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GENERALIZED MANIFOLDS, NORMAL INVARIANTS, AND $\mathbb{L}$-HOMOLOGY

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Dedicated to the memory of Professor Erik Kjær Pedersen (1946–2020)

Abstract Let X^n be an oriented closed generalized n-manifold, $n \geq 5$. In our recent paper (Proc. Edinb. Math. Soc. (2) 63 (2020), no. 2, 597–607), we have constructed a map $t : \mathcal{N}(X^n) \to H^{st}_n(X^n; \mathbb{L}^+)$ which extends the normal invariant map for the case when X^n is a topological n-manifold. Here, $\mathcal{N}(X^n)$ denotes the set of all normal bordism classes of degree one normal maps $(f, b) : M^n \to X^n$, and $H^{st}_*(X^n; \mathbb{E})$ denotes the Steenrod homology of the spectrum $\mathbb{E}$. An important non-trivial question arose whether the map t is bijective (note that this holds in the case when X^n is a topological n-manifold). It is the purpose of this paper to prove that the answer to this question is affirmative.

Keywords: Generalized manifold; Steenrod $\mathbb{L}$-homology; Poincaré duality complex; normal invariant of degree one map; periodic surgery spectrum $\mathbb{L}$; fundamental complex; Spivak fibration; Pontryagin–Thom construction; Spanier–Whitehead duality; absolute neighbourhood retract

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1. Introduction

Throughout the paper, n will denote an integer ≥ 5. A *generalized manifold* X^n *(without boundary)* of dimension $n \in \mathbb{N}$ is a *Euclidean neighbourhood retract (ENR)* (i.e. X^n is an n-dimensional locally compact separable metrizable absolute neighbourhood retract (ANR)), satisfying the *local Poincaré duality* (i.e. the local homology of X^n is like that of $\mathbb{R}^n$).

In this paper, we shall consider only *oriented connected compact* generalized manifolds. Clearly, every oriented *closed* (i.e. connected, compact and without boundary) topological manifold is such a space (cf. Cavicchioli, Hegenbarth and Repovš [3]).

For every generalized n-manifold X^n, there exists an embedding $\varphi : X^n \hookrightarrow \mathbb{R}^m$ into $\mathbb{R}^m$, for a sufficiently large $m \geq n \in \mathbb{N}$, so that the boundary $\partial N^m \subset \mathbb{R}^m$ of a neighbourhood $N^m \subset \mathbb{R}^m$ of $\varphi(X^n)$ in $\mathbb{R}^m$ is homotopy equivalent to a spherical fibration ν_{X^n}, called the *Spivak fibration*, with fibre homotopy equivalent to S^{m-n-1} (cf. Browder [1]). We shall consider only the oriented case and we shall denote also its classifying map by $\nu_{X^n} : X^n \to BSG$.

A systematic construction of generalized manifolds was given by Bryant, Ferry, Mio and Weinberger [2] (for a comprehensive treatment see Cavicchioli, Hegenbarth and Repovš [3] and Hegenbarth and Repovš [8], and the references therein). It was proved by Ferry and Pedersen [6] that there is a canonical lift $\xi_0 : X^n \to BSTOP$ of ν_{X^n}, i.e. the composition $X^n \xrightarrow{\xi_0} BSTOP \xrightarrow{\mathcal{J}} BSG$ is homotopic to ν_{X^n}. It gives rise to the *canonical surgery problem,* denoted by (f_0, b_0), via the Pontryagin–Thom construction.

Here, $f_0 : M_0^n \to X^n$ is a degree one map, where M_0^n is a closed topological n-manifold and $b_0 : \nu_{M_0^n} \to \xi_0$ is a bundle map, covering the map f_0 (by slightly abusing the notation, we shall denote by $\nu_{M_0^n}$ also the stable normal $\mathbb{R}^{m-n}$-bundle of an embedding $M_0^n \hookrightarrow \mathbb{R}^m$, not just its associated spherical fibration). The canonical surgery problem (f_0, b_0) is unique up to normal bordism.

Let us denote the set of all normal bordism classes of normal degree one maps (f, b) by $\mathcal{N}(X^n)$, where $f : M^n \to X^n$ is a map of degree one, $b : \nu_{M^n} \to \xi$ is a bundle map covering f, and $\xi : X^n \to BSTOP$ is a TOP-reduction of ν_{X^n} (i.e. $\mathcal{J} \circ \xi$ is homotopic to ν_{X^n}).

In the case when X^n is a closed n-manifold, one associates with (f, b) and element in $H_n(X^n; \mathbb{L}^+)$, where $\mathbb{L}^+ = \mathbb{L} < 1 >$ is the (semi-simplicial) *connected surgery spectrum* (cf. Kühl, Macko and Mole [12], Nicas [17], and Ranicki [20, Chapter 18]).

In the case when X^n is a topological n-manifold, this element in $H_n(X^n; \mathbb{L}^+)$ is obtained by decomposing (f, b) into adic pieces, using a transversality structure on the manifold X^n (cf. Ranicki [20, Chapter 16]). This defines a map $t : \mathcal{N}(X^n) \to H_n(X^n; \mathbb{L}^+)$ which is bijective. The image of (f, b) is called the *normal invariant* of the normal degree one map (f, b).

This construction does not carry over to generalized manifolds X^n. If X^n is not homotopy equivalent to a topological n-manifold, there is no transversality structure on X^n. Moreover, what does $\mathbb{L}^+$-homology mean in the class of compact ENR's? In our recent paper, we have proved the following result.

Theorem 1.1 (Hegenbarth-Repovš [9, Theorem 5.1]). *Let X^n be an oriented closed generalized n-manifold, $n \geq 5$. Then one can construct a map*

$$t : \mathcal{N}(X^n) \to H_n^{st}(X^n; \mathbb{L}^+)$$

which extends the normal invariant map in the case when X^n is a topological n-manifold.

Here, $H_*^{st}(X^n; \mathbb{E})$ denotes the *Steenrod homology* of the spectrum $\mathbb{E}$. We refer to Ferry [5], Kahn, Kaminker and Schochet [10], and Milnor [15] for the construction and properties.

As it was already pointed out above, the map $t : \mathcal{N}(X^n) \to H_n^{st}(X^n; \mathbb{L}^+)$ in Theorem 1.1 is bijective for topological n-manifolds X^n. Therefore, it is very natural to ask if perhaps bijectivity of t also holds for generalized n-manifolds X^n? The main goal of the present paper is to show that the answer to this question is affirmative.

Theorem 1.2. *Let X^n be an oriented closed generalized n-manifold, $n \geq 5$. Then the map $t : \mathcal{N}(X^n) \to H_n^{st}(X^n; \mathbb{L}^+)$ in Theorem 1.1 is also a bijection.*

We outline the plan how we shall prove Theorem 1.2. In § 2, we shall recall the construction of the map $t : \mathcal{N}(X^n) \to H_n^{st}(X^n; \mathbb{L}^+)$ from Hegenbarth and Repovš [9]. In § 3, we shall prove that the map $t : \mathcal{N}(X^n) \to H_n^{st}(X^n; \mathbb{L}^+)$ is the composition of maps in the following commutative diagram

$$\begin{array}{ccc} \mathcal{N}(M_0^n) & \xrightarrow{t_0} & H_n(M_0^n; \mathbb{L}^+) \\ \uparrow & & \downarrow (f_0)_* \\ \mathcal{N}(X^n) & \xrightarrow{t} & H_*^{st}(X^n; \mathbb{L}^+) \end{array} \tag{1.1}$$

There are canonical identifications of $\mathcal{N}(M_0^n)$ with $H^0(M_0^n; \mathbb{L}^+)$ and $\mathcal{N}(X^n)$ with $H^0(X^n; \mathbb{L}^+)$ such that $\mathcal{N}(X^n) \to \mathcal{N}(M_0^n)$ corresponds to

$$(f_0)^* : H^0(X^n; \mathbb{L}^+) \to H^0(M_0^n; \mathbb{L}^+).$$

A precise definition will be given at the beginning of § 3.

Here, (f_0, b_0) is the canonical surgery problem mentioned above. It is well known that the composed map

$$H^0(M_0^n; \mathbb{L}^+) \xrightarrow{\cong} \mathcal{N}(M_0^n) \xrightarrow{t_0} H_n(M_0^n; \mathbb{L}^+)$$

is equal to the following composition of isomorphisms

$$H^0(M_0^n; \mathbb{L}^+) \xrightarrow{\cong} \widetilde{H}^{m-n}(T(\nu_{M_0^n}); \mathbb{L}^+) \xrightarrow{SD} H_n(M_0^n; \mathbb{L}^+),$$

where $T(\nu_{M_0^n})$ denotes the *Thom space* of the normal bundle of an embedding $M_0^n \hookrightarrow \mathbb{R}^m$ and the first map is the *Thom isomorphism*. The second map SD denotes the *S-duality* (i.e. the *Spanier-Whitehead duality*) isomorphism (cf. Kühl, Macko, and Mole [12, Chapter 14, p.259] and Ranicki [20, Chapter 17, p.193]).

The same isomorphisms hold for X^n (cf. Ranicki [20, Proposition 16.1 (v), p.175],

$$H^0(X^n; \mathbb{L}^+) \xrightarrow{\cong} \widetilde{H}^{m-n}(T(\nu_{X^n}); \mathbb{L}^+),$$

where we assume $X^n \hookrightarrow \mathbb{R}^m$, and the existence of the isomorphism

$$\widetilde{H}^{m-n}(T(\nu_{X^n}); \mathbb{L}^+) \underset{\cong}{\xrightarrow{SD}} H_n^{st}(X^n; \mathbb{L}^+)$$

follows from Kahn, Kaminker and Schochet [10, Theorem B, p.205].

Finally, in § 4, we shall show that since (f_0, b_0) is a normal degree one map, the following diagram commutes (cf. diagram 4.1 in § 4)

$$\begin{array}{ccccc} H^0(M_0^n;\mathbb{L}^+) & \longrightarrow & \widetilde{H}^{m-n}(T(\nu_{M_0^n});\mathbb{L}^+) & \xrightarrow{SD} & H_n(M_0^n;\mathbb{L}^+) \\ {\scriptstyle (f_0)^*}\big\uparrow & & {\scriptstyle (T(b_0))^*}\big\uparrow & & \big\downarrow{\scriptstyle (f_0)_*} \\ H^0(X^n;\mathbb{L}^+) & \longrightarrow & \widetilde{H}^{m-n}(T(\nu_{X^n});\mathbb{L}^+) & \xrightarrow{SD} & H^{st}_n(X^n;\mathbb{L}^+) \end{array} \tag{1.2}$$

The bottom isomorphism is therefore equal to the composite map

$$H^0(X^n;\mathbb{L}^+) \cong \mathcal{N}(X^n) \to \mathcal{N}(M_0^n) \xrightarrow{t_0} H_n(M_0^n;\mathbb{L}^+) \xrightarrow{(f_0)_*} H^{st}_n(X^n;\mathbb{L}^+).$$

Now the commutativity of diagram (1.1) implies that the map $t : \mathcal{N}(X^n) \to H^{st}_n(X^n;\mathbb{L}^+)$ is indeed bijective, as asserted in Theorem 1.2. Details will be given in the forthcoming sections.

Remark 1.3. In the epilogue (cf. § 5), we shall give an outlook for comparing the exact sequence of a map $q : X^n \to B$, where B is a compact metric space, with the controlled surgery sequence, determined by the map q (cf. Bryant, Ferry, Mio and Weinberger [2]). We are grateful to the referee for suggesting to also include a discussion of this interesting problem.

2. Construction of the map $t : \mathcal{N}(X^n) \to H^{st}_n(X^n;\mathbb{L}^+)$

We recall the construction of the map $t : \mathcal{N}(X^n) \to H^{st}_n(X^n;\mathbb{L}^+)$ from Hegenbarth and Repovš [9, Section 4]. So let us fix an oriented closed generalized n-manifold X^n of dimension $n \geq 5$. If $\mathcal{U}$ is a covering of X^n by open sets, we denote its *nerve* by $N(\mathcal{U})$. If the covering $\mathcal{U}' \prec \mathcal{U}$ is a refinement of $\mathcal{U}$, then there is a simplicial map $s : N(\mathcal{U}') \to N(\mathcal{U})$.

Proposition 2.1. *There exists a sequence of open coverings* $\{\mathcal{U}_j\}_{j\in\mathbb{N}}$ *with the following properties*:

(a) *for every* $j \in \mathbb{N}, \mathcal{U}_{j+1} \prec \mathcal{U}_j$, *and there exists a simplicial map* $s_j : N(\mathcal{U}_{j+1}) \to N(\mathcal{U}_j)$;

(b) *for every* $j \in \mathbb{N}$, *there exist maps* $\varphi_j : X^n \to N(\mathcal{U}_j)$, $\psi_j : N(\mathcal{U}_j) \to X^n$ *such that* $\psi_j \circ \varphi_j : X^n \to X^n$ *is an* ε_j−equivalence, *where* $\lim_{j\to\infty} \varepsilon_j = 0$;

(c) $\varprojlim_j N(\mathcal{U}_j) = X^n$; *and*

(d) *the following diagram is homotopy commutative*

$$\begin{array}{ccccc} & & N(\mathcal{U}_{j+1}) & & \\ & \overset{\varphi_{j+1}}{\nearrow} & & \overset{\psi_{j+1}}{\searrow} & \\ X^n & & \big\downarrow{\scriptstyle s_j} & & X^n \\ & \underset{\varphi_j}{\searrow} & & \underset{\psi_j}{\nearrow} & \\ & & N(\mathcal{U}_j) & & \end{array} \tag{2.1}$$

Proof. See Hegenbarth and Repovš [9, Sections 2 and 3] for verification of properties (a), (b), (d), and Milnor [15, Lemma 2] for property (c). □

Let $M(s_j) = N(\mathcal{U}_{j+1}) \times I \underset{s_j}{\cup} N(\mathcal{U}_j)$ be the mapping cylinder of the map $s_j : N(\mathcal{U}_{j+1}) \to N(\mathcal{U}_j)$. Using property Proposition 2.1 (d), we can form the mapping telescope $F_0 = \underset{j\in\mathbb{N}}{\cup} M(s_j)$ and the obvious maps

$$X^n \times [j, j+1] \xrightarrow{\varphi_j \times Id_{[j,j+1]}} M(s_j) \xrightarrow{\psi_j \times Id_{[j,j+1]}} X^n \times [j, j+1]$$

fit together to give the map $X^n \times \mathbb{R}_+ \xrightarrow{\Gamma} F_0 \xrightarrow{\Lambda} X^n \times \mathbb{R}_+$.

Here, F_0 is a locally finite complex which can be completed to give a complex F such that (cf. Hegenbarth and Repovš [9, Section 3] for details):

(i) at the ∞-end, we add

$$\underset{\overleftarrow{j}}{\lim} N(\mathcal{U}_j) = X^n;$$

(ii) at the 0-end, we add a cone with the cone point c_0.

The complex F_0 (respectively F) is an open (respectively closed) fundamental complex of the (compact metric) space X^n. If $\mathbb{E}$ is an arbitrary spectrum and $H_*^{lf}(F_0; \mathbb{E})$ denotes the locally finite homology of F_0, then the Steenrod homology satisfies the following axiom

$$H_*^{lf}(F_0; \mathbb{E}) \cong H_*^{st}(F, X^n, \{c_0\}; \mathbb{E}).$$

Note that F is contractible, hence we have the following isomorphism

$$H_m^{st}(F, X^n, \{c_0\}; \mathbb{E}) \underset{\cong}{\xrightarrow{\partial}} H_{m-1}^{st}(X^n; \mathbb{E}).$$

We can now outline the construction of the map $t : \mathcal{N}(X^n) \to H_n^{st}(X^n; \mathbb{L}^+)$ (cf. Hegenbarth and Repovš [9, Section 4]). Let (f, b) be a normal degree one map, i.e. $f : M^n \to X^n$ is of degree one and $b : \nu_{M^n} \to \xi$ is a bundle map covering f. As before, (f_0, b_0) denotes the canonical map, i.e. $f_0 : M_0^n \to X^n$, $b_0 : \nu_{M_0^n} \to \xi_0$. Consider the following bundles over F_0: $\eta = \Lambda^*(\xi \times \mathbb{R}_+)$, $\eta_0 = \Lambda^*(\xi_0 \times \mathbb{R}_+)$. Then $\Gamma^*(\eta) \cong \xi \times \mathbb{R}_+$, $\Gamma^*(\eta_0) \cong \xi_0 \times \mathbb{R}_+$, since $\Lambda \circ \Gamma$ is homotopic to $Id_{X^n \times \mathbb{R}_+}$.

One obtains bundle maps (Φ, B) and (Φ_0, B_0) from the following compositions

$$\Phi : M^n \times \mathbb{R}_+ \xrightarrow{f \times Id_{\mathbb{R}_+}} X^n \times \mathbb{R}_+ \xrightarrow{\Gamma} F_0,$$

$$B : \nu_{M^n} \times \mathbb{R}_+ \xrightarrow{b \times Id_{\mathbb{R}_+}} \xi \times \mathbb{R}_+ \xrightarrow{\Gamma} \eta,$$

$$\Phi_0 : M_0^n \times \mathbb{R}_+ \xrightarrow{f_0 \times Id_{\mathbb{R}_+}} X^n \times \mathbb{R}_+ \xrightarrow{\Gamma} F_0,$$

$$B_0 : \nu_{M_0^n} \times \mathbb{R}_+ \xrightarrow{b_0 \times Id_{\mathbb{R}_+}} \xi_0 \times \mathbb{R}_+ \xrightarrow{\Gamma} \eta_0.$$

Their mapping cylinders $M(\Phi, B)$ (respectively $M(\Phi_0, B_0)$) are normal spaces with boundaries $(M^n \times \mathbb{R}_+) \amalg F_0$ (respectively $(M_0^n \times \mathbb{R}_+) \amalg F_0$). Gluing them along F_0 yields the normal space

$$N = M(F,B) \underset{F_0}{\cup} -M(F_0,B_0), \quad \partial N = M^n \times \mathbb{R}_+ \underset{F_0}{\cup} M_0^n \times \mathbb{R}_+,$$

where the minus sign denotes the opposite orientation on $M(F_0, B_0)$.

This normal space N can be decomposed into adic pieces to define an element in $H^{lf}_{n+2}(F_0; \Omega^{NSTOP})$, where Ω^{NSTOP} is the semi-simplicially defined spectrum of adic normal spaces with manifold boundary (cf. Kühl, Macko and Mole [12, Section 11] for the precise definition).

There is a similar spectrum Ω^{NPD}, where the boundaries are Poincaré duality spaces, and there exists an obvious map $\Omega^{NSTOP} \to \Omega^{NPD}$. Moreover, there is a map of spectra $\Omega^{NPD} \to \mathbb{L}^+$ (cf. Ranicki [19, p.287]), inducing isomorphisms in homology theory (cf. Hausmann and Vogel [7], Levine [13], Quinn [18]). The composition $\Omega^{NSTOP} \to \Omega^{NPD} \to \mathbb{L}^+$ is called $sign^{\mathbb{L}}$ in Kühl, Macko and Mole [12, p.232].

A word about notation: we shall denote the element represented by $M(\Phi, B) \underset{F_0}{\cup} -M(\Phi_0, B_0)$ by

$$\{f, b\} - \{f_0, b_0\} \in H^{lf}_{n+2}(F_0; \Omega^{NSTOP})$$

and its image under

$$H^{lf}_{n+2}(F_0; \Omega^{NSTOP}) \xrightarrow{\cong} H^{st}_{n+2}(F, X^n, \{c_0\}; \Omega^{NSTOP})$$
$$\underset{\cong}{\xrightarrow{\partial}} H^{st}_{n+1}(X^n; \Omega^{NSTOP}) \xrightarrow{sign^{\mathbb{L}}} H^{st}_n(X^n; \mathbb{L}^+)$$

will be denoted by $[f, b] - [f_0, b_0]$.

Finally, one can then show that the map $t : \mathcal{N}(X^n) \to H^{st}_n(X^n; \mathbb{L}^+)$ sending (f, b) to $[f, b] - [f_0, b_0]$, is well defined (cf. Hegenbarth and Repovš [9, Theorem 5.1]).

3. Factorization of the map $t : \mathcal{N}(X^n) \to H^{st}_n(X^n; \mathbb{L}^+)$

This section is devoted to studying diagram (1.2).

I. First, one has to define the map $\mathcal{N}(X^n) \to \mathcal{N}(M_0^n)$. We shall keep the notation from § 2, so (f_0, b_0) denotes the canonical surgery problem for an oriented closed generalized n-manifold X^n with $f_0 : M_0^n \to X^n$, $b_0 : \nu_{M_0^n} \to \xi_0$.

Let (f, b) represent an element in $\mathcal{N}(X^n)$, where $f : M^n \to X^n$, $b : \nu_{M^n} \to \xi$. We shall also write $\xi_0, \xi : X^n \to BSTOP$ for the corresponding classifying maps. Their compositions with $\mathcal{J} : BSTOP \to BSG$ are homotopic.

Consider now the bundles $(f_0)^*(\xi_0)$ and $(f_0)^*(\xi)$ over M_0^n. Observe that $(f_0)^*(\xi_0) = \nu_{M_0^n}$ and that $(f_0)^*(\xi)$ is fibre homotopy equivalent to $\nu_{M_0^n}$. In other words, $(f_0)^*(\xi)$ is a TOP-reduction of the Spivak fibration of the manifold M_0^n.

Therefore $(f_0)^*(\xi)$ defines a surgery problem $f' : M'^n \to M_0^n$, $b' : \nu_{M'^n} \to (f_0)^*(\xi)$, which we shall denote by (f', b'). These are well-known constructions (cf. Browder [1, Section II.4], Madsen and Milgram [14, Chapter 2], Wall [22, Chapter 10]).

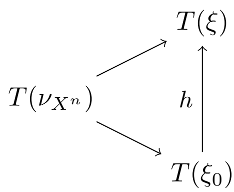
Lemma 3.1. *The composition of the normal maps*

$$M'^n \xrightarrow{f'} M_0^n \xrightarrow{f_0} X^n, \quad \nu_{M'^n} \xrightarrow{b'} (f_0)^*(\xi) \xrightarrow{\tilde{f}_0} \xi,$$

where $\tilde{f}_0$ *is the obvious bundle map covering the map* f_0, *is normally bordant to* (f, b).

Proof. For the proof, we have to describe (f_0, b_0), (f, b), and (f', b') in more details. Suppose that X^n is embedded into S^m, for some sufficiently large $m \geq n$, with a regular neighbourhood $W^m \subset S^m$ and a retraction $r : W^m \twoheadrightarrow X^n$. Thus, $r|_{\partial W^m} : \partial W^m \twoheadrightarrow X^n$ is homotopy equivalent to the spherical fibration ν_{X^n}, giving rise to $\beta : S^m \to W^m/\partial W^m \to T(\nu_{X^n})$.

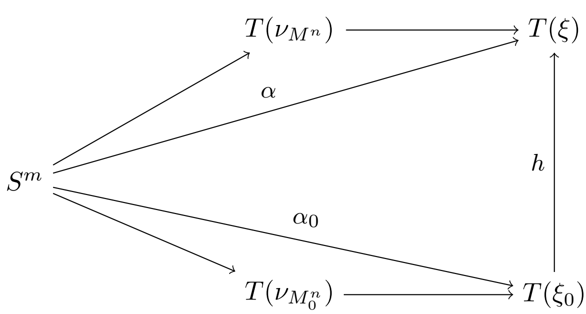
The TOP-reductions ξ_0 and ξ of ν_{X^n} then yield the following homotopy commutative diagram



(3.1)

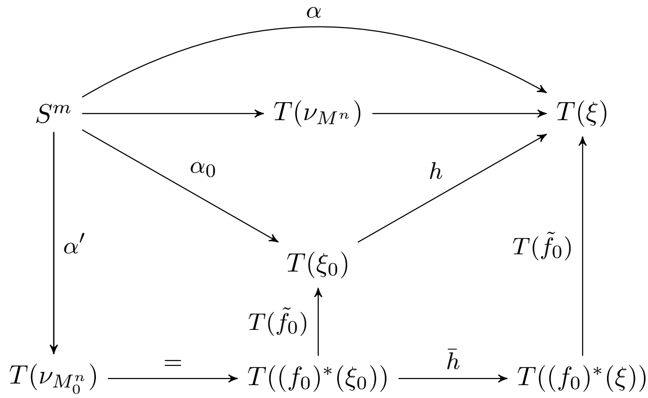
Note that $h : T(\xi_0) \to T(\xi)$ is induced by a fibre homotopy equivalence $\dot{\xi}_0 \sim \nu_{X^n} \sim \dot{\xi}$, where $\dot{\xi}_0$ (respectively $\dot{\xi}$) denotes the sphere bundles of ξ_0 (respectively ξ).

Denote the compositions with β by $\alpha_0 : S^m \to T(\xi_0)$, $\alpha : S^m \to T(\xi)$. They can be made transverse to $X^n \subset T(\xi_0)$ (respectively $T(\xi)$) in order to obtain $\alpha_0^{-1}(X^n) = M_0^n$ (respectively $\alpha^{-1}(X^n) = M^n$), and b_0 (respectively b) are the obvious maps from their normal bundles in S^m. Moreover, α_0 (respectively α) factor as $S^m \to T(\nu_{M_0^n}) \to T(\xi_0)$ (respectively $S^m \to T(\nu_{M^n}) \to T(\xi)$) and we have the following homotopy commutative diagram



(3.2)

Note that $h : T(\xi_0) \to T(\xi)$ induces a homotopy equivalence $\bar{h} : T((f_0)^*(\xi_0)) \to T((f_0)^*(\xi))$. However, $(f_0)^*(\xi_0) = \nu_{M_0^n}$, so we get the following homotopy commutative diagram



(3.3)

Here, $\tilde{f}_0 : (f_0)^*(\xi) \to \xi$ (respectively $\tilde{f}_0 : (f_0)^*(\xi_0) \to \xi_0$) are the obvious bundle maps over $f_0 : M_0^n \to X^n$ (for simplicity we use the same symbol $\tilde{f}_0$ for both maps), and $T(\tilde{f}_0)$ is the induced map between the Thom spaces, so $T(\tilde{f}_0)^{-1}(X^n) = M_0^n$, similarly for $T(\tilde{f}_0) : T((f_0)^*(\xi_0)) \to T(\xi_0)$. Note that h and $\bar{h}$ are not induced by bundle maps.

By making the composition

$$S^m \xrightarrow{\alpha'} T(\nu_{M_0^n}) = T((f_0)^*(\xi_0)) \xrightarrow{\bar{h}} T((f_0)^*(\xi))$$

transverse to M_0^n, one obtains the surgery problem

$$M'^n = (\bar{h} \circ \alpha')^{-1}(M_0^n) \to M_0^n, \quad b' : \nu_{M'^n} \to (f_0)^*(\xi).$$

Homotopy commutativity of diagram (3.3) then implies that

$$M'^n \xrightarrow{f'} M_0^n \xrightarrow{f_0} X^n, \quad \nu_{M'^n} \xrightarrow{b'} (f_0)^*(\xi) \xrightarrow{\tilde{f}_0} \xi$$

is normally bordant to (f, b). To see this, observe that (f, b) is obtained from the upper arrow α, whereas the composition $(f_0, b_0) \circ (f', b')$ is obtained from the composition of the arrows $\downarrow\longrightarrow\uparrow$, that is $T(\tilde{f}_0) \circ \bar{h} \circ \alpha'$. Note that $T(\tilde{f}_0)$ produces (f_0, b_0) and $\bar{h} \circ \alpha'$ gives (f', b'). This completes the proof of Lemma 3.1. ☐

Remark 3.2. One might expect that homotopy commutativity of diagram (3.3) implies that (f_0, b_0) and (f, b) are normally bordant. However, this is not the case since h (respectively $\bar{h}$) are not induced by TOP-bundle maps.

The association $(f, b) \to (f', b')$ defines a map $\mathcal{N}(X^n) \to \mathcal{N}(M_0^n)$. It depends on the fixed surgery problems (f_0, b_0), and $Id_{M_0^n} : M_0^n \xrightarrow{\cong} M_0^n$, $Id_{\nu_{M_0^n}} : \nu_{M_0^n} \xrightarrow{\cong} \nu_{M_0^n}$. We shall relate this map using the following identifications (cf. Kühl, Macko and Mole [12, Chapter 14, in particular Section 14.23])

$$\mathcal{N}(X^n) \to [X^n, G/TOP], \quad \mathcal{N}(M_0^n) \to [M_0^n, G/TOP].$$

Given $f : M^n \to X^n$, $b : \nu_{M^n} \to \xi$, we know that $\xi \oplus (-\xi_0) : X^n \to BTOP$ classifies the Whitney sum of ξ and $-\xi_0$. The composition with $\mathcal{J} : BTOP \to BSG$ is homotopic to the constant map, hence it yields a map $X^n \to G/TOP$. This defines a bijection $\mathcal{N}(X^n) \to [X^n, G/TOP]$, depending on (f_0, b_0).

Let us denote the image of $(f, b) \in \mathcal{N}(X^n)$ in $[X^n, G/TOP]$ by $[\xi - \xi_0]$. Similarly, $\mathcal{N}(M_0^n) \to [M_0^n, G/TOP]$ can be defined using $Id_{M_0^n} : M_0^n \xrightarrow{\cong} M_0^n$, $Id_{\nu_{M_0^n}} : \nu_{M_0^n} \xrightarrow{\cong} \nu_{M_0^n}$. The construction above then implies the following corollary.

Corollary 3.3. *The diagram*

$$\begin{array}{ccc} \mathcal{N}(M_0^n) & \longrightarrow & [M_0^n, G/TOP] \\ \uparrow & & \uparrow (f_0)^* \\ \mathcal{N}(X^n) & \longrightarrow & [X^n, G/TOP] \end{array} \tag{3.4}$$

commutes. Moreover, $(f_0)^*([\xi - \xi_0]) = [(f_0)^*(\xi) - \nu_{M_0^n}]$.

II. Next, we shall show how (f', b') can be used to calculate $t(f, b) \in H_n^{st}(X^n; \mathbb{L}^+)$. By crossing (f', b') with $\mathbb{R}_+$, one gets a normal map

$$f' \times Id_{\mathbb{R}_+} : M'^n \times \mathbb{R}_+ \to M_0^n \times \mathbb{R}_+, \quad b' \times Id_{\mathbb{R}_+} : \nu_{M'^n} \times \mathbb{R}_+ \to (f_0)^*(\xi) \times \mathbb{R}_+,$$

denoted by $(f', b') \times Id_{\mathbb{R}_+}$. The mapping cylinder $M((f', b') \times Id_{\mathbb{R}_+})$ of the map $(f', b') \times Id_{\mathbb{R}_+}$ is a normal space with manifold boundary, hence it defines an element

$$M((f', b') \times Id_{\mathbb{R}_+}) \in H_{n+2}^{lf}(M_0^n \times \mathbb{R}_+; \Omega^{NSTOP}).$$

Lemma 3.4. *Let* $\Gamma_0 : M_0^n \times \mathbb{R}_+ \to F_0$ *be defined as the composition of the maps*

$$f_0 \times Id_{\mathbb{R}_+} : M_0^n \times \mathbb{R}_+ \to X^n \times \mathbb{R}_+, \quad \Gamma : X^n \times \mathbb{R}_+ \to F_0.$$

Then Γ_0 *induces a homomorphism*

$$(\Gamma_0)_* : H_{n+2}^{lf}(M_0^n \times \mathbb{R}_+; \Omega^{NSTOP}) \to H_{n+2}^{lf}(F_0; \Omega^{NSTOP}),$$

such that

$$(\Gamma_0)_*([M((f', b') \times Id_{\mathbb{R}_+})]) = \{f, b\} - \{f_0, b_0\}.$$

Proof. The element $(\Gamma_0)_*([M((f', b') \times Id_{\mathbb{R}_+})])$ is represented by the mapping cylinder

$$(M'^n \times \mathbb{R}_+) \times I \bigcup_{f' \times Id_{\mathbb{R}_+}} M_0^n \times \mathbb{R}_+,$$

but decomposed according to the dissection given by $\Gamma_0 : M_0^n \times \mathbb{R}_+ \to F_0$. The element $\{f, b\} - \{f_0, b_0\}$ is represented by

$$(M'^n \times \mathbb{R}_+) \times I \bigcup_{\Phi} F_0 \bigcup_{F_0} -(M_0^n \times \mathbb{R}_+) \times I \bigcup_{\Phi_0} F_0,$$

as described in § 2. By Lemma 3.1, it is equivalent to the mapping cylinder construction based on the composition of the normal maps $(f_0, b_0) \circ (f', b')$. It gives the following

$$(M'^n \times \mathbb{R}_+) \times I \bigcup_{f' \times Id_{\mathbb{R}_+}} M_0^n \times \mathbb{R}_+ \cup (M_0^n \times \mathbb{R}_+) \times I \bigcup_{\Gamma_0} F_0 \cup -(M_0^n \times \mathbb{R}_+) \times I \bigcup_{\Gamma_0} F_0,$$

(cf. Ferry [4, Proposition 8.10] for the mapping cylinder calculations).

This is obviously bordant to

$$(M'^n \times \mathbb{R}_+) \times I \underset{f' \times Id_{\mathbb{R}_+}}{\cup} M_0^n \times \mathbb{R}_+$$

since

$$(M_0^n \times \mathbb{R}_+) \times I \underset{\Gamma_0}{\cup} F_0 \cup -(M_0^n \times \mathbb{R}_+) \times I \underset{\Gamma_0}{\cup} F_0$$

is 0-bordant. This completes the proof of Lemma 3.4. □

Now (f', b') is a normal degree one map between manifolds, so it defines an element $[f', b'] \in H_n(M_0^n; \mathbb{L}^+)$, namely its normal invariant.

Corollary 3.5. *Consider the homomorphism* $(f_0)_* : H_n(M_0^n; \mathbb{L}^+) \to H_n^{st}(X^n; \mathbb{L}^+)$. *Then* $(f_0)_*([f', b']) = [f, b] - [f_0, b_0]$.

Remark 3.6. If X^n happens to be a topological n-manifold, then this is the Ranicki composition formula (cf. Ranicki [21, Proposition 2.7]).

Proof. The assertion follows from the following diagram

$$\begin{array}{ccc}
H^{lf}_{n+2}(M_0^n \times \mathbb{R}_+; \Omega^{NSTOP}) & \xrightarrow{(\Gamma_0)_*} & H^{lf}_{n+2}(F_0; \Omega^{NSTOP}) \\
\downarrow \cong & & \downarrow \cong \\
H^{st}_{n+2}(M_0^n \times [0,\infty], M_0^n \times \{\infty\}, M_0^n \times \{0\}; \Omega^{NSTOP}) & \xrightarrow{(\bar{\Gamma}_0)_*} & H^{st}_{n+2}(F, X^n, \{c_0\}; \Omega^{NSTOP}) \\
\downarrow & & \downarrow \\
H_{n+1}(M_0^n; \Omega^{NSTOP}) & \xrightarrow{(f_0)_*} & H^{st}_{n+1}(X^n; \Omega^{NSTOP}) \\
\downarrow (sign^{\mathbb{L}^+})_* & & \downarrow (sign^{\mathbb{L}^+})_* \\
H_n(M_0^n; \mathbb{L}^+) & \xrightarrow{(f_0)_*} & H^{st}_n(X^n; \mathbb{L}^+)
\end{array} \tag{3.5}$$

Note that the element $[M((f', b') \times Id_{\mathbb{R}_+})] \in H^{lf}_{n+2}(M_0^n \times \mathbb{R}_+; \Omega^{NSTOP})$ maps to $[f', b']$ under the left vertical arrow of morphisms. The completion of Γ_0 then gives the map $\bar{\Gamma}_0 : M_0^n \times [0, \infty] \to F$. This completes the proof of Corollary 3.5. □

III. Summary: Let X^n be an oriented closed generalized manifold of dimension $n \geq 5$, and $f_0 : M_0^n \to X^n$, $b_0 : \nu_{M_0^n} \to \xi_0$ a surgery problem according to a $BSTOP$-reduction of ν_{X^n}. Then the map $t : \mathcal{N}(X^n) \to H_n^{st}(X^n; \mathbb{L}^+)$, defined in § 2, fits into the following commutative diagram

$$\begin{array}{ccc} \mathcal{N}(M_0^n) & \xrightarrow{t_0} & H_n(M_0^n; \mathbb{L}^+) \\ \uparrow & & \downarrow (f_0)_* \\ \mathcal{N}(X^n) & \xrightarrow{t} & H_n^{st}(X^n; \mathbb{L}^+) \end{array} \tag{3.6}$$

Here, t_0 sends a normal degree one map with target M_0^n to its normal invariant. Moreover, under the identification of Corollary 3.3, diagram (3.6) can be redrawn as follows

$$\begin{array}{ccc} [M_0^n, G/TOP] & \longrightarrow & H_n(M_0^n; \mathbb{L}^+) \\ (f_0)^* \uparrow & & \downarrow (f_0)_* \\ [X^n, G/TOP] & \longrightarrow & H_n^{st}(X^n; \mathbb{L}^+) \end{array} \tag{3.7}$$

4. Proof of Theorem 1.2

The essence of the proof was already given in § 1, by comparing diagrams (1.1) and (1.2). In this section, we present the details.

Let $\mathbb{L}^\bullet$ denote the symmetric $\mathbb{L}$-spectrum (cf. Ranicki [20, Chapter 13]). It is a ring spectrum and $\mathbb{L}^+$ is a $\mathbb{L}^\bullet$-module spectrum. Hence, the cup product constructions $H^q(Z, A; \mathbb{L}^\bullet) \times H^p(Z; \mathbb{L}^+) \to H^{p+q}(Z, A; \mathbb{L}^+)$ are well defined.

Considering an oriented closed generalized n-manifold, embedded in $X^n \subset S^m$, for some $m \geq n$, its Spivak fibration ν_{X^n} has a canonical orientation (cf. Ranicki [20, Chapter 16]), i.e. a Thom class

$$\mathcal{U}_{\nu_{X^n}} \in H^{m-n}(E(\nu_{X^n}), \partial E(\nu_{X^n}); \mathbb{L}^\bullet) \cong \tilde{H}^{m-n}(T(\nu_{X^n}); \mathbb{L}^\bullet),$$

inducing the Thom isomorphism (here, $E(\nu_{X^n})$ is the associated disk fibration)

$$H^0(X^n; \mathbb{L}^+) = H^0(E(\nu_{X^n}); \mathbb{L}^+) \xrightarrow{\cup\, \mathcal{U}_{\nu_{X^n}}} \tilde{H}^{m-n}(T(\nu_{X^n}); \mathbb{L}^+).$$

Recall that *canonical* means that it is constructed via the canonical reduction ξ_0 of ν_{X^n}. Hence, the Thom class $\mathcal{U}_{\xi_0} \in \tilde{H}^{m-n}(T(\xi_0); \mathbb{L}^\bullet)$, corresponds to $\mathcal{U}_{\nu_{X^n}}$ under the homotopy equivalence between $T(\xi_0)$ and $T(\nu_{X^n})$.

The existence of $\mathcal{U}_{\xi_0}$ is guaranteed (cf. Ranicki [20, Chapter 16]). Moreover, since $(f_0)^*(\xi_0) \cong \nu_{M_0^n}$, it follows that $f_0 : M_0^n \to X^n$, $b_0 : \nu_{M_0^n} \to \xi_0$ induces $T(b_0) : T(\nu_{M_0^n}) \to T(\xi_0)$, so that under

$$(T(b_0))^* : H^{m-n}(T(\xi_0); \mathbb{L}^\bullet) \to H^{m-n}(T(\nu_{M_0^n}); \mathbb{L}^\bullet),$$

$\mathcal{U}_{\xi_0}$ is mapped to $\mathcal{U}_{\nu_{M_0^n}}$, the Thom class of $\nu_{M_0^n}$. This implies commutativity of the following diagram

$$\begin{array}{ccc} H^0(M_0^n;\mathbb{L}^+) & \xrightarrow{\ .\cup\,\mathcal{U}_{\nu_{M_0^n}}\ } & \tilde{H}^{m-n}(T(\nu_{M_0^n});\mathbb{L}^+) \\ {\scriptstyle (f_0)^*}\uparrow & & \uparrow{\scriptstyle (T(b_0))^*} \\ H^0(X^n;\mathbb{L}^+) & \xrightarrow{\ .\cup\,\mathcal{U}_{\nu_{X^n}}\ } & \tilde{H}^{m-n}(T(\xi_0);\mathbb{L}^+) \end{array} \tag{4.1}$$

The Thom isomorphisms are now composed with the S-duality isomorphisms:

$$\tilde{H}^{m-n}(T(\nu_{M_0^n});\mathbb{L}^+) \cong H_n^{st}(M_0^n;\mathbb{L}^+) \cong H_n(M_0^n;\mathbb{L}^+)$$

and

$$\tilde{H}^{m-n}(T(\nu_{X^n});\mathbb{L}^+) \cong H_n^{st}(X^n;\mathbb{L}^+).$$

For the generalized manifold X^n, this follows from Kahn, Kaminker and Schochet [10, Theorem B], which asserts that

$$H^{m-n-1}(S^m \setminus X^n;\mathbb{L}^+) \cong H_n^{st}(X^n;\mathbb{L}^+).$$

Since for every $m \geq n$,

$$H^{m-n-1}(S^m;\mathbb{L}^+) = L_{m-1}, \quad H^{m-n}(S^m;\mathbb{L}^+) = L_m,$$

where $L_q = \pi_q(G/TOP)$, the exact sequence of the pair $(S^m,\, S^m \setminus X^n)$ then implies that

$$\begin{aligned} H^{m-n-1}(S^m \setminus X^n;\mathbb{L}^+) &\cong \tilde{H}^{m-n}(S^m, S^m \setminus X^n;\mathbb{L}^+) \\ &\cong \tilde{H}^{m-n}(T(\nu_{X^n});\mathbb{L}^+) \cong \tilde{H}^{m-n}(T(\xi_0);\mathbb{L}^+). \end{aligned}$$

This also applies to M_0^n.

The proof of Kahn, Kaminker and Schochet [10, Theorem B] shows that the following diagram commutes

$$\begin{array}{ccc} \tilde{H}^{m-n}(T(\nu_{M_0^n});\mathbb{L}^+) & \xrightarrow{\cong} & H_n^{st}(M_0^n;\mathbb{L}^+) = H_n(M_0^n;\mathbb{L}^+) \\ {\scriptstyle (T(b_0))^*}\uparrow & & \downarrow{\scriptstyle (f_0)_*} \\ \tilde{H}^{m-n}(T(\xi_0);\mathbb{L}^+) & \xrightarrow{\cong} & H_n^{st}(X^n;\mathbb{L}^+) \end{array} \tag{4.2}$$

Briefly, this follows since the Spanier–Whitehead duality isomorphism comes from the slant product constructions, using the map

$$X_+^n \wedge T(\xi_0) \cong X_+^n \wedge T(\nu_{X^n}) \to S^m,$$

i.e. it comes from the element in $H^m(X_+^n \wedge T(\xi_0);\mathbb{L}^+)$ which it defines. This construction is natural for the normal map $(f_0,\, b_0)$. Since X^n is not a complex, $T(\nu_{X^n})$ is replaced by a certain function space which leads to the Steenrod homology (cf. Kahn, Kaminker and Schochet [10, Section 4]).

Summary: The following diagram commutes

$$\begin{array}{ccccc} H^0(M_0^n;\mathbb{L}^+) & \xrightarrow{\cong} & \tilde{H}^{m-n}(T(\nu_{M_0^n});\mathbb{L}^+) & \xrightarrow{SD} & H_n(M_0^n;\mathbb{L}^+) \\ \uparrow (f_0)^* & & \uparrow (T(b_0))^* & & \downarrow (f_0)_* \\ H^0(X^n;\mathbb{L}^+) & \xrightarrow{\cong} & \tilde{H}^{m-n}(T(\xi_0);\mathbb{L}^+) & \xrightarrow{SD} & H_n^{st}(X^n;\mathbb{L}^+) \end{array} \tag{4.3}$$

The composition of the upper row isomorphisms is known to be

$$.\cap [M_0^n]_{\mathbb{L}^\bullet} : H^0(M_0^n;\mathbb{L}^+) \xrightarrow{\cong} H_n(M_0^n;\mathbb{L}^+),$$

where $[M_0^n]_{\mathbb{L}^\bullet} \in H_n(M_0^n;\mathbb{L}^\bullet)$ is the $\mathbb{L}^\bullet$-coefficient fundamental class of M_0^n (cf. Ranicki [20, Proposition 18.3]). Finally, we can identify

$$[M_0^n, G/TOP] = H^0(M_0^n;\mathbb{L}^+), \quad [X^n, G/TOP] = H^0(X^n;\mathbb{L}^+),$$

according to the equivalence $G/TOP \xrightarrow{\cong} \mathbb{L}^+$ (cf. Kirby and Siebenmann [11, Essay 5, Appendix C], Ranicki [20, Proposition 16.1]).

Combining this with Corollary 3.3 and diagram (3.6), we obtain the following diagram

$$\begin{array}{ccccccc} & & & .\cap [M_0^n]_{\mathbb{L}^+} \ (\text{from } H^0(M_0^n;\mathbb{L}^+) \text{ to } H_n(M_0^n;\mathbb{L}^+)) & & & \\ H^0(M_0^n;\mathbb{L}^+) & \xrightarrow{\cong} & [M_0^n, G/TOP] & \xrightarrow{\cong} & \mathcal{N}(M_0^n) & \xrightarrow{t_0} & H_n(M_0^n;\mathbb{L}^+) \\ \uparrow (f_0)^* & & \uparrow (f_0)^* & & \uparrow & & \downarrow (f_0)_* \\ H^0(X^n;\mathbb{L}^+) & \xrightarrow{\cong} & [X^n, G/TOP] & \xrightarrow{\cong} & \mathcal{N}(X^n) & \xrightarrow{t} & H_n^{st}(X^n;\mathbb{L}^+) \\ & & & \cong \ (\text{from } H^0(X^n;\mathbb{L}^+) \text{ to } H_n^{st}(X^n;\mathbb{L}^+)) & & & \end{array} \tag{4.4}$$

Commutativity of the outer diagram (cf. diagram (4.3)) and each square imply that

$$\mathcal{N}(X^n) \to \mathcal{N}(M_0^n) \xrightarrow{t_0} H_n(M_0^n;\mathbb{L}^+) \xrightarrow{(f_0)_*} H_n^{st}(X^n;\mathbb{L}^+)$$

is an isomorphism, hence by diagram (3.6), this composition is t. This completes the proof of Theorem 1.2.

Remark 4.1. In particular, the proof of Theorem 1.2 also shows that the $\mathbb{L}$-duality isomorphism for generalized manifold X^n factors over $t : \mathcal{N}(X^n) \to H_n^{st}(X^n;\mathbb{L}^+)$.

5. Epilogue

We shall conclude this paper by a brief outlook for further studies, following a very interesting suggestion of the referee. In this paper, we have proved that there exists a bijective map $t : \mathcal{N}(X^n) \to H^{st}_n(X^n; \mathbb{L}^+)$ from normal degree one bordisms to the Steenrod homology of the spectrum $\mathbb{L}^+$.

The Steenrod homology is known to behave well on the category of compact metric spaces. In particular, if $q : X^n \to B$ is any morphism, then there exists a long exact sequence

$$\cdots \to H^{st}_{n+1}(B; \mathbb{L}^+) \to H^{st}_{n+1}(B, X^n; \mathbb{L}^+) \xrightarrow{\partial_*} H^{st}_n(X^n; \mathbb{L}^+) \xrightarrow{q_*} H^{st}_n(B; \mathbb{L}^+) \to \dots \quad (5.1)$$

On the other hand, if $q : X^n \to B$ is a UV^1-map, then there is a controlled surgery sequence (cf. Bryant, Ferry, Mio and Weinberger [2], Mio [16], and Nicas [17]),

$$H^{st}_{n+1}(B; \mathbb{L}) \to \mathcal{S}^c \begin{pmatrix} X^n \\ \downarrow q \\ B \end{pmatrix} \to \mathcal{N}(X^n) \xrightarrow{\sigma^c} H^{st}_n(B; \mathbb{L}) \quad (5.2)$$

Here, $\mathbb{L}$ denotes the 4-periodic spectrum with $\mathbb{L}_0 = \mathbb{Z} \times G/TOP$, σ^c is the controlled surgery obstruction map, and

$$\mathcal{S}^c \begin{pmatrix} X^n \\ \downarrow q \\ B \end{pmatrix} \quad (5.3)$$

is the controlled structure set. This controlled surgery sequence 5.2 makes sense if the controlled structure set 5.3 is nonempty (cf. Mio [16, Theorem 3.8]).

It is natural to ask if sequences (5.1) and (5.2) are related via the map $t : \mathcal{N}(X^n) \to H^{st}_n(X^n; \mathbb{L}^+)$. First, one notes that two spectra $\mathbb{L}^+$ and $\mathbb{L}$ are involved, where $\mathbb{L}^+ \xrightarrow{i} \mathbb{L}$ is considered as the covering spectrum over the Eilenberg–MacLane spectrum $K(\mathbb{Z}, 0)$, i.e. $\mathbb{L}^+ \xrightarrow{i} \mathbb{L} \to K(\mathbb{Z}, 0)$ is a fibration of spectra.

In order to compare sequences (5.1) and (5.2), we consider the composite map

$$q_* \circ i_* : H^{st}_n(X^n; \mathbb{L}^+) \xrightarrow{i_*} H^{st}_n(X^n; \mathbb{L}) \xrightarrow{q_*} H^{st}_n(B; \mathbb{L}),$$

and obtain the following diagram

$$\begin{array}{ccc} H^{st}_n(X^n; \mathbb{L}^+) & \xrightarrow{q_* \circ i_*} & H^{st}_n(B; \mathbb{L}) \\ t \uparrow & & \uparrow = \\ \mathcal{N}(X^n) & \xrightarrow{\sigma^c} & H^{st}_n(B; \mathbb{L}) \end{array} \quad (5.4)$$

The first step would be to prove commutativity of diagram (5.4). However, this is not enough, since one also needs a map between $H^{st}_{n+1}(B, X^n; \mathbb{L}^+)$ and the set (5.3), compatible with $t : \mathcal{N}(X^n) \to H^{st}_n(X^n; \mathbb{L}^+)$. This can all be done if X^n is a topological n-manifold (cf. Hegenbarth and Repovš [8]). In the case when X^n is a generalized n-manifold, this is still an unsolved problem.

For the second step, one is led to "refining" the map $t : \mathcal{N}(X^n) \to H_n^{st}(X^n; \mathbb{L}^+)$ to a map

$$\bar{t} : \mathcal{S}^c\left(\begin{array}{c} X^n \\ \downarrow q \\ B \end{array}\right) \to H_{n+1}^{st}(B, X^n; \mathbb{L}^+)$$

so that the following diagram is commutative

$$\begin{CD} H_{n+1}^{st}(B;\mathbb{L}^+) @>>> H_{n+1}^{st}(B,X^n;\mathbb{L}^+) @>>> H_n^{st}(X^n;\mathbb{L}^+) @>{q_* \circ i_*}>> H_n^{st}(B;\mathbb{L}) \\ @V{i_*}VV @A{\bar{t}}AA @A{t}AA @V{=}VV \\ H_{n+1}^{st}(B;\mathbb{L}) @>>> \mathcal{S}^c @>>> \mathcal{N}(X^n) @>{\sigma^c}>> H_n^{st}(B;\mathbb{L}) \end{CD} \tag{5.5}$$

where $\mathcal{S}^c$ denotes the set (5.3).

Since $\dim X^n = n$, we may assume that $\dim B \le n$. In this case, it follows from the Atiyah–Hirzebruch spectral sequence (which holds for the Steenrod homology, cf. Hegenbarth and Repovš [9, p. 206]) that $H_{n+1}^{st}(B;\mathbb{L}^+) \xrightarrow{i_*} H_{n+1}^{st}(B;\mathbb{L})$ is an isomorphism. In this case, the map

$$\bar{t} : \mathcal{S}^c\left(\begin{array}{c} X^n \\ \downarrow q \\ B \end{array}\right) \to H_{n+1}^{st}(B, X^n; \mathbb{L}^+)$$

is bijective. However, the existence of such a map $\bar{t}$ is at present still a conjecture.

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