



Controlled Surgery and \mathbb{L} -Homology

Friedrich Hegenbarth and Dušan Repovš

Dedicated to the memory of Professor Andrew Ranicki (1948–2018).

Abstract. This paper presents an alternative approach to controlled surgery obstructions. The obstruction for a degree one normal map $(f, b) : M^n \rightarrow X^n$ with control map $q : X^n \rightarrow B$ to complete controlled surgery is an element $\sigma^c(f, b) \in H_n(B, \mathbb{L})$, where M^n, X^n are topological manifolds of dimension $n \geq 5$. Our proof uses essentially the geometrically defined \mathbb{L} -spectrum as described by Nicas (going back to Quinn) and some well-known homotopy theory. We also outline the construction of the algebraically defined obstruction, and we explicitly describe the assembly map $H_n(B, \mathbb{L}) \rightarrow L_n(\pi_1(B))$ in terms of forms in the case $n \equiv 0(4)$. Finally, we explicitly determine the canonical map $H_n(B, \mathbb{L}) \rightarrow H_n(B, L_0)$.

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Introduction

To solve a surgery problem, one encounters an obstruction being an element of the Wall group [20]. If one does controlled surgery with respect to a control map over B , the obstruction belongs to a controlled version of Wall groups. Both groups are constructed in a purely algebraic way as equivalence classes of certain forms or formations. The principal result (cf. Theorem 3.3 in Sect. 3) of the present paper shows that controlled obstructions are elements of $H_n(B, \mathbb{L})$, where \mathbb{L} is the geometrically defined surgery spectrum as described by Nicas [13]. The basic idea of our proof is that controlled surgeries are done in small regions of the manifold when projecting it onto B (and this fits well with \mathbb{L} -homology of B). The proof is given in Sect. 3.

In Sect. 1, we review the algebraic construction of controlled surgery obstructions for the case $n \equiv 0(4)$ in terms of forms. In Proposition 1.1 we

show how to obtain from this the Hermitian form of the uncontrolled surgery obstruction.

In Sect. 2 we introduce relevant surgery spaces and \mathbb{L} -spectra. We follow the Nicas description [13] (which goes back to Quinn [15]). The surgery spaces and spectra are defined semi-simplicially, i.e. by adic surgery problems. According to the targets of the surgery problems, one obtains spectra denoted by \mathbb{L} , resp. \mathbb{L}^{PD} . Here, the targets in \mathbb{L}^{PD} are adic Poincaré duality complexes, whereas in \mathbb{L} they are adic manifolds.

Then we prove that the natural inclusion $\mathbb{L} \rightarrow \mathbb{L}^{\text{PD}}$ is a homotopy equivalence (cf. Proposition 2.2). In particular,

$$\pi_n(\mathbb{L}) \cong \pi_n(\mathbb{L}^{\text{PD}}),$$

and as shown by Wall [20],

$$\pi_n(\mathbb{L}^{\text{PD}}) \cong L_n(\{1\}),$$

the Wall group of the trivial group. We note that this problem was not addressed by Nicas [13].

In Sect. 2.2, we describe elements of the \mathbb{L} -homology group. The spectrum \mathbb{L} is not connected, in fact,

$$\pi_0(\mathbb{L}) = L_0 \cong \mathbb{Z}.$$

There is a fiber sequence of spectra

$$\mathbb{L}\langle 1 \rangle \rightarrow \mathbb{L} \rightarrow K(\mathbb{Z}, 0)$$

with $\mathbb{L}\langle 1 \rangle$ the connected covering of \mathbb{L} , and $K(\mathbb{Z}, 0)$ the Eilenberg–Mac Lane spectrum. We study the induced map

$$H_n(B, \mathbb{L}) \rightarrow H_n(B, L_0)$$

and give an explicit formula in Sect. 2.3 (cf. Corollary 2.6). It has particular significance when determining the resolution invariant of Quinn [16, 17].

In Sect. 3, we treat $H_n(B, \mathbb{L})$ as the controlled Wall group and we present the main result of this paper—an alternative proof that $H_n(B, \mathbb{L})$ is the obstruction group for controlled surgery problems (cf. Theorem 3.3). Finally, in Epilogue we discuss controlled Wall realizations of elements in $H_{n+1}(B, \mathbb{L})$ on n -manifolds X .

1. Controlled and Uncontrolled Surgery Obstructions

I. In this section, we denote by B a finite connected polyhedron with fundamental group $\pi = \pi_1(B)$, giving rise to the group ring $\Lambda = \mathbb{Z}[\pi]$. We shall restrict ourselves only to the oriented situation, i.e. when the usual orientation map $\pi \rightarrow \{\pm 1\}$ is 1. More precisely, we shall work in the category of oriented topological manifolds and topological bundles. Normal degree one maps

$$(f, b) : M^n \rightarrow X^n$$

are defined as in Wall [20] (here, M in X are n -manifolds, possibly with nonempty boundary ∂M and ∂X , respectively).

We add to this a reference map $q : X \rightarrow B$. In the controlled case, it serves as the control map, where B is equipped by a metric given by an embedding $B \subset \mathbb{R}^m$ as a subcomplex, for a sufficiently large m . For controlled surgery, we assume that q is a UV^1 -map, i.e. for each contractible open set $U \subset B$, $\pi_1(q^{-1}(U)) = 0$ (cf. e.g. Ferry [4]).

For $\dim X \geq 5$, it was proved by Bestvina that q is homotopic to a UV^1 -map (cf. [1, Theorem 4.4]). In the case when $\partial X \neq \emptyset$, one must also assume that

$$q|_{\partial X} : \partial X \rightarrow B \text{ is } UV^1,$$

so in this case one must have $\dim X \geq 6$. Suppose that f restricts to a simple homotopy equivalence on the boundary ∂X . The map f can be made highly connected.

To complete the surgery in the middle dimension, a surgery obstruction $\sigma(f, b)$, belonging to the Wall group $L_n(\pi)$, must vanish. Here, we may assume without loss of generality that

$$q_* : \pi_1(X) \xrightarrow{\cong} \pi_1(B).$$

Of course, this holds if q is UV^1 . If $\sigma(f, b) = 0$, then we get a simple homotopy equivalence $M' \rightarrow X$ relative the boundary, if $n \geq 5$, which is normally cobordant to $M \rightarrow X$.

Controlled surgery is much more delicate (cf. [2]). One can define an obstruction $\sigma^c(f, b)$, belonging to the controlled Wall group $L_n(B, \varepsilon, \delta)$ (in the notations of Pedersen, Quinn and Ranicki [14]). Here, $\varepsilon > 0$ is smaller than a certain $\varepsilon_0 > 0$ which depends on B and $\dim X$, and $\delta > 0$ is determined by ε .

When q is UV^1 and $n \geq 4$, the following holds: If $\sigma^c(f, b) = 0$ then $(f, b) : M \rightarrow X$ is normally cobordant to a δ -homotopy equivalence $M' \xrightarrow{f'} X$ over B . The map $f' : M' \rightarrow X$ is unique up to ε -homotopy.

This means that there exist a homotopy inverse $g' : X \rightarrow M'$ and homotopies

$$h_t : f' \circ g' \sim \text{Id}_X, \quad g_t : g' \circ f' \sim \text{Id}_{M'}$$

such that the tracks of the homotopies

$$q \circ h_t, \quad q \circ f' \circ g_t$$

are smaller than δ , measured in the metric of B . If $\partial X \neq \emptyset$, one has to additionally assume that $f|_{\partial M}$ is already a δ -homotopy equivalence, and f' is then a δ -homotopy equivalence relative to the boundary.

There is an obvious morphism

$$L_n(B, \varepsilon, \delta) \rightarrow L_n(\pi),$$

forgetting the control, also considered as the assembly map. This is because controlled surgeries are done in small pieces which can be glued together to obtain the global result. We shall come back to this point in Sect. 3.

Here, we point out how one can obtain the Wall obstruction $\sigma(f, b)$ from the controlled obstruction $\sigma^c(f, b)$ (cf. Part IV below). We shall do this for $n \equiv 0(4)$. This is the case which is interesting for the resolution obstruction.

II. Let now $n = 2k$, where k is even. If $f : M \rightarrow X$ is highly connected then one is left with the following exact sequence:

$$0 \rightarrow K_k(f, \Lambda) \rightarrow H_k(M, \Lambda) \rightarrow H_k(X, \Lambda) \rightarrow 0.$$

By duality and the Hurewicz–Whitehead theorems, one has to kill

$$K_k(f, \Lambda) \cong \pi_{k+1}(X, M)$$

by surgeries. Here, $K_k(f, \Lambda)$ is a stable free-based Λ -module, finitely generated, and carrying a Hermitian Λ -bilinear form

$$\lambda : K_k(f, \Lambda) \times K_k(f, \Lambda) \rightarrow \Lambda$$

which is refined by a quadratic form μ , deduced from the bundle map b . In Wall [20, p. 47], this is called a special Hermitian form. Equivalence classes of such special Hermitian forms constitute the Wall group $L_{2k}(\pi)$ (cf. Wall [20, Chapter 5] for precise constructions). Hence,

$$\sigma(f, b) = [K_k(f, \Lambda), \lambda, \mu] \in L_{2k}(\pi).$$

III. We are now going to describe the controlled surgery obstructions. It was Quinn who explicitly constructed them (cf. Quinn [16, Sect. 2]). His aim was to prove the existence of resolutions of generalized manifolds. For this purpose, it was not necessary to construct controlled Wall groups (cf. also Quinn [17]). A detailed construction can be found in Ferry [5]. To obtain controlled results one has to work with the chain complex $C_{\#}(X, M)$ instead of homology. Here are the main steps:

Step 1. $(f, b) : M \rightarrow X$ is normally cobordant to $(\bar{f}, \bar{b}) : \bar{M} \rightarrow X$ so that $C_g(X, \bar{M}) = 0$ for $j \leq k$. This can be obtained for any surgery problem. To continue, we recall that manifolds \bar{M} satisfy the controlled Poincaré duality, i.e. the cap product with a fundamental cycle is a δ -chain equivalence $C^{\#}(\bar{M}) \rightarrow C_{n-\#}(\bar{M})$, and this implies a δ -chain equivalence

$$C^{\#}(X, \bar{M}) \rightarrow C_{n+1-\#}(X, \bar{M})$$

for arbitrary $\delta > 0$.

Step 2. Using the δ -chain equivalence

$$C^{\#}(X, \bar{M}) \rightarrow C_{n+1-\#}(X, \bar{M})$$

and controlled cell trading, one proves that $C_{\#}(X, \bar{M})$ is δ -chain equivalent to a chain complex of the type

$$0 \rightarrow D_{k+1} \rightarrow D_k \rightarrow 0.$$

By doing surgery on small k -spheres in \bar{M} , according to the basis of D_k , one obtains a chain complex of the type

$$0 \rightarrow A_{k+1} \rightarrow 0.$$

Let M' be the result of this surgery.

Step 3. By Quinn [16, Proposition 2.4], the pair (X, M') is δ -homotopy equivalent to a pair (X', M') such that

$$C_{\#}(X', M') = \begin{cases} A_{k+1} & \# = k + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since the chain equivalence in Step 1 is a δ -equivalence for arbitrary small δ , we have the same situation in Step 3. So the composition

$$q' : X' \xrightarrow{\sim} X \xrightarrow{g} B$$

is a $UV^1(\delta)$ -map. This will be sufficient for our purpose (cf. e.g. Ferry [5], Quinn [16], Yamasaki [21] for the concept of geometric algebra of chain complexes, $UV^1(\delta)$, and δ -chain equivalences).

By Step 3, our original surgery problem $M \rightarrow X$ is replaced by a normal degree one map

$$(f', b') : M' \rightarrow X',$$

where b' is a bundle map between the normal bundle $\nu_{M'}$ of M' and the bundle ξ over X' , induced by the map $X' \rightarrow X$ from the normal bundle ν_X of X .

The result is a finitely generated geometric \mathbb{Z} -module $C_{k+1}(X', M')$, with obvious intersection number

$$\lambda_{\mathbb{Z}} : C_{k+1}(X', M') \times C_{k+1}(X', M') \rightarrow \mathbb{Z},$$

refined by a quadratic form $\mu_{\mathbb{Z}}$, determined by the normal data, such that the radius of $\lambda_{\mathbb{Z}}$ is δ -small: for basis elements

$$a, b \in C_{k+1}(X', M')$$

one has

$$\lambda_{\mathbb{Z}}(a, b) = 0 \quad \text{provided that } d(q'(a), q'(b)) > \delta.$$

The equivalence class of

$$[C_{k+1}(X', M'), \lambda_{\mathbb{Z}}, \mu_{\mathbb{Z}}] \in L_n(B, \varepsilon, \delta)$$

is the controlled surgery obstruction of the surgery problem

$$(f', b') : M' \rightarrow X'.$$

One notes that the Wall obstructions $\sigma(f, b)$ and $\sigma(f', b')$ in $L_n(\pi)$ coincide.

IV. The map $L_n(B, \varepsilon, \delta) \rightarrow L_n(\pi)$.

We are given $\sigma(f, b) \in L_n(\pi)$ which we represent by the triple $(K_k(f', \Lambda), \lambda, \mu)$. One first notes that

$$K_k(f', \Lambda) = C_{k+1}(X', M') \otimes_{\mathbb{Z}} \Lambda.$$

Let

$$a_1, \dots, a_r \in C_{k+1}(X', M')$$

be a \mathbb{Z} -basis. Then

$$\tilde{a}_i = a_i \otimes 1, \quad i = 1, \dots, r$$

is a Λ -basis of $K_k(f', \Lambda)$. To calculate $\lambda_{\mathbb{Z}}(a_i, a_j)$, one observes that the a_i s are represented by small maps

$$(D^{k+1}, S^k) \rightarrow (X', M'),$$

where $\partial a_i : S^k \rightarrow M'$ are framed immersions in general position. Let

$$\partial a_i \cap \partial a_j = \{p_1, \dots, p_m\}.$$

Then

$$\lambda_{\mathbb{Z}}(a_i, a_j) = \sum_{i=1}^m \varepsilon_i, \quad \text{where } \varepsilon_i = \pm 1$$

is the usual algebraic intersection number at the point p_i .

The elements

$$\tilde{a}_1, \dots, \tilde{a}_r \in K_k(f', \Lambda) \cong C_{k+1}(\tilde{X}', \tilde{M}')$$

are considered as liftings of $\partial a_1, \dots, \partial a_r$ in the universal covering \tilde{M}' of M' . Alternatively, $\tilde{a}_1, \dots, \tilde{a}_r$ are immersed spheres in M' together with connecting paths to a base point of M' . We state our observation in the following:

Proposition 1.1. *With the above assumptions and notations, we have*

$$\lambda(\tilde{a}_i, \tilde{a}_j) = \lambda_{\mathbb{Z}}(a_i, a_j)g_{ij} \in \Lambda,$$

where $g_{ij} \in \pi$ is determined by the paths connecting \tilde{a}_i, \tilde{a}_j to the base point.

Proof. Since the radius of $\lambda_{\mathbb{Z}}$ is as small as we want, and the immersed spheres are small, we may assume that their images in B are contained in a contractible subset. By the UV^1 property we conclude that

$$\tilde{a}_i(S^k) \cup \tilde{a}_j(S^k) \subset U \subset M' \quad \text{with } \pi_1(U) = \{1\}.$$

Calculating $\lambda(\tilde{a}_i, \tilde{a}_j)$ as in the proof in Wall [20, Theorem 5.2], one obtains the claim. □

The case when π is the fundamental group of the n -torus, this was first proved by Mio and Ranicki [12, Sect. 10.1]. Since any surgery problem $(f, b) : M^n \rightarrow X^n$ between n -manifolds without boundaries can be considered as a controlled problem over $\text{Id} : X \rightarrow X$, we can get the following:

Corollary 1.2. *Let $n \equiv 0(4)$. Then*

$$\sigma(f, b) \in L_n(\pi_1(X))$$

has a representation (G, λ, μ) with G a free Λ -module with basis b_1, \dots, b_r such that

$$\lambda(b_i, b_j) = n_{ij}g_{ij}, \quad n_{ij} \in \mathbb{Z}, \quad \text{and } g_{ij} \in \pi_1(X).$$

Remark 1.3. If $\partial M, \partial X$ are nonempty, the restriction $f|_{\partial M}$ has to be a δ -controlled homotopy equivalence. In the case of $\text{Id} : X \rightarrow X$ as the control map this implies that $f|_{\partial M}$ is a homeomorphism. However, if $f|_{\partial M}$ is a δ -homotopy equivalence for some UV^1 -map $q : X \rightarrow B$, then the proof goes through.

2. \mathbb{L} -Spectra and \mathbb{L} -Homology

2.1. On the Geometric Construction of the \mathbb{L} -Spectrum

The geometric \mathbb{L} -spectrum was introduced in Quinn [15] as a semi-simplicial Ω -spectrum. Details can also be found in Nicas [13] which we shall follow. We define surgery spaces $\mathbb{L}_r(B)$, where B is a polyhedron. We are only interested in the case $B = \{*\}$ and we shall write $\mathbb{L}_r = \mathbb{L}_r\{*\}$.

An s -simplex $\sigma \in \mathbb{L}_r$ is a normal degree one map between $(r + s)$ -dimensional oriented $(s + 3)$ -ads of manifolds

$$(M, \partial_0 M, \dots, \partial_s M, \partial_{s+1} M) \rightarrow (X, \partial_0 X, \dots, \partial_s X, \partial_{s+1} X)$$

such that f restricted to $\partial_{s+1} M$ is a homotopy equivalence. To each σ belongs a reference map of $(s + 3)$ -ads

$$(X, \partial_0 X, \dots, \partial_s X, \partial_{s+1} X) \rightarrow (\Delta^s, \partial_0 \Delta^s, \dots, \partial_s \Delta^s, \Delta^s)$$

to the standard s -simplex Δ^s . Note that the last face $\partial_{s+1} X$ maps to the interior of Δ^s , and plays a special role in the constructions.

Let $\mathbb{L}_r(s)$ be the set of s -simplices. Then \mathbb{L}_r is a pointed semisimplicial complex with base points the empty problem and there is a homotopy equivalence to the simplicial loop space of \mathbb{L}_{r-1} (cf. Nicas [13, Proposition 2.2.2]):

$$\mathbb{L}_r \rightarrow \Omega \mathbb{L}_{r-1}.$$

The collection of surgery spaces $\{\mathbb{L}_r, r \in \mathbb{Z}\}$ defines a spectrum \mathbb{L}^+ such that its homotopy groups $\pi_n(\mathbb{L}^+)$ are the Wall groups $L_n(1)$. In the notation of [18], $\mathbb{L}^+ = \mathbb{L}\langle 1 \rangle$, whereas \mathbb{L} denotes the periodic \mathbb{L} -spectrum with the 0-term $= \mathbb{Z} \times G/TOP$.

To do this we have to address two problems. The first one comes from the following easily proved (and well known) lemma.

Lemma 2.1. *The surgery space \mathbb{L}_0 defined above satisfies $\pi_0(\mathbb{L}_0) = \{0\}$.*

Proof. Recall, that we are working in the simplicial category. A typical element $\sigma \in \mathbb{L}_0(0)$ is a map of degree one of the type $\{\pm y_1, \dots, \pm y_k\} \rightarrow \{x\}$. By the degree one property one can reorder it as follows:

$$\{y_1, +y_2, -y_2, \dots, +y_l, -y_l\} \rightarrow \{x\}.$$

The 1-simplex $\{I_1, \dots, I_l\} \rightarrow J$, with I_j denoting the interval with $\partial I_j = \{y_j, -y_j\}$, shows that σ is equivalent to $(\{y_1\} \rightarrow \{x\})$. Here we view J as a degenerate 1-simplex consisting of a single point. Moreover, $(\{y\} \rightarrow \{x\})$ is equivalent to the empty set. Therefore, $\pi_0(\mathbb{L}_0) = 0$. □

The second problem arises from comparison with the Wall groups in Wall [20, Chapter 9] (cf. the proof of Nicas [13, Proposition 2.2.4]). The point is that in Wall [20], Poincaré duality spaces are used as targets, whereas in [13] manifolds are used. This point was not addressed in Nicas [13]. It might be not the same for a generic polyhedron B , but it gives the same result when $B = \{*\}$.

To see this, we introduce the surgery spaces \mathbb{L}_r^{PD} in the same way as \mathbb{L}_r , but Poincaré-ads as targets (this was used in Quinn [15]). One also proves that

\mathbb{L}_r^{PD} is homotopy equivalent to $\Omega\mathbb{L}_{r-1}^{\text{PD}}$. There is an obvious map $\mathbb{L}_r \rightarrow \mathbb{L}_r^{\text{PD}}$, and

$$\pi_0(\mathbb{L}_r) \cong \pi_0(\mathbb{L}_r^{\text{PD}}) = \{0\}.$$

We can define Ω -spectra \mathbb{L}^+ and \mathbb{L}^{PD} using this.

To match up with the usual notation, we write

$$\mathbb{L}^+ = \{\mathbb{L}_{-r}, r \geq 0\}, \quad \mathbb{L}^{\text{PD}} = \{\mathbb{L}_{-r}^{\text{PD}}, r \geq 0\}.$$

Both spectra are connected and \mathbb{L}^+ becomes $\mathbb{L}\langle 1 \rangle$ in the notations of Ranicki [18].

Proposition 2.2. *The map $\mathbb{L}^+ \rightarrow \mathbb{L}^{\text{PD}}$ is a homotopy equivalence.*

Proof. We shall show that the induced morphism

$$\pi_n(\mathbb{L}^+) \rightarrow \pi_n(\mathbb{L}^{\text{PD}})$$

is an isomorphism for $n \geq 0$. The assertion will then follow by the Whitehead theorem.

Observe that

$$\pi_n(\mathbb{L}^{\text{PD}}) \cong \pi_{n+r}(\mathbb{L}_{-r}^{\text{PD}}) \cong \pi_n(\mathbb{L}_0^{\text{PD}}) \cong \pi_0(\Omega^n \mathbb{L}_{-n}^{\text{PD}}).$$

However, the last one coincides with the group $L_n^1(\{*\})$, considered by Wall [20, Chapter 9]. We begin with the higher dimensional case.

Case I: $n \geq 5$. Wall defines a restricted set

$$L_n^2(\{*\}) \subset L_n^1(\{*\})$$

consisting of simply connected surgery problems (an adic version of this was considered by Nicas [13, Chapter 2]). He shows that

$$L_n^2(\{*\}) \rightarrow L_n^1(\{*\})$$

is bijective for $n \geq 4$ (cf. Wall [20, Theorem 9.4], for the adic case cf. Nicas [13, Proposition 2.2.7]). A corollary of this is that the surgery obstruction map

$$\Theta : L_n^1(\{*\}) \rightarrow L_n \quad (= \text{Wall group of } \pi_1 = \{1\})$$

is an isomorphism for $n \geq 5$ (cf. [20, Corollary 9.4.1.]). Since the composition

$$L_n = \pi_n(\mathbb{L}^+) \rightarrow \pi_n(\mathbb{L}^{\text{PD}}) \cong L_n^1(\{*\}) \xrightarrow{\Theta} L_n$$

is the identity, this proves that we indeed have an isomorphism

$$\pi_n(\mathbb{L}^+) \xrightarrow{\cong} \pi_n(\mathbb{L}^{\text{PD}})$$

for all $n \geq 5$.

Case II: $n = 4$. The surgery obstruction map Θ is defined for $n = 4$ and the composition

$$L_4 = \pi_4(\mathbb{L}^+) \rightarrow \pi_4(\mathbb{L}^{\text{PD}}) \cong L_4^1(\{*\}) \xrightarrow{\Theta} L_4$$

is the identity. Therefore,

$$\pi_4(\mathbb{L}^+) \rightarrow \pi_4(\mathbb{L}^{\text{PD}})$$

is injective. Since,

$$L_4^2(\{*\}) \xrightarrow{\cong} L_4^1(\{*\}),$$

we can represent an element in $\pi_4(\mathbb{L}^{\text{PD}})$ by

$$(f, b) : M \rightarrow X \text{ with } \pi_1(X) = \{1\}.$$

Assume first that $\partial X = \emptyset$. Then $G = H_2(X, \mathbb{Z})$ is \mathbb{Z} -free and the intersection form

$$\lambda_X : G \times G \rightarrow \mathbb{Z}$$

is unimodular. By Freedman [6, Theorem 1.5], there is a simply-connected 4-manifold M' realizing (G, λ_X) . However, by Milnor [11], M' is homotopically equivalent to X ; therefore,

$$(f, b) : M \rightarrow X$$

is equivalent to the surgery problem

$$(f', b') : M \rightarrow M'$$

arising from $\pi_4(\mathbb{L}^+)$. Now assume that $\partial X \neq \emptyset$. Then

$$f|_{\partial M} : \partial M \rightarrow \partial X$$

is a homotopy equivalence. We obtain a closed surgery problem by glueing

$$\text{Id} : M \rightarrow M \text{ and } f : M \rightarrow X$$

along the boundary

$$N = M \cup_{\text{Id}} M \xrightarrow{\text{Id} \cup f} M \cup_{f|_{\partial M}} X = Y.$$

By the van Kampen theorem, $\pi_1(Y) = \{1\}$. It is now easy to see that the class of $N \rightarrow Y$ represents the same as the classes of

$$(f, b) : M \rightarrow X \text{ and } \text{Id} : M \rightarrow M$$

in $L_4^1(\{*\})$ (cf. Supplement below). However, $\text{Id} : M \rightarrow M$ represents the trivial class, so we are back in the closed case.

Case III: $n = 3$. (See also a short proof in Supplement below.) Let

$$(f, b) : M^3 \rightarrow X^3$$

be given. As in the case $n = 4$, we may assume that $\partial X = \emptyset$. There is a commutative diagram of well-known isomorphisms of Hurewicz maps between cobordism groups

$$\begin{array}{ccc} \Omega_3(X) & \xrightarrow{\mu} & \Omega_3^{\text{PD}}(X) \\ & \searrow & \swarrow \\ & H_3(X, \mathbb{Z}) & \end{array}$$

It follows that μ is an isomorphism and since f is of degree one, M is PD -cobordant to X over X .

Let $q : Z \rightarrow X$ be a PD_4 -complex over X with

$$q|_X = \text{Id} \quad \text{and} \quad q|_M = f.$$

The Spivak fibration ν_Z of Z restricts to the Spivak fibration ν_X and ν_M , and we have the maps of the m -sphere into the Thom spaces

$$(S^m \times I, S^m \times \{0\}, S^m \times \{1\}) \rightarrow (T\nu_Z, T\nu_X, T\nu_M).$$

Since M is a manifold, let us for simplicity write ν_M also for the stable normal bundle of $M \subset S^m$, i.e.

$$b : \nu_M \rightarrow \xi,$$

where ξ is a certain topological reduction of ν_X . □

Claim. *If ν_Z has a topological reduction ω which restricts to ξ on X , then*

$$(f, b) : M \rightarrow X$$

is equivalent to a normal degree one map

$$(f'', b'') : M'' \rightarrow M, \quad \text{where } b'' : \nu_{M''} \rightarrow \eta \quad \text{and} \quad \eta = \omega|_M.$$

This is obtained by taking the transverse inverse images of the composition of (Z, X, M) :

$$(S^m \times I, S^m \times \{0\}, S^m \times \{1\}) \rightarrow (T\nu_Z, T\nu_X, T\nu_M) \xrightarrow{h} (T\omega, T\xi, T\eta),$$

where h comes from the reduction ω of ν_Z .

Now, the obstructions to existence of such ω belong to

$$H^{r+1}(Z, X, \pi_r(G/TOP));$$

hence, there is only one in

$$H^3(Z, X, \pi_2(G/TOP)) \cong H^3(Z, X, \mathbb{Z}_2).$$

Since $X \subset Z \xrightarrow{q} X$ is the identity, the homomorphism

$$H^r(Z, \mathbb{Z}_2) \rightarrow H^r(X, \mathbb{Z}_2)$$

is surjective, i.e. the short cohomology sequence

$$0 \rightarrow H^3(Z, X, \mathbb{Z}_2) \rightarrow H^3(Z, \mathbb{Z}_2) \rightarrow H^3(X, \mathbb{Z}_2) \rightarrow 0$$

is exact.

The image of the obstruction in $H^3(Z, \mathbb{Z}_2)$ is 0 because ν_z has topological reduction (cf. Hambleton [7]). Therefore, such ω exists which proves the surjectivity of

$$\{0\} = \pi_3(\mathbb{L}^+) \rightarrow \pi_3(\mathbb{L}^{PD}),$$

i.e. $\pi_3(\mathbb{L}^{PD}) = \{0\}$.

Case IV: $n = 1, 2$. These two cases are obvious since for $n = 1, 2$ all PD -complexes are manifolds.

This completes the proof of Proposition 2.2. □

Supplement. We add two remarks here.

1. In the case $n = 4$ and $\partial X \neq \emptyset$, a normal cobordism between

$$N = M \underset{\text{Id}}{\cup} M \rightarrow M \underset{f|_{\partial M}}{\cup} X, \quad M \xrightarrow{(f,b)} X, \quad \text{and} \quad \text{Id} : M \rightarrow M$$

can be constructed as follows: replace X by

$$X' = X \underset{f|_{\partial M}}{\cup} \partial M \times I$$

being homotopy equivalent to X with a collared boundary $\partial M \subset X'$. Then glue

$$M \times I \dot{\cup} X' \times I \quad \text{at} \quad M \times \{0\} \cup X' \times \{0\}$$

along the collar

$$\partial M \times [1 - \varepsilon, 1] \subset M \cap X'.$$

This gives a PD_5 -complex V^5 . A similar construction on

$$M \times M \dot{\cup} M \times I$$

gives a 5-manifold W^5 . An obvious degree one normal map can be constructed from Id_M and (f, b) . Note that

$$\partial W = M \dot{\cup} M \dot{\cup} M \underset{\text{Id}}{\cup} M \quad \text{and} \quad \partial V = X \dot{\cup} M \dot{\cup} M \underset{f|_{\partial M}}{\cup} X.$$

2. In the case $n = 3$ it seems that one can replace the PD_4 -complex Z by Z' with $\partial Z' = \partial Z$ and $\pi_1(Z') = \{1\}$ by Poincaré surgeries. The obstruction to finding a reduction ω of $\nu_{Z'}$ such that

$$\omega|_X = \xi \quad \text{and} \quad \omega|_M = \nu_M$$

belongs to

$$H^3(Z', M \dot{\cup} X, L_2) \cong H_1(Z', L_2) = 0.$$

Then we get a normal bordism between

$$(f, b) : M \rightarrow X \quad \text{and} \quad \text{Id} : M \rightarrow M;$$

hence, the class of (f, b) is trivial.

2.2. Concerning the Elements of $H_n(B, \mathbb{L})$

We shall write as before \mathbb{L} for the periodic spectrum $\mathbb{L}\langle 0 \rangle$, and $\mathbb{L}^+ = \mathbb{L}\langle 1 \rangle$ for its connective covering spectrum. Recall the fibration sequence (cf. Ranicki [18, Sect. 15])

$$\mathbb{L}^+ \rightarrow \mathbb{L} \rightarrow K(L_0, 0),$$

where $K(L_0, 0)$ is the Eilenberg–Mac Lane spectrum. We shall study the homology of this sequence in Sect. 2.3.

Here, we want to describe elements $x \in H_n(B, \mathbb{L})$, where $B \subset S^m$ is a finite polyhedron. We follow Ranicki [18, Sect. 12], to represent x by a cycle, using a dual-cell decomposition of S^m . This is justified by Ranicki [18, Remark 12.5].

If σ is a simplex of S^m , let $D(\sigma, S^m)$ be its dual cell. It has a canonical $(m - |\sigma| + 3)$ -ad structure, where $|\sigma| = \dim \sigma$ and

$$m - |\sigma| = \dim D(\sigma, S^m).$$

The element x is then represented by a simplicial map

$$(S^m, S^m \setminus B) \rightarrow (\mathbb{L}_{n-m}, \emptyset)$$

(one should merely replace $S^m \setminus B$ with the supplement of B , as done in Ranicki [18]). Let us first consider the case when

$$x : (S^m, S^m \setminus B) \rightarrow (\mathbb{L}_{n-m}^+, \emptyset)$$

represents an element of $H_n(B, \mathbb{L}^+)$, i.e.

$$x(\sigma) \in \mathbb{L}_{n-m}^+(m - |\sigma|).$$

However, this is the surgery space described above, i.e. $x(\sigma)$ is a degree one normal map

$$(f_\sigma, b_\sigma) : M_\sigma^{n-|\sigma|} \rightarrow X_\sigma^{n-|\sigma|}$$

between $(n - |\sigma|)$ -dimensional $(m - |\sigma| + 3)$ -ads with a reference map $X_\sigma^{n-|\sigma|} \rightarrow D(\sigma, S^m)$. The cycle condition implies that they can be assembled (the colimit) to a degree one normal map $(f, b) : M^n \rightarrow X^n$ with boundaries $\partial M, \partial X$, so that $f|_{\partial M}$ is a homotopy equivalence, together with a reference map $X \rightarrow B$. Note that $x(\sigma) = \emptyset$ if $\sigma \notin B$, and $X \rightarrow B$ is the colimit of all

$$X_\sigma^{n-|\sigma|} \rightarrow D(\sigma, S^m) \subset S^m$$

with a retraction onto B (cf. Nicas [13, Theorem 3.3.2], or Laures and McClure [10, Proposition 6.6]). Moreover, the boundary map $\partial M \rightarrow \partial X$ is the colimit of the various homotopy equivalences

$$\partial_{m-|\sigma|+1} M_\sigma^{n-|\sigma|} \rightarrow \partial_{m-|\sigma|+1} X_\sigma^{n-|\sigma|}.$$

To consider the general case $x \in H_n(B, \mathbb{L})$ we recall two properties:

- (a) (Periodicity): Suppose that $\dim B - 1 \leq r$. Then there is a natural isomorphism $H_r(B, \mathbb{L}) \rightarrow H_{r+4}(B, \mathbb{L})$ (cf. Ranicki [18, p. 289-290]);
- (b) If $\dim B < r$, then $H_r(B, \mathbb{L}^+) \xrightarrow{\cong} H_r(B, \mathbb{L})$.

Both properties also easily follow from the Atiyah–Hirzebruch spectral sequence

$$H_p(B, \pi_q(\mathbb{L})) \xrightarrow{p+q=r} H_r(B, \mathbb{L}),$$

and the periodicity of the \mathbb{L} -spectrum:

$$\mathbb{L}_r \cong \mathbb{L}_s \quad \text{if } r - s \equiv 0(4).$$

To represent $x \in H_n(B, \mathbb{L})$, we choose r sufficiently large with $r - n \equiv 0(4)$, and represent x as an element of $H_r(B, \mathbb{L}) \cong H_r(B, \mathbb{L}^+)$ as above. Assembling (colimit) then gives a degree one normal map $(f, b) : P^r \rightarrow Q^r$ with the reference map $q : Q^r \rightarrow B$, and $f|_{\partial P}$ a homotopy equivalence.

A specific construction of the degree one normal map $P^r \rightarrow Q^r$ is given using the identification $H_n(B, \mathbb{L})$ with the controlled Wall group $L_n(B, \varepsilon, \delta)$, as established by Pedersen, Quinn and Ranicki [14]. Here are some details. Suppose that also $n \equiv 0(4)$. Then x corresponds to a triple $\{G, \lambda_{\mathbb{Z}}, \mu_{\mathbb{Z}}\}$ as described in Sect. 1. It can be considered as an element of $L_r(B, \varepsilon, \delta)$ by the periodicity, $r - n \equiv 0(4)$, and it can be realized in a controlled way, in the sense of Wall on the boundary ∂N of a regular neighbourhood $N \subset \mathbb{R}^r$ of $B \subset \mathbb{R}^r$.

We obtain P_0^r which can be written as

$$P_0^r = N \cup \partial N \times I \cup \left\{ \cup_k D^{\frac{r}{2}} \times D^{\frac{r}{2}} \right\}.$$

Here, $k = \text{rank } G$, and $\lambda_{\mathbb{Z}}, \mu_{\mathbb{Z}}$ are realized as framed immersions

$$S^{\frac{r}{2}} \times I \rightarrow \partial N \times I.$$

The handles $D^{\frac{r}{2}} \times D^{\frac{r}{2}}$ are attached to the top along the framed embeddings. By the controlled Hurewicz–Whitehead theorem and the α -approximation theorem one gets a degree one normal map $P_0^r \rightarrow N$ of r -manifolds with boundary, such that $\partial P_0^r \rightarrow \partial N$ is a homeomorphism. Then we can close this in the usual way to get

$$P^r = P_0^r \cup_{\partial} N \rightarrow N \cup_{\partial} N = Q^r.$$

It is more convenient to consider $P_0^r \rightarrow N$ and we shall denote it by $P^r \rightarrow N$ with $\partial P^r \rightarrow \partial N$ a homeomorphism. Let $q : N \rightarrow B$ be the retraction. It can be made transverse to the dual cell decomposition, the map $P^r \rightarrow N$ is in the natural way a surgery mock bundle (cf. Nicas [13, Sect. 3.2])

Remark 2.3. If conversely, we are given a degree one normal map $(f, b) : P^r \rightarrow Q^r$ with the reference map $q : Q^r \rightarrow B$, one can define an element $x \in H_r(B, \mathbb{L}^+)$ by splitting (f, b) into pieces using transversality of q with respect to the dual cell-decomposition of $B \subset S^m$.

2.3. The Homomorphism $H_n(B, \mathbb{L}) \rightarrow H_n(B, L_0)$

Without loss of generality we may assume that $\dim B = n$. Let $B^{(n-1)}$ be the $(n - 1)$ -skeleton of B . This implies that

$$H_n(B, \mathbb{L}) \cong Z_n(B) \otimes L_0 \hookrightarrow C_n(B) \otimes L_0 \cong H_n(B, B^{(n-1)}, L_0)$$

is injective. Here, $Z_n(B)$ are the n -cycles of B and $C_n(B)$ are the n -chains. Moreover, from the Atiyah–Hirzebruch spectral sequence one easily gets that

$$H_n(B, B^{(n-1)}, \mathbb{L}) \xrightarrow{\cong} H_n(B, B^{(n-1)}, L_0).$$

Lemma 2.4. *The natural map*

$$H_n(B, \mathbb{L}) \rightarrow H_n(B, B^{(n-1)}, \mathbb{L})$$

factorizes as

$$H_n(B, \mathbb{L}) \rightarrow H_n(B, L_0) \subset H_n(B, B^{(n-1)}, L_0) \cong H_n(B, B^{(n-1)}, \mathbb{L}).$$

Proof. This follows by the commutativity of the diagram:

$$\begin{array}{ccccc}
 \longrightarrow & H_n(B, \mathbb{L}) & \longrightarrow & H_n(B, B^{(n-1)}, \mathbb{L}) & \longrightarrow \\
 & \downarrow & & \downarrow \cong & \\
 \longrightarrow & H_n(B, L_0) & \longrightarrow & H_n(B, B^{(n-1)}, L_0) & \longrightarrow
 \end{array}$$

induced by the map of spectra $\mathbb{L} \rightarrow K(L_0, 0)$. □

To prepare the next lemma we must study the spectral sequence

$$E_{pq}^2 \cong H_p(B, L_q) \xrightarrow[p+q=m]{} H_m(B, \mathbb{L})$$

in more detail. First, we note that

$$E_{n,m-n}^\infty \subset E_{n,m-n}^2,$$

since, $H_p(B, L_q) = 0$ for $p > n$. Moreover,

$$E_{n,m-n}^\infty = F_{n,m-n} / F_{n-1,m-n+1},$$

where

$$F_{n,m-n} = \text{Im}(H_m(B^{(n)}, \mathbb{L}) \rightarrow H_m(B, \mathbb{L})) \cong H_m(B, \mathbb{L}).$$

We consider the composite map

$$\alpha : H_m(B, \mathbb{L}) \rightarrow E_{n,m-n}^\infty \subset E_{n,m-n}^2 \cong H_n(B, L_{m-n}) \cong Z_n(B) \otimes L_{m-n}.$$

Lemma 2.5. *Let $B \subset S^m$, $\dim B = n$, and $m - n \equiv 0(4)$. Then*

$$\begin{array}{ccc}
 H_n(B, \mathbb{L}) & \longrightarrow & H_n(B, L_0) \cong Z_n(B) \otimes L_0 \\
 \downarrow \cong & & \cong \downarrow \beta \\
 H_m(B, \mathbb{L}) & \xrightarrow{\alpha} & Z_n(B) \otimes L_{m-n}
 \end{array}$$

commutes. Here,

$$H_n(B, \mathbb{L}) \xrightarrow{\cong} H_m(B, \mathbb{L})$$

and

$$\beta : Z_n(B) \otimes L_0 \xrightarrow{\cong} Z_n(B) \otimes L_{m-n}$$

are isomorphisms induced by periodicity.

The proof follows by the spectral sequences. □

We now describe the image of $x \in H_n(B, \mathbb{L})$ in

$$H_n(B, L_0) \cong Z_n(B) \otimes L_0 \subset C_n(B) \otimes L_0.$$

It can be written as $\sum k_\tau \cdot \tau$, where τ ranges over the n -simplices of B .

Step 1. Consider $x \in H_m(B, \mathbb{L}^+) \cong H_m(B, \mathbb{L}) \cong H_n(B, \mathbb{L})$.

Step 2. Represent x as the cycle $x : (S^m, S^m \setminus B) \rightarrow (\mathbb{L}_0, \emptyset)$.

Step 3. Consider $x(\tau) : (f_\tau, b_\tau) : P_\tau^{m-n} \rightarrow Q_\tau^{m-n}$ for $\tau < B$, $|\tau| = n$.

One observes that $\partial Q_\tau^{m-n} = \partial P_\tau^{m-n} = \emptyset$ because its boundaries are composed of elements $x(\rho)$, with $|\rho| > n$ (because the boundary $\partial D(\tau, S^m)$ is formed from cells of type $D(\rho, S^m)$, $|\rho| > n$). Now $\dim B = n$, so (f_τ, b_τ) is a closed surgery problem.

To summarize, we have obtained

Corollary 2.6. *Let $\dim B = n$, $B \subset S^m$, with $m - n \equiv 0(4)$. An element $x \in H_n(B, \mathbb{L})$ has the image in*

$$H_n(B, L_0) \cong Z_n(B) \otimes L_0 \cong Z_n(B) \otimes L_{m-n}$$

equal to

$$\sum_{\tau < B^{(n)}} n_\tau \tau$$

with $n_\tau = \text{image of } \sigma(f_\tau, b_\tau) \text{ under}$

$$L_{m-n}(\pi_1(Q_\tau^{m-n})) \rightarrow L_{m-n} \cong L_0.$$

Supplement to Lemma 2.5 and Corollary 2.6.

The diagram in Lemma 2.5 can be rewritten as

$$\begin{array}{ccc} H_n(B, \mathbb{L}) & \longrightarrow & H_n(B, L_0) \\ \downarrow \cong & & \downarrow \cong \\ H_m(B, \mathbb{L}) & \longrightarrow & H_n(B, L_{m-n}), \end{array}$$

where the map

$$H_m(B, \mathbb{L}) \rightarrow H_n(B, L_{m-n})$$

is the composition of

$$H_m(B, \mathbb{L}) \cong H_m(B, \mathbb{L}\langle m - n \rangle)$$

(cf. Ranicki [18, p. 156]) and

$$H_m(B, \mathbb{L}\langle m - n \rangle) \rightarrow H_n(B, L_{m-n})$$

(cf. Ranicki [18, p. 289]). Note also the following commutativity:

$$\begin{array}{ccc} L_n(B, \varepsilon, \delta) \cong H_n(B, \mathbb{L}) & & \\ \downarrow \cong & & \downarrow \cong \\ L_m(B, \varepsilon, \delta) \cong H_m(B, \mathbb{L}) & & \end{array}$$

The above calculation resulting in Corollary 2.6 follows from the compositions

$$H_n(B, \mathbb{L}) \rightarrow H_m(B, \mathbb{L}) \rightarrow H_n(B, L_{m-n})$$

of the above diagrams.

For the other composition, one has to determine the map $H_n(B, \mathbb{L}) \rightarrow H_n(B, L_0)$. This was done by Ranicki [18]. In Prop. 15.3(II) therein an explicit formula is established using, however, the algebraic version of the \mathbb{L} -spectrum. In fact, Proposition 15.3(II) is the formula for the case of the symmetric \mathbb{L} -spectrum, but it is similar for the quadratic \mathbb{L} -spectrum.

3. $H_n(B, \mathbb{L})$ as the Controlled Wall Group

We mentioned in Sect. 1 the controlled Wall group $L_n(B, \varepsilon, \delta)$. It can be defined for any $n \geq 0$. As before, we assume that B is a finite polyhedron.

Based on the work of Yamasaki [22], Quinn, Pedersen and Ranicki [14] proved the following result.

Theorem 3.1. *For finite dimensional ANRs there is a morphism $H_n(B, \mathbb{L}) \rightarrow L_n(B, \varepsilon, \delta)$ which is an isomorphism for suitable $\varepsilon > 0$ and $\delta > 0$.*

Remark 3.2. In the paper by Pedersen, Quinn and Ranicki [14], \mathbb{L} is the spectrum of quadratic algebraic Poincaré ads, and the morphism mentioned above is an assembling map. The proof of the theorem consists of showing that an element of $L_n(B, \varepsilon, \delta)$ can be split into pieces giving an element of $H_n(B, \mathbb{L})$. Now, the algebraic \mathbb{L} -spectrum is homotopy equivalent to the geometric one (cf. Ranicki [18]), so $H_n(B, \mathbb{L})$ can be considered as the controlled Wall group.

As in the classical surgery theory, the controlled version leads to the controlled surgery sequence (cf. Ferry [5, Theorem 1.1.]). This involves the controlled structure set for which one needs the "stability properties" as proved in Ferry [5, Theorem 10.2].

We shall now present the main result of this paper— an alternative proof that $H_n(B, \mathbb{L})$ is the obstruction group for controlled surgery problems.

Theorem 3.3. *Let $(f, b) : M^n \rightarrow X^n$ be a degree one normal map between manifolds, $n \geq 5$, and $\pi : X^n \rightarrow B$ a UV^1 -map. Then an element*

$$\sigma^c(f, b) \in H_n(B, \mathbb{L})$$

is defined so that $\sigma^c(f, b) = 0$ if and only if (f, b) is normally cobordant to a δ -homotopy equivalence, uniquely up to ε -homotopy.

Remark 3.4. Note that the UV^1 -condition for π is no restriction when $n \geq 5$. The theorem holds for $n = 4$, if the UV^1 -condition is satisfied.

Proof. The map $\pi : X \rightarrow B$ can be assumed to be transverse to the dual cells of B (cf. Cohen [3]), i.e.

$$\pi^{-1}(D(\sigma, B)) = X_\sigma^{n-|\sigma|}$$

is an $(n - |\sigma|)$ -dimensional submanifold. If we embed $B \subset S^m$, for m sufficiently large, we have

$$\pi^{-1}(D(\sigma, B)) = \pi^{-1}(D(\sigma, S^m)),$$

and $X_\sigma^{n-|\sigma|}$ has the corresponding $(m - |\sigma| + 3)$ -ad structure. By transversality we define

$$M_\sigma^{n-|\sigma|} = f^{-1}(X_\sigma^{n-|\sigma|}).$$

The restrictions of b gives a family

$$\{(f_\sigma, b_\sigma) : M_\sigma^{n-|\sigma|} \rightarrow X_\sigma^{n-|\sigma|} \mid \sigma \subset B\}$$

which obviously defines a cycle

$$z : (S^m, S^m \setminus B) \rightarrow (\mathbb{L}_{n-m}, \emptyset),$$

i.e. an element

$$[z] = \sigma^c(f, b) \in H_n(B, \mathbb{L}).$$

We now suppose that $[z] = 0$, i.e. there is a simplicial map

$$w : (S^m, S^m \setminus B) \times \Delta^1 \rightarrow (\mathbb{L}_{n-m}, \emptyset)$$

with $w(0) = z$, and $w(1) = \emptyset$ (cf. Ranicki [18, Sect. 12]). This means that the various $(m - |\sigma| + 3)$ -ads $M_\sigma^{n-|\sigma|} \rightarrow X_\sigma^{n-|\sigma|}$ normally bound. Since π is UV^1 , we can assume that these are simply connected surgery problems. If

$$f_\sigma|_{\partial M_\sigma} : \partial M_\sigma \rightarrow \partial X_\sigma$$

is already a homotopy equivalence, it follows that (f_σ, b_σ) is normally cobordant to a homotopy equivalence. The proof now proceeds by induction on $n - |\sigma|$.

Let

$$X_q = \bigcup_{|\sigma| \geq q} X_\sigma^{n-|\sigma|} \quad \text{and} \quad M_q = \bigcup_{|\sigma| \geq q} M_\sigma^{n-|\sigma|},$$

hence

$$X_n \subset X_{n-1} \subset \dots \subset X_1 \subset X_0 = X,$$

similarly for M .

The induction hypothesis The restriction f to M_q is a homotopy equivalence with the inverse $\bar{f} : X_q \rightarrow M_q$ such that the homotopies of

$$f \circ \bar{f} \sim \text{Id}_{X_q} \quad \text{and} \quad \bar{f} \circ f \sim \text{Id}_{M_q}$$

are controlled, i.e. when restricted onto $X_\sigma^{n-|\sigma|}$ (resp. $M_\sigma^{n-|\sigma|}$) they have tracks over $D(\sigma, B)$ when projected down to B . More precisely,

$$f|_{M_\sigma} : M_\sigma^{n-|\sigma|} \rightarrow X_\sigma^{n-|\sigma|}$$

is a homotopy equivalence with the inverse

$$\bar{f}|_{X_\sigma^{n-|\sigma|}} : X_\sigma^{n-|\sigma|} \rightarrow M_\sigma^{n-|\sigma|},$$

and the homotopies above restrict to homotopies of

$$f|_{M_\sigma} \circ \bar{f}|_{X_\sigma} \sim \text{Id}_{X_\sigma} \quad \text{and} \quad \bar{f}|_{X_\sigma} \circ f|_{M_\sigma} \sim \text{Id}_{M_\sigma}$$

over $D(\sigma, B)$.

The inductive step Suppose we are given $\tau \subset B$ with $|\tau| = q - 1$, i.e. $\dim X_\tau = \dim M_\tau = n - q + 1$, and

$$\partial M_\tau = \bigcup_{\sigma} M_\sigma, \quad \partial X_\tau = \bigcup_{\sigma} X_\sigma$$

with $|\sigma| = q$, and σ a face of τ . By the inductive hypothesis, $f|_{M_\sigma}$ is a homotopy equivalence. These can be glued together by the well known homotopy

theory (cf. Hatcher [8], or Sullivan [19, Lemma H]) to give a homotopy equivalence $f|_{\partial M_\tau} : \partial M_\tau \rightarrow \partial X_\tau$. So let

$$F_\tau : (V_\tau, M_\tau, M'_\tau) \rightarrow (X_\tau \times I, X_\tau \times 0, X_\tau \times 1)$$

be a normal cobordism as explained above such that $F_\tau|_{M_\tau} = f_\tau$, $F_\tau|_{M'_\tau} = f'_\tau$ are homotopy equivalences, and because surgery was done in the interior of M_τ , we have that

$$F_\tau|_{\partial V_\tau} : \partial V_\tau = M_\tau \cup \partial M_\tau \times I \cup M'_\tau \rightarrow X_\tau \times \{0\} \cup \partial X_\tau \times I \cup X_\tau \times \{1\}$$

coincides with

$$f_\tau \cup (f_\tau \times I) \cup f'_\tau$$

(note that $f'_\tau|_{\partial M_\tau} = f_\tau|_{\partial M_\tau}$).

We denote by $\bar{f}'_\tau : X_\tau \rightarrow M_\tau$ a homotopy inverse of f'_τ . In our construction we add the cylinders $\partial M_\tau \times I$ and $\partial X_\tau \times I$ to M'_τ and $X_\tau \times 1$, and again denote them by M'_τ and $X_\tau \times 1$. Then f and f'_τ can be glued to give a homotopy equivalence

$$f \cup f'_\tau : M_q \cup M'_\tau \rightarrow X_q \cup X_\tau.$$

This can be done for every $\tau \subset B$ with $|\tau| = q - 1$. If $M_\tau \cap M_{\tau'}$ are nonempty, they intersect in a common face M_σ , resp. X_σ , where we have the map f . Glued together they give a homotopy equivalence $f' : M_{q-1} \rightarrow X_{q-1}$.

Lemma 3.5. *There is a homotopy inverse $\bar{f}' : X_{q-1} \rightarrow M_{q-1}$ such that $\bar{f}'|_{X_q} = \bar{f}$, and $\bar{f}'|_{X_\tau}$ is a homotopy inverse of f'_τ for every $\tau \subset B$ with $|\tau| = q - 1$.*

Proof. We fix $\tau \subset B$, $|\tau| = q - 1$. First note that $\bar{f}|_{\partial X_\tau} \sim \bar{f}'_\tau|_{\partial X_\tau}$ (where \bar{f}'_τ is the above introduced inverse of f'_τ). This can be seen as follows:

$$f \circ \bar{f}|_{\partial X_\tau} \sim \text{Id}_{\partial X_\tau} \quad \text{and} \quad f_\tau \circ \bar{f}'_\tau|_{\partial X_\tau} \sim \text{Id}_{\partial X_\tau}$$

implies

$$f_\tau \circ \bar{f}|_{\partial X_\tau} = f \circ \bar{f}|_{\partial X_\tau} \sim f_\tau \circ \bar{f}'_\tau|_{\partial X_\tau}.$$

However, f_τ is a homotopy equivalence, hence $\bar{f}|_{\partial X_\tau} \sim \bar{f}'_\tau|_{\partial X_\tau}$.

Let

$$H_t : \partial X_\tau \rightarrow \partial M'_\tau = \partial M_\tau$$

be a homotopy such that

$$H_0 = \bar{f}|_{\partial X_\tau} \quad \text{and} \quad H_1 = \bar{f}'_\tau|_{\partial X_\tau}.$$

By the Homotopy Extension Property we obtain a homotopy $\tilde{H}_t : X_\tau \times I \rightarrow M'_\tau$ such that

$$\begin{array}{ccc} X_\tau \times I & & \\ \cup & \searrow^{\tilde{H}_t} & \\ \partial X_\tau \times I \cup X_\tau \times \{1\} & \xrightarrow{\tilde{f}'_\tau} & \partial M'_\tau \cup M'_\tau = M'_\tau \\ & H_t \cup \tilde{f}'_\tau & \end{array}$$

commutes. Hence,

$$\tilde{f}_\tau = \tilde{H}_0 : X_\tau \rightarrow M'_\tau$$

is a homotopy equivalence such that $\tilde{f}_\tau|_{\partial X_\tau} = \bar{f}|_{\partial X_\tau}$. Hence,

$$\bar{f} \cup \tilde{f}_\tau : X_q \cup X_\tau \rightarrow M_q \cup M'_\tau$$

is a homotopy inverse of f' and it has the desired property. Since at the intersection $X_\tau \cap X_{\tau'}$ the maps $\tilde{f}_\tau, \tilde{f}_{\tau'}$ coincide with \bar{f} , we can glue them together to get $\bar{f}' : X_{q-1} \rightarrow M_{q-1}$ as claimed. \square

In order to complete the proof of Theorem 3.3, it remains to prove that there are homotopies of $f' \circ \bar{f}' \sim \text{Id}_{X_{q-1}}$, and $\bar{f}' \circ f' \sim \text{Id}_{M_{q-1}}$ with small tracks. We shall construct such a homotopy for $f' \circ \bar{f}' \sim \text{Id}_{X_{q-1}}$. The other case is similar.

We let $H_t : X_q \times I \rightarrow X_q$ be the homotopy of $f \circ \bar{f} \sim \text{Id}_{X_q}$ given by the inductive hypothesis, so $h_t = H_t|_{\partial X_\tau}$ is a homotopy of $f \circ \bar{f}|_{\partial X_\tau} \sim \text{Id}|_{\partial X_\tau}$. Recall that $X_q \cap X_\tau = \partial X_\tau$, so $f'_\tau \circ \bar{f}'_\tau$ coincides with $h_0 = f_\tau \circ \bar{f}_\tau$ on ∂X_τ . We consider

$$h_t \cup f'_\tau \circ \bar{f}'_\tau : \partial X_\tau \times I \cup X_\tau \times \{0\} \rightarrow X_\tau$$

and apply the homotopy extension property to obtain $h'_t = X_\tau \times I \rightarrow X_\tau$ such that

$$\begin{array}{ccc} X_\tau \times I & & \\ \cup & \searrow^{h'_t} & \\ \partial X_\tau \times I \cup X_\tau \times \{0\} & \longrightarrow & X_\tau \end{array}$$

The map $h'_1 : X_\tau \rightarrow X_\tau$ is homotopic to Id_{X_τ} , since $h'_0 = f'_\tau \circ \bar{f}'_\tau$ and it satisfies $h'_1|_{\partial X_\tau} = h_1 = \text{Id}_{\partial X_\tau}$.

It follows from Hatcher [8, Proposition 0.19] that h'_1 is homotopic relative ∂X_τ to Id_{X_τ} by a homotopy h''_t (note that here Id_{X_τ} is a homotopy inverse of h'_1). We can therefore compose the homotopies h'_t and h''_t in the usual way to get a homotopy

$$(h' * h'')_t : X_\tau \times I \rightarrow X_\tau$$

which coincides with H_t on $X_q \cap X_\tau$, giving a homotopy

$$H_t \cup (h' * h'')_t : (X_q \cup X_\tau) \times I \rightarrow X_q \cup X_\tau$$

between $(f \circ \bar{f}) \cup (f'_\tau \circ \bar{f}'_\tau)$ and Id.

If $\tau, \tau' \subset B$ are $(q+1)$ -simplices such that $X_\tau \cap X_{\tau'} \neq \emptyset$, they intersect in a common face $\sigma, |\sigma| = q$, so the above-constructed homotopies coincide with H_t , i.e. we can glue them together to get the desired controlled homotopies.

One notes that the tracks can be arbitrary small (measured in B) if we use an arbitrary small cell decomposition of B . This proves the inductive step.

We have in particular to consider the low-dimensional cases $n, n - 1$, and $n - 3$, because surgery does not apply (note that in dimension 4 one has to apply Freedman's result).

By the degree one property we can assume that $M_n = X_n$. For $n - i, 1 \leq i \leq 3$, the pieces

$$(f_\tau, b_\tau) : M_\tau^j \rightarrow X_\tau^j, \quad 1 \leq j \leq 3,$$

are special, namely ∂X_τ^j is a $(j - 1)$ -sphere, because π is UV^1 . We can close ∂X_τ^j by a j -disk to get a closed simply connected j -manifold, i.e. a j -sphere. By the inductive hypothesis, ∂M_τ^j must also be a $(j - 1)$ -sphere so M_τ^j can be closed.

The closed problem $M_\tau^j \rightarrow X_\tau^j$ bounds a problem $W_\tau^{j+1} \rightarrow V_\tau^{j+1}$ (because $\sigma^c(f, b) = 0$). Deleting the $(j + 1)$ -disks one obtains a normal cobordism between

$$M_\tau^j \rightarrow X_\tau^j \quad \text{and} \quad M_\tau^j = S^j \xrightarrow{\cong} X_\tau^j = S^j.$$

We can now choose a degree one map

$$(V_\tau^{j+1} \setminus \mathring{D}^{j+1}, X_\tau^j, S^j) \rightarrow (S^j \times I, S^j \times \{0\}, S^j \times \{1\})$$

and obtain a composition

$$F_\tau : (W_\tau^{j+1} \setminus \mathring{D}^{j+1}, M_\tau^j, S^j) \rightarrow (X_\tau^j \times I, X_\tau^j \times \{0\}, X_\tau^j \times \{1\}).$$

With this map F_τ , the proof proceeds as above, and Theorem 3.3 is finally proved. □

Epilogue

We shall conclude this paper by a remark on the controlled Wall realization. In our earlier paper [9], we showed that the controlled structure set of a manifold X with control map $q : X \rightarrow B$ is a subgroup of $H_{n+1}(B, X, \mathbb{L})$. The controlled Wall action of $H_{n+1}(B, \mathbb{L})$ on it is then nothing but the canonical map

$$H_{n+1}(B, \mathbb{L}) \rightarrow H_{n+1}(B, X, \mathbb{L})$$

of \mathbb{L} -homology groups.

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Friedrich Hegenbarth
Dipartimento di Matematica “Federigo Enriques”
Università degli studi di Milano
20133 Milan
Italy
e-mail: friedrich.hegenbarth@unimi.it

Dušan Repovš
Faculty of Education
University of Ljubljana
1000 Ljubljana
Slovenia

and

Faculty of Mathematics and Physics
University of Ljubljana
1000 Ljubljana
Slovenia
e-mail: dusan.repovs@guest.arnes.si

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