of Λ). The intersection form λ_X^Λ is obtained from this by passing to $H_2(X; \Lambda) \otimes H_2(X; \Lambda)$ via Poincaré duality. We will identify λ_X^Λ with the cup product. By our main result of Section 3 we have that the first k-invariants k_M and k_{M_1} of M and M_1, respectively, are the same. In fact, $\psi : M^{(3)} \to M_1^{(3)}$ defines an isomorphism of the algebraic 2-types $[\pi_1(M), \pi_2(M), k_M]$ and $[\pi_1(M_1), \pi_2(M_1), k_{M_1}]$. In other words, we have a 2-stage Postnikov system $p : D \to B\pi_1$, and maps $\varphi : M \to D$ and $\varphi_1 : M_1 \to D$ inducing isomorphisms on π_1 and π_2. Note that $\widetilde{D} = K(\pi_2, 2)$ and $\Gamma(\pi_2) = H_4(D; \Lambda)$. There is a natural map

$$F : H_4(D; \mathbb{Z}) \to \operatorname{Hom}_{\Lambda-\overline{\Lambda}}\big(H^2(D; \Lambda) \otimes H^2(D; \Lambda), \Lambda\big)$$

defined by $F(z)(x \otimes y) := (x \cup y) \cap z$. As above, it is Λ-linear in the first component, and anti-Λ-linear (i.e., $\overline{\Lambda}$-linear) in the second one. We can identify λ_M^Λ and $\lambda_{M_1}^\Lambda$ with $F(\varphi_*[M])$ and $F((\varphi_1)_*[M_1])$, respectively. The map F can be defined on the chain level by using an equivariant chain approximation to the diagonal

$$\delta : C_*\big(\widetilde{D}\big) \to C_*\big(\widetilde{D}\big) \otimes_{\mathbb{Z}} C_*\big(\widetilde{D}\big).$$

If $w \in C_4(\widetilde{D})$ represents z, and a and b represent x and y, respectively, then F is induced from

$$\overline{F}(w)(a, b) := \sum a(w')\overline{b(w'')},$$

where $\delta(w) = \sum w' \otimes w''$. Therefore, the map F factorizes over the canonical map

$$H_2(D;\Lambda) \otimes_\Lambda H_2(D;\Lambda) \xrightarrow{\varepsilon} \mathrm{Hom}_{\Lambda-\overline{\Lambda}}\big(H^2(D;\Lambda) \otimes H^2(D;\Lambda), \Lambda\big)$$

defined by $\varepsilon(z_1 \otimes z_2)(x \otimes y) := \langle x, z_1\rangle \overline{\langle y, z_2\rangle}$. We will prove that the obstruction for extending ψ belongs to $H_2(D;\Lambda) \otimes_\Lambda H_2(D;\Lambda)$. We first note that, as a space, D can be obtained from M by attaching cells of dimension $q \geqslant 4$. So we can identify

$$H_2(D;\Lambda) = H_2\big(D^{(3)};\Lambda\big) = H_2\big(M^{(3)};\Lambda\big) \underset{\cong}{\xrightarrow{\psi_*}} H_2\big(M_1^{(3)};\Lambda\big).$$

The Poincaré complex $M_1 = P \# M'$ is obtained from $M_1^{(3)} \simeq P^{(3)} \vee (M')^{(3)}$ by attaching one 4-cell D_1^4 along $[\partial D_1^4] \in \pi_3(M_1^{(3)})$. Similarly, M is obtained from $M^{(3)}$ by attaching a 4-cell D^4 along $[\partial D^4] \in \pi_3(M^{(3)})$. The obstruction to extending $\psi : M^{(3)} \to M_1^{(3)}$ belongs to

$$H^4\big(M;\pi_3(M_1)\big) \cong H_0\big(M;\pi_3(M_1)\big) \cong \pi_3(M_1) \otimes_\Lambda \mathbb{Z}.$$

Obviously, it is equal to

$$i_*\psi_*\big[\partial D^4\big] \otimes_\Lambda 1,$$

where $i : M_1^{(3)} \to M_1$ is the inclusion map. We prefer to analyze the element

$$\psi_*\big[\partial D^4\big] \otimes_\Lambda 1 - \big[\partial D_1^4\big] \otimes_\Lambda 1 = \xi \in \pi_3\big(M_1^{(3)}\big) \otimes_\Lambda \mathbb{Z},$$

or even more

$$\tilde{\xi} = \psi_*\big[\partial D^4\big] - \big[\partial D_1^4\big] \in \pi_3\big(M_1^{(3)}\big).$$

Obviously, $\tilde{\xi} = 0$ implies the vanishing of the obstruction. To state the next lemma we recall that

$$\Gamma\big(\pi_2\big(M_1^{(3)}\big)\big) = \Gamma\big(\pi_2\big(P^{(3)}\big)\big) \oplus \pi_2\big(P^{(3)}\big) \otimes G \oplus \Gamma(G) \subset \pi_3\big(M_1^{(3)}\big).$$

Lemma 4.1. *The element $\tilde{\xi}$ belongs to $\pi_2(P^{(3)}) \otimes G \oplus \Gamma(G)$.*

Proof. The claim follows immediately from the following diagrams of Whitehead's sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\pi_2(M^{(3)})) & \longrightarrow & \pi_3(M^{(3)}) & \longrightarrow & H_3(M^{(3)};\Lambda) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{**}} & & \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} & & \\
0 & \longrightarrow & \Gamma(\pi_2(M_1^{(3)})) & \longrightarrow & \pi_3(M_1^{(3)}) & \longrightarrow & H_3(M_1^{(3)};\Lambda) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\pi_2(M^{(3)})) & \longrightarrow & \pi_3(M^{(3)}) & \longrightarrow & H_3(M^{(3)};\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & \Gamma(\pi_2(P^{(3)})) & \longrightarrow & \pi_3(P^{(3)}) & \longrightarrow & H_3(P^{(3)};\Lambda) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \cong & & \\
0 & \longrightarrow & \Gamma(\pi_2(M_1^{(3)})) & \longrightarrow & \pi_3(M_1^{(3)}) & \longrightarrow & H_3(M_1^{(3)};\Lambda) & \longrightarrow & 0
\end{array}$$

The vertical maps are induced by the map $f : M \to P$ and the collapsing map $c : P \# M' \to P$. The morphisms from the last to the first rows are derived from the map $\psi : M^{(3)} \to M_1^{(3)}$, constructed in Section 3. The isomorphisms $H_3(M^{(3)};\Lambda) \to H_3(P^{(3)};\Lambda)$ and $H_3(M_1^{(3)};\Lambda) \to H_3(P^{(3)};\Lambda)$ are induced by the isomorphisms $H_3(M;\Lambda) \to H_3(P;\Lambda)$ and $H_3(M_1;\Lambda) \to H_3(P;\Lambda)$, respectively, as explained in Section 3. □

It follows from Lemma 2.2 of [9] that $\Gamma(G) \otimes_\Lambda \mathbb{Z} \subset G \otimes_\Lambda G$. Hence we have the following corollary.

Corollary 4.2. *There is a well-defined element $\xi \in \pi_2(P^{(3)}) \otimes_\Lambda G \oplus G \otimes_\Lambda G$ which vanishing implies the extension of ψ.*

As always, tensor products of right (left-)Λ-modules over Λ are formed by using the canonical anti-involution of Λ.

Let us write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \pi_2(P^{(3)}) \otimes_\Lambda G$ and $\xi_2 \in G \otimes_\Lambda G$.

Lemma 4.3. *If $\lambda_G^\Lambda : G \otimes G \to \Lambda$ is extended from $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda \mathbb{Z}}$, then $\xi_2 = 0$.*

Proof. Under the homomorphism

$$\varepsilon : H_2(D;\Lambda) \otimes_\Lambda H_2(D;\Lambda) \to \mathrm{Hom}_{\Lambda-\overline{\Lambda}}\big(H^2(D;\Lambda) \otimes H^2(D;\Lambda), \Lambda\big)$$

the element ξ_2 maps to the difference of λ_G^Λ and the restriction of the pairing $\lambda^\Lambda_{M_1} : H_2(M_1;\Lambda) \times H_2(M_1;\Lambda) \to \Lambda$ to G. But $\lambda^\Lambda_{M_1}$ restricted to G is the Λ-extension of $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda\mathbb{Z}}$ (see Lemma 2.2). It is now obvious that $G \otimes_\Lambda G \subset H_2(D;\Lambda) \otimes_\Lambda H_2(D;\Lambda)$ and $\varepsilon|_{G\otimes_\Lambda G}$ is injective. The claim now follows. □

Lemma 4.4. *Suppose that $H^2(B\pi_1;\Lambda) \cong 0$. Then we have $\xi_1 = 0$.*

Proof. Recall the exact sequence (see [1])

$$\begin{aligned} 0 &\to H^2(B\pi_1;\Lambda) \to H^2(X;\Lambda) \to \mathrm{Hom}_\Lambda\big(H_2(X;\Lambda), \Lambda\big) \\ &\to H^3(B\pi_1;\Lambda) \to H^3(X;\Lambda), \end{aligned}$$

where X can be P, D, M, or M_1. Applied to P, we obtain

$$0 \to H^2(P;\Lambda) \to \mathrm{Hom}_\Lambda\big(H_2(P;\Lambda), \Lambda\big).$$

By Poincaré duality we get that the canonical map $H_2(P;\Lambda) \to \mathrm{Hom}_\Lambda(H^2(P;\Lambda),\Lambda)$ is injective. Since $G \cong \bigoplus_1^r \Lambda$, we obtain an injection

$$H_2(P;\Lambda)\otimes_\Lambda G \to \mathrm{Hom}_\Lambda\big(H^2(P;\Lambda),G\big) \xrightarrow[\cong]{T} \mathrm{Hom}_{\Lambda-\overline{\Lambda}}\big(H^2(P;\Lambda)\otimes G^*,\Lambda\big).$$

Here the isomorphism

$$T:\mathrm{Hom}_\Lambda\big(H^2(P;\Lambda),G\big) \to \mathrm{Hom}_{\Lambda-\overline{\Lambda}}\big(H^2(P;\Lambda)\otimes G^*,\Lambda\big)$$

is defined by

$$T(\eta)(x\otimes y) := \overline{y\big(\eta(x)\big)}.$$

The composition

$$H_2(P;\Lambda)\otimes_\Lambda G \to \mathrm{Hom}_{\Lambda-\overline{\Lambda}}\big(H^2(P;\Lambda)\otimes G^*,\Lambda\big)$$

is the restriction of ε, hence $\varepsilon|_{H_2(P;\Lambda)\otimes_\Lambda G}$ is injective. On the other hand, $\varepsilon(\xi_1)$ is the difference of the intersection Λ-forms (cup products) on $H^2(P;\Lambda)\otimes G^*$. But for both intersection Λ-forms, $H_2(P;\Lambda)$ and G are orthogonal submodules. Therefore, $\varepsilon(\xi_1)=0$, hence $\xi_1 = 0$. □

So far we have used the intersection Λ-form to detect the obstruction. The next lemma gives an example where the integral intersection form detects ξ_1.

Lemma 4.5. *Suppose that $H_2(P;\Lambda)$ is Λ-trivial* (*in the sense of Theorem A, part* (2)) *and without torsion, that is, $H_2(P;\Lambda)\cong \bigoplus_1^s \mathbb{Z}$. Then we have $\xi_1 = 0$.*

Proof. By hypothesis, there is an isomorphism

$$H_2(P;\Lambda)\otimes_\Lambda G \cong H_2(P;\Lambda)\otimes_{\mathbb{Z}} (G\otimes_\Lambda \mathbb{Z}),$$

and the map

$$\varepsilon: H_2(P;\Lambda)\otimes_{\mathbb{Z}} (G\otimes_\Lambda \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}\big(H^2(P;\Lambda)\otimes \big(G^*\otimes_\Lambda \mathbb{Z}\big),\mathbb{Z}\big)$$

is injective. As above, $\varepsilon(\xi_1)$ is the difference of the integral intersection forms (cup products) restricted to $H_2(P;\Lambda)\otimes_{\mathbb{Z}}(G\otimes_\Lambda\mathbb{Z})$. But $H_2(P;\Lambda)$ and $G\otimes_\Lambda\mathbb{Z}$ are orthogonal with respect to both intersection forms. Hence we have $\varepsilon(\xi_1)=0$, which implies that $\xi_1=0$. See also [11] for other results. □

Example. Let F be a closed connected aspherical surface. If $P = F\times\mathbb{S}^2$, then $H_2(P;\Lambda)\cong\mathbb{Z}$. Suppose $\pi_1(M)\cong\pi_1(F)$. It was shown in [4] that there exists a degree 1 map $f:M\to P$ such that $f_*:\pi_1(M)\to\pi_1(P)$ is an isomorphism. Let $G=\mathrm{Ker}(f_*:H_2(M;\Lambda)\to H_2(P;\Lambda))$. Then M is homotopy equivalent to $P\#M'$ if and only if λ_G^Λ is extended from $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda\mathbb{Z}}$.

Summarizing we have proved the following result.

Theorem 4.6. *Let M^4 be a closed connected oriented topological 4-manifold with a CW-decomposition and $\pi_1(M)$ infinite. Suppose $M = M^{(3)} \cup_\varphi D^4$, and let $G \subset H_2(M;\Lambda)$ be a Λ-free submodule so that $\lambda_G^\Lambda : G \times G \to \Lambda$ is extended from $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda \mathbb{Z}}$. If $H^2(B\pi_1;\Lambda) \cong 0$ or $H_2(M;\Lambda)/G$ is a Λ-trivial module, then M is homotopy equivalent to a connected sum $P \# M'$, where P is a Poincaré 4-complex with $\pi_1(P) \cong \pi_1(M)$ and M' is a closed simply-connected topological 4-manifold with $H_2(M';\mathbb{Z}) \cong G \otimes_\Lambda \mathbb{Z}$.*

Proof. If λ_G^Λ is extended from $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda \mathbb{Z}}$, then $\hat{\lambda}_G^\Lambda : G \to G^*$ is an isomorphism. So by previous lemmata there is an extension $\psi : M \to M_1 = P \# M'$. Since $\pi_1(M)$ is infinite, the map ψ is of degree 1. This implies that ψ is a homotopy equivalence. □

5. Application of surgery theory and proof of Theorem A

We assume that $\pi_1(M)$ is a good fundamental group (see, for example, [7]) and $w_2(G \otimes_\Lambda \mathbb{Z}) = 0$. Hence, for a Λ-basis $e_1, \dots, e_r$ of G, we have trivializations

$$t_i : e_i^*(\nu_M) \to \mathbb{S}^2 \times D^{N-4},$$

where ν_M is the normal bundle of $M \subset \mathbb{R}^N$. By using the t_i's we obtain the bundle ν_P over P and a canonical bundle map $b : \nu_M \to \nu_P$ over $f : M \to P$.

Remark. Since M is orientable, the second Stiefel–Whitney class of ν_M coincides with that of M.

The degree 1 normal map (f, b) has a surgery obstruction $\sigma(f,b) \in L_4(\pi_1(M))$. It is represented by $(G, \lambda_G^\Lambda, \mu_G^\Lambda)$, where μ_G^Λ is the self-intersection number defined by the t_i's (see [19, Chapter 5], for more details). The trivializations $t_1, \dots, t_r$ are also used in [19] to define the intersection numbers geometrically. However, they coincide with the algebraic definition via cup product and Poincaré duality. Let us assume that λ_G^Λ is extended from $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda \mathbb{Z}}$ and let the signature of $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda \mathbb{Z}}$ be zero. Then we find a basis of G of type $\{u_1, v_1, u_2, v_2, \dots, u_s, v_s\}$, $2s = r$, with $\lambda_G^\Lambda(u_i, v_i) = 1$, and $\lambda_G^\Lambda(x, y) = 0$ otherwise. It follows from the relations between λ_G^Λ and μ_G^Λ (see [19, Theorem 5.2]) that $\mu_G^\Lambda(u_i) = \mu_G^\Lambda(v_i) = 0$. Since $\pi_1(M)$ is good, surgeries on $\{u_1, v_1, u_2, v_2, \dots, u_s, v_s\}$ can be performed to get a homotopy equivalence $f' : P' \to P$. If the signature of $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda \mathbb{Z}}$ is not zero, then we can form the connected sum of the normal map $f : M \to P$ with an appropriate degree 1 normal map $f'' : M'' \to \mathbb{S}^4$ to get the above situation.

In summary, we have proved the following result which completes the proof of Theorem A.

Theorem 5.1. *If $w_2(G \otimes_\Lambda \mathbb{Z}) = 0$ and λ_G^Λ is extended from $\lambda^{\mathbb{Z}}_{G\otimes_\Lambda \mathbb{Z}}$, then there is a degree 1 normal map $\bar{f} : \overline{M} \to P$ with trivial surgery obstruction. If $\pi_1(P) \cong \pi_1(M)$ is good, then there is a closed connected topological 4-manifold homotopy equivalent to P.*

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