Virtual Special Issue – In memory of Professor Sibe Mardešić

The relationship of generalized manifolds to Poincaré duality complexes and topological manifolds

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A R T I C L E   I N F O

A B S T R A C T

The primary purpose of this paper concerns the relation of (compact) generalized manifolds to finite Poincaré duality complexes (PD complexes). The problem is that an arbitrary generalized manifold X is always an ENR space, but it is not necessarily a complex. Moreover, finite PD complexes require the Poincaré duality with coefficients in the group ring Λ (Λ-complexes). Standard homology theory implies that X is a $\mathbb{Z}$-PD complex. Therefore by Browder's theorem, X has a Spivak normal fibration which in turn, determines a Thom class of the pair $(N, \partial N)$ of a mapping cylinder neighborhood of X in some Euclidean space. Then X satisfies the Λ-Poincaré duality if this class induces an isomorphism with Λ-coefficients. Unfortunately, the proof of Browder's theorem gives only isomorphisms with $\mathbb{Z}$-coefficients. It is also not very helpful that X is homotopy equivalent to a finite complex K, because K is not automatically a Λ-PD complex. Therefore it is convenient to introduce Λ-PD structures. To prove their existence on X, we use the construction of 2-patch spaces and some fundamental results of Bryant, Ferry, Mio, and Weinberger. Since the class of all Λ-PD complexes does not contain all generalized manifolds, we appropriately enlarge this class and then describe (i.e. recognize) generalized manifolds within this enlarged class in terms of the Gromov–Hausdorff metric.

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1. Introduction

This paper deals with compact oriented generalized manifolds, mostly without boundary, and of dimension ≥ 5. Topological manifolds belong to this class. Conversely, by the well-known characterization theorem of Edwards [14] and the resolution theorem of Quinn [24,25], topological manifolds can be recognized in the class of generalized manifolds. This is briefly described in Section 2, thereby stabilizing basic notations used later on.

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A generalized manifold X^n has the fundamental class $[X] \in H_n(X, \mathbb{Z})$ and it satisfies the Poincaré duality (PD) with respect to $\mathbb{Z}$ coefficients. However, it is not a Poincaré duality complex in the sense of Wall [30] (see also Appendix). Therefore it is appropriate to introduce the concept of the *simple Λ-PD structure* on X, where $\Lambda = \mathbb{Z}[\pi_1(X)]$ denotes the integral group ring of $\pi_1(X)$. It consists of a (simple) symmetric algebraic Poincaré chain complex $(D_\#, \Phi)$ together with a (simple) chain equivalence

$$\alpha : (D_\#, \Phi) \ \to \ (S_\#(\widetilde{X}), \Delta[X]),$$

where $S_\#(\widetilde{X})$ denotes the singular chain complex of the universal cover $\widetilde{X}$ of X. We introduce this in Section 3.2.

To construct such a structure the following is convenient: Let N be a mapping cylinder neighborhood of an embedding $X \subset \mathbb{R}^m$, and $r : \partial N \to X$ the retraction. Then X is a $\mathbb{Z}$-PD space, but by Browder's theorem [1, Theorem A], the map $r : \partial N \to X$ has the structure of a spherical fibration, hence there is a Thom class, represented by a cycle

$$[\mathcal{U}] \in C^{m-n}(N, \partial N, \mathbb{Z}).$$

Here, $C^\#(\cdot, G)$ stands for the cellular chain complex with coefficients in G. Let

$$[\Sigma] = [\mathcal{U}] \cap [N] \in C_n(N, \mathbb{Z}),$$

where $[N] \in C_m(N, \partial N, \mathbb{Z})$ is the fundamental cycle of the manifold N. It is not obvious that $C_\#(N, \Lambda) = C_\#(\widetilde{N})$, together with $[\Sigma]$, determines a symmetric Poincaré chain complex (see Remark 3.7 in Section 3.2). In Section 3.3 we present a different approach, based on constructions used in [4,5].

Roughly speaking, one approximates X by Poincaré duality complexes K, obtained by gluing manifolds W^n and V^n along a controlled homotopy equivalence $y : \partial W \to \partial V$ between their boundaries. Applying Chapman's extensions of the Whitehead torsion theory to ANR spaces [9], one obtains symmetric simple Λ-PD chain complexes $(D_\#, \Phi)$ with simple chain equivalences

$$\alpha : (D_\#, \Phi) \to (S_\#(\widetilde{X}), \Delta[X]).$$

In Section 3.4 these are called *simple Λ-Poincaré duality types* on X.

Section 4 is an attempt to distinguish generalized manifolds. This requires a class of spaces containing all generalized manifolds, and then describe generalized manifolds in this class. In Section 4.2 we introduce an appropriate class $\mathcal{B}$ consisting of compact separable metric ENR's B satisfying the $\mathbb{Z}$-Poincaré duality. Moreover, these spaces B come equipped by simple Λ-PD types. Of course, this class contains the class of finite Λ-PD complexes in the sense of Wall.

The main theorem of [5, Theorem 9.1] leads to the following important characterization of generalized manifolds:

Characterization Theorem. *If $X \in \mathcal{B}$ has formal dimension ≥ 6 then X is a generalized n-manifold if the following is satisfied:*

(1) either X is the limit of an inverse sequence

$$X = \varprojlim \{K_{\varepsilon_1} \leftarrow K_{\varepsilon_2} \leftarrow \ldots \leftarrow K_{\varepsilon_i} \leftarrow \ldots\}$$

of ε_i-controlled Λ-PD complexes of dimension n and controlled homotopy equivalences $K_{\varepsilon_{i+1}} \to K_{\varepsilon_i}$
(2) or X is the cell-like-image $f : Y \to X$ of a generalized manifold Y of type (1).

In Section 4.1 we recall some definitions and facts about controlled PD spaces, controlled homotopy lifting properties and approximate fibrations. In summary, this leads to the formulation that "generalized manifolds are controlled Λ-Poincaré complexes", which however is not appropriate.

It follows from the Daverman and Husch theorem [13] that X is a generalized manifold if $r : \partial N \to X$ is an approximate fibration, where ∂N is the boundary of a mapping cylinder neighborhood N of $X \subset \mathbb{R}^m$ (see also [24, Example 2.3]). Hence a necessary and sufficient condition for X to be a generalized manifold is that the spherical Spivak fibration ν_X of X reduces to an approximate fibration in a controlled manner, i.e. that there is a controlled homotopy equivalence $\partial E\nu_X \to \partial N$ over X. This leads to the following:

Recognition Criterion. *Suppose that B belongs to the class $\mathcal{B}$. Then B is a generalized manifold if B admits an ε-Λ-PD structure for all $\varepsilon > 0$.*

In Section 4.3 we characterize generalized manifolds as isolated limits in the metric Gromov–Hausdorff space. If B belongs to $\mathcal{B}$ then its isometry class is an element of this space. Given two elements B, B' of $\mathcal{B}$, the Gromov–Hausdorff distance $d_G(B, B') \in \mathbb{R}_+$ is well-defined.

The approximation of generalized manifolds by 2-patch spaces (see Lemma 3.8 below) leads to the following criterion:

Gromov–Hausdorff Limit Criterion. *Any neighborhood of a generalized manifold B contains non-generalized manifolds. Moreover, any generalized manifold B is the limit of 2-patch spaces.*

Note that both limit criteria, the inverse limit criterion and the Gromov–Hausdorff limit criterion, are consequences of the constructions in [5], specifically Lemma 7.2 therein.

In a slightly modified spirit of Mardešić and Segal [19,20], one can state the following: An ANR space X is a generalized n-manifold if X is a $\mathcal{C}_n$-space, where $\mathcal{C}_n$ is the class of n-dimensional 2-patch spaces. A more precise description can be given in terms of the Gromov–Hausdorff space which contains $\mathcal{C}_n$. It is a complete metric space with respect to the Gromov–Hausdorff metric d_G.

We prove in Section 4.3 that a compact generalized n-manifold is a limit of elements of $\mathcal{C}_n$ with respect to the metric d_G, i.e. the frontier of $\mathcal{C}_n$ in the Gromov–Hausdorff space consists of compact generalized n-manifolds (here $n \geq 6$, as usual).

2. Manifolds and generalized manifolds

2.1. Preliminaries

Definition 2.1. Let X be a nonempty separable metric space and $k \geq 0$ any integer. Then

- X is said to have *dimension at most k*, $\dim X \leq k$, if for any open covering $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{J}}$ of X there is an open refinement $\{\mathcal{V}_i\}_{i \in \mathcal{L}}$ such that any $k + 1$ elements of $\{\mathcal{V}_i\}_{i \in \mathcal{L}}$ have empty intersection;
- X is said to be *k-dimensional*, $\dim X = k$, if $\dim X \leq k$ and $\dim X \nleq k - 1$.
- X is said to be *infinite-dimensional*, $\dim X = \infty$, if $\dim X \nleq k$ for every $k \geq 0$.

Definition 2.2. A topological space X is called a *Euclidean neighborhood retract* (ENR) if X embeds in some m-dimensional Euclidean space $\mathbb{R}^m$ as a closed subset so that there is a neighborhood $N \subset \mathbb{R}^m$ of X which retracts onto X.

It's well-known that a separable metric space X is an ENR if X is locally contractible and $\dim X < \infty$.

Definition 2.3 *(see [8]).* A topological space X is called a *generalized n-manifold*, $n \in \mathbb{N}$, if it satisfies the following properties:

(i) X is an n-dimensional separable metric ENR; and
(ii) for every $x \in X$, $H_*(X, X \setminus \{x\}, \mathbb{Z}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}, \mathbb{Z})$.

Furthermore, X is called a *generalized n-manifold with boundary* $\partial X \subset X$, if ∂X is also an ENR, and the boundary ∂X of X is characterized by the following property: $H_*(X, X \setminus \{x\}, \mathbb{Z}) \cong 0$ for every $x \in \partial X$ (see [22]).

Instead of ENR generalized manifolds, one often considers ANR generalized manifolds.

Remark 2.4 *(see [8]).* For a separable metric X with $\dim X < \infty$, the following conditions are equivalent:

(1) X is an ANR.
(2) X is locally contractible.
(3) $\dim X = k$ and X is locally k-connected.

Remarkable properties of compact ENR's, hence of compact generalized manifolds, are expressed by the following result.

Theorem 2.5. *The following properties are equivalent:*

(a) X is an ENR.
(b) For some m, there exist an embedding $\varphi : X \hookrightarrow \mathbb{R}^m$ and a mapping cylinder neighborhood $N \subset \mathbb{R}^m$ of $\varphi(X)$ in $\mathbb{R}^m$.
(c) For some n, X is the cell-like image of a compact manifold M^n (possibly with boundary ∂M^n).

Here, (b) means that N is homeomorphic to the mapping cylinder of a map $r : \partial N \to X$, denoted as $\partial N \times I \underset{r}{\cup} X$. The homeomorphism $N \cong \partial N \times I \cup X$ is the identity on ∂N and X. To explain (c), we recall the notion of a cell-like map and some of its properties (see for instance [15,18]).

Definition 2.6. A compact subset $C \subset \mathrm{Int} M^n$, where M^n is a topological n-manifold, is said to be *cellular* in M^n if it can be represented as follows:

$$C = \bigcap_{i=1}^{\infty} B_i, \quad \text{where} \quad B_i \subset \mathrm{Int} M^n \quad \text{are} \quad n\text{-balls such that} \quad B_{i+1} \subset \mathrm{Int} B_i, \quad \text{for} \quad i = 1, 2, \ldots.$$

Definition 2.7. A surjective map $f : Y \to X$ is said to be *cell-like* if for every $x \in X$, the preimage $f^{-1}(x)$ is a cell-like set, i.e. there exists an embedding

$$\varphi : f^{-1}(x) \hookrightarrow \mathrm{Int} M^n$$

into some topological n-manifold M^n such that $\varphi(f^{-1}(x))$ is cellular in M^n.
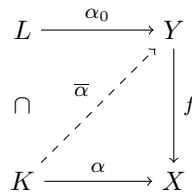
The following characterization is very useful (see [18]).

Theorem 2.8. *Let X and Y be ENR spaces and $f : Y \to X$ a surjective proper map. The following properties are equivalent:*

(a) *The map f is cell-like.*
(b) *For all open contractible subsets $K \subset X$, $f^{-1}(K) \subset Y$ is contractible.*
(c) *For all open subsets $U \subset X$, the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a proper homotopy equivalence.*

The properties of cell-like maps are related to controlled topology, and this is extremely important, because it links the resolution problem [14] to Quinn's invariant [25]. There is extensive literature on the subject, let us mention [3,8,15,16,18,23,28]. Here are some definitions and properties.

Definition 2.9. A mapping $f : Y \to X$ between compact ENR's Y and X is said to have:

- the $UV^k(\varepsilon)$-*property*, where $k \in \mathbb{N}$ and $\varepsilon > 0$, if for every pair (K, L) of complexes $L \subset K$ of dimension $\leq k+1$ and every pair of maps $(\alpha, \alpha_0) : (K, L) \to (X, Y)$, the commutative diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\ \alpha_0\ } & Y \\
\cap & \overset{\overline{\alpha}}{\nearrow} & \downarrow f \\
K & \xrightarrow[\ \alpha\ ]{} & X
\end{array}
$$

can be completed by a map $\overline{\alpha} : K \to Y$ such that $\overline{\alpha}|_L = \alpha_0$ and there is a homotopy $H : K \times I \to X$ between $f \circ \overline{\alpha}$ and α with tracks $\{H(x, t) | t \in I\}$ of diameter $< \varepsilon$.
- the UV^k-*property*, where $k \in \mathbb{N}$, if it has the $UV^k(\varepsilon)$-property for all $\varepsilon > 0$.
- the $UV^\infty(\varepsilon)$-*property*, where $\varepsilon > 0$, if it has the $UV^k(\varepsilon)$-property for all $k \in \mathbb{N}$.
- the UV^∞-*property*, if it has the UV^k-property for all $k \in \mathbb{N}$.

Definition 2.10. The homotopy $H : K \times I \to X$ in Definition 2.9 is called an ε-*homotopy*. It is now obvious what an ε-homotopy equivalence means.

The following was proved in [18]:

Theorem 2.11. *Let $f : Y \to X$ be a surjective map between compact ENR's. Then the following properties are equivalent:*

(i) *$f : Y \to X$ is a cell-like map.*
(ii) *$f : Y \to X$ is an ε-homotopy equivalence for all $\varepsilon > 0$.*
(iii) *$f : Y \to X$ is a UV^∞-map.*

2.2. Recognizing topological manifolds among generalized manifolds

Obviously, every topological n-manifold is a generalized n-manifold. An answer to the converse problem was given by Edwards [14]:

Definition 2.12. A metric space X is said to have the *disjoint disks property* (DDP) if for any $\varepsilon > 0$ and any maps $\alpha_1, \alpha_2 : D^2 \to X$ of the 2-disk D^2 into X, there exist maps $\beta_1, \beta_2 : D^2 \to X$ such that $\mathrm{dist}(\alpha_i(x), \beta_i(x)) < \varepsilon$ for all $x \in D^2, i \in \{1, 2\}$, and $\beta_1(D^2) \cap \beta_2(D^2) = \emptyset$.

Theorem 2.13. *Let X^n be a generalized n-manifold, $n \geq 5$, and suppose that X^n satisfies the DDP. Then X^n is homeomorphic to a topological n-manifold if there exist a is a topological n-manifold M^n and a cell-like-map $f : M^n \to X^n$. In fact, such f can then be approximated by homeomorphisms.*

At this point, Quinn invented controlled surgery theory to construct maps $f_\varepsilon : M^n_\varepsilon \to X$ which are ε-homotopy equivalences (over X) between n-manifolds M^n_ε, $n \geq 5$. Choosing a sequence $\{\varepsilon_i\}$ with $\varepsilon_i \to 0$, one can construct a "telescope"-manifold N^{n+1} with an end-manifold M^n (applying End Theorem [24]), and a map being an ε-homotopy equivalence for all $\varepsilon > 0$. Hence $f : M \to X$ is a cell-like map.

The manifolds M^n_ε and the maps $f_\varepsilon : M^n_\varepsilon \to X$ are constructed by controlled surgery ([16,24,25], revised in [7], see also [8]), starting from a controlled surgery problem $(g, b) : V^n \to X^n$, where b is an appropriate bundle map.

As usually, there are obstructions to complete surgery in the middle dimension to obtain an ε-homotopy equivalence. It turns out that there is only an integer obstruction $i(X)$ if one starts with an appropriate problem $(g, b) : V^n \to X$ (see [24,25]). Actually, the proof given in [24] (and later corrected in [25]) is of local nature. The existence of a canonical controlled surgery problem $(g, b) : V^n \to X$ follows from results in [17].

3. Poincaré duality structures on generalized manifolds and simple type

A generalized manifold X^n is not à priori a Poincaré duality complex in the sense of Wall [30]. First of all, X is not a CW complex. Even if we know that X^n is homotopy equivalent to a finite CW complex K [31], it is not at all clear that K is a Poincaré duality complex with respect to local coefficients, i.e. that K satisfies the Poincaré duality with coefficients in $\Lambda = \mathbb{Z}[\pi_1(K)]$ (shortly, Λ-PD complex).

It appears that the notion of "Λ-PD complex structure" is appropriate in this context. Moreover, Wall's definition requires that the PD isomorphism is a simple equivalence on the chain complex level, i.e. its Whitehead torsion vanishes. For details we refer to [9–11,15,16,21,29]. In the sequel, X will denote a compact oriented generalized n-manifold without boundary (if necessary, $n \geq 5$).

3.1. The Poincaré duality over $\mathbb{Z}$

Sheaf-theoretical methods imply that there is a fundamental class $[X] \in H_n(X, \mathbb{Z})$ such that

$$\cap[X] : H^q(X, \mathbb{Z}) \to H_{n-q}(X, \mathbb{Z})$$

is an isomorphism. A representing cycle of $[X]$ (also denoted $[X]$) defines a chain equivalence $S^q(X) \to S_{n-q}(X)$ on the singular (co-)chain level.

At this point we know that there is an embedding $X \subset \mathbb{R}^m$, for m sufficiently large, a mapping cylinder neighborhood N of X, and a map $r : \partial N \to X$, which is homotopy equivalent to a spherical fibration (Spivak fibration) $\partial E\nu_X \to X$ (see [1, Theorem A]).

3.2. Λ-PD complex structures

The neighborhood N of X is a (smooth) manifold with boundary, hence a simplicial complex. Let $C_\#(N, \Lambda)$ be the cellular chain complex of the universal cover $\widetilde{N}$ of N, and

$$C^\#(N, \Lambda) = \mathrm{Hom}_\Lambda(C_\#(N, \Lambda), \Lambda)$$

the cellular cochain complex. Similarly, $C_\#(N, \partial N, \Lambda)$ and

$$C^{\#}(N, \partial N, \Lambda), \Lambda = \mathbb{Z}[\pi_1(X)].$$

Note that $C^{\#}(N, \Lambda)$ is equivalent to $C_c^{\#}(\widetilde{N})$, the cochain complex of $\widetilde{N}$ with compact support ([6, pp. 358–360]).

Let $[N] \in C_m(N, \partial N)$ represent the fundamental class, and let $[\mathcal{U}] \in C^{m-n}(N, \partial N)$ represent the Thom class in $H^{m-n}(N, \partial N, \mathbb{Z})$ coming from the spherical fibration $\partial E\nu_X \to X$. Then we have the following obvious diagram

$$
\begin{array}{ccccc}
S_c^{\#}(\widetilde{X}) & \xrightarrow{\ r^{\#}\ } & C^{\#}(N, \Lambda) & \xrightarrow{\ \cap [N]\ } & C_{m-\#}(N, \partial N, \Lambda) \\
\big\downarrow{\scriptstyle \bullet \cap [X]} & & & {\scriptstyle \cap [\Sigma]}\searrow & \big\downarrow{\scriptstyle \cap [\mathcal{U}]} \\
S_{n-\#}(\widetilde{X}) & \xleftarrow{\ r_{\#}\ } & & & C_{n-\#}(N, \Lambda)
\end{array}
\tag{1}
$$

(see Remark 3.7 at the end of Section 3.2). Here, $[\Sigma] \in C_n(N)$ is the image of $[N] \in C_m(N, \partial N)$, under the map $C_m(N, \partial N) \to C_n(N)$.

Lemma 3.1 *(see [27]). The pair $(C_{\#}(N, \Lambda), [\Sigma])$ is a symmetric (algebraic) complex of dimension n.*

Proof. For details we refer to [27]. The symmetric property of the chain equivalence

$$\cap [\Sigma] : C^{\#}(N, \Lambda) \to C_{n-\#}(N, \Lambda)$$

is defined as the image of $[\Sigma]$ under the usual diagonal approximation

$$\Delta : C_n(N) \to W \underset{\mathbb{Z}[\mathbb{Z}/2]}{\otimes} (C_{\#}(N, \Lambda) \underset{\Delta}{\otimes} C_{\#}(N, \Lambda)),$$

where W denotes the free $\mathbb{Z}[\mathbb{Z}/2]$-module resolutions of $\mathbb{Z}$ (the generator of $\mathbb{Z}/2$ acts on $\mathbb{Z}$ by multiplication by -1). $\square$

Remark 3.2. $C_{\#}(N, \Lambda)$ is Λ-free and finitely generated, hence

$$W \underset{\mathbb{Z}[\mathbb{Z}/2]}{\otimes} (C_{\#}(N, \Lambda) \underset{\Lambda}{\otimes} C_{\#}(N, \Lambda)) \cong \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, \mathrm{Hom}_{\Lambda}(C^{\#}(N, \Lambda), C_{\#}(N, \Lambda))).$$

The image of $[N]$ therefore gives a sequence of Λ-chain-maps

$$\Phi_s : C^r(N, \Lambda) \to C_{n-r+s}(N, \Lambda), \quad s = 0, 1, 2, \ldots,$$

such that $\Phi_0 = \bullet \cap [\Sigma]$.

Definition 3.3. A *symmetric algebraic Λ-chain complex* is a couple $(D_{\#}, \Phi)$, where $D_{\#}$ is a free Λ-complex, and $\Phi = (\phi_s)$ is an element in

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, \mathrm{Hom}_{\Lambda}(D^{\#}, D_{\#})).$$

It is a symmetric (algebraic) Λ-Poincaré complex if ϕ_0 is a Λ-chain equivalence. If ϕ_0 is a simple Λ-chain equivalence, $(D_{\#}, \Phi)$ is called a *simple* symmetric Λ-Poincaré complex.

Definition 3.4. Let X be a generalized manifold as above. A *(simple)* Λ-PD *structure* on X is a commutative diagram

$$
\begin{array}{ccc}
S_c^\#(\widetilde{X}) & \xrightarrow{\ \alpha^\#\ } & D^\# \\
\big\downarrow{\scriptstyle \bullet\,\cap\,[X]} & & \big\downarrow{\scriptstyle \Phi_0} \\
S_{n-\#}(\widetilde{X}) & \xleftarrow{\ \alpha\ } & D_{n-\#}
\end{array}
$$

with $(D_\#, \Phi)$ a (simple) symmetric Λ-Poincaré complex and α a chain equivalence.

Lemma 3.5. *Under the conditions as above, the restriction map $r : N \to X$ and $(C_\#(N, \Lambda), [\Sigma])$ determine a symmetric Λ-Poincaré structure on X. It is unique up to an equivalence.*

A proof of Lemma 3.5 will be given in Section 4.2.

Definition 3.6. Two (simple) Λ-PD structures $(D_\#, \Phi)$, $(D'_\#, \Phi')$ on X are said to be *equivalent* if there is a chain equivalence $\gamma : D_\# \to D'_\#$ such that

$$
\begin{array}{ccccc}
& & \gamma^\# & & \\
D^\# & \xleftarrow{\ \alpha^\#\ } & S_c^\# & \xrightarrow{\ \alpha'^\#\ } & D'^\# \\
\big\downarrow{\scriptstyle \Phi_0} & & \big\downarrow{\scriptstyle \cap[X]} & & \big\downarrow{\scriptstyle \Phi'_0} \\
D_{n-\#} & \xrightarrow{\ \alpha\ } & S_{n-\#}(\widetilde{X}) & \xleftarrow{\ \alpha'\ } & D'_{n-\#} \\
& & \gamma & &
\end{array}
$$

commutes, and γ respects Φ and Φ' (for details see [27]).

Uniqueness of $(C_\#(N, \Lambda), [\Sigma])$ is due to the stability of the homotopy equivalence

$$(N, \partial N) \sim (E\nu_X, \partial E\nu_X),$$

where $E\nu_X$ is the mapping cylinder of $\partial E\nu_X \to X$.

The question if $(C_\#(N, \Lambda), [\Sigma])$ is a simple Λ-PD structure reduces to whether

$$\bullet \cap [\mathcal{U}] : C_\#(N, \partial N, \Lambda) \to C_{\#-(m-n)}(N, \Lambda)$$

is a simple Λ-chain equivalence. This will follow from an alternative approach presented in the following subsections.

Remark 3.7 *(Comments on [1, Theorem A]).* In Diagram (1), the maps $r_\#$ and $r^\#$ are chain equivalences, and we want that $\bullet \cap [\Sigma]$ is also a chain equivalence. However, this cannot be deduced from [1, Theorem A].

The strategy of the proof is to embed X into a simply-connected Poincaré space W^{n+1} of dimension $n + 1$ (more precisely, $X \times I$ is embedded in W^{n+1}), and then apply Spivak's result to W, to get the fibration ν_W. Then restricting it to X, one gets the Spivak fibration over X. Hence the Thom class of ν_X is the restriction of the Thom class of ν_W, so the cap product with it inherits only a $\mathbb{Z}$-chain equivalence, and not a Λ-equivalence.

3.3. Simple Λ-PD structures on X

The construction here uses ideas and results from [4,5]. More specifically, [5, Lemma 7.2] contains the following fact:
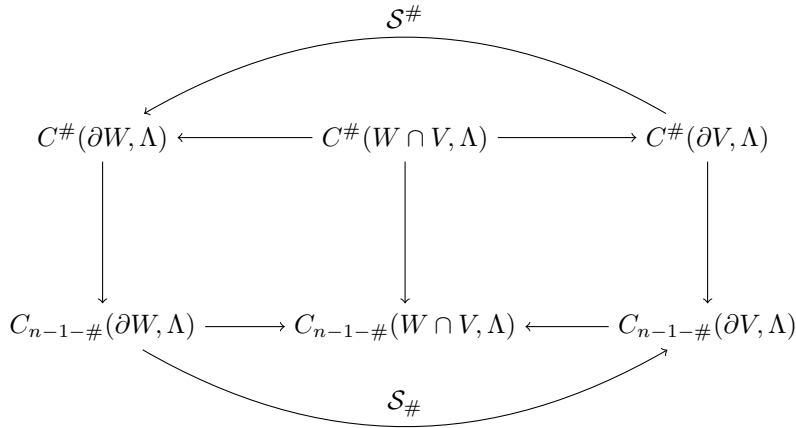
Lemma 3.8. *Given X as above with $\dim X = n \geq 6$, and given $\varepsilon > 0$, there is a space X_ε and an ε-homotopy equivalence $X_\varepsilon \to X$ over X, where $X_\varepsilon = W \underset{\mathcal{S}}{\cup} V$ is patched together from n-manifolds with boundary ∂W and ∂V along an ε-homotopy equivalence $\mathcal{S} : \partial W \to \partial V$ over X (see[5]).*

It is of course, required that $\pi_1(\partial W) \cong \pi_1(W)$ and $\pi_1(\partial V) \cong \pi_1(V)$. We may assume that W, V are simplicial complexes and $\mathcal{S} : \partial W \to \partial V$ is simplicial, hence X_ε is a complex. It is proved that X_ε is an ε-controlled PD complex. We are only interested if X_ε is a simple Λ-PD complex, which follows from the standard Mayer–Vietors argument and a vanishing Whitehead torsion. For convenience we write down the relevant diagrams:

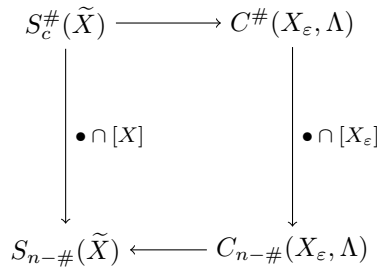Here, we consider $W, V \subset X_\varepsilon$. The left vertical isomorphism fits into the diagram of Λ-PD isomorphisms

$$\mathcal{S}^{\#}$$

$$C^{\#}(\partial W, \Lambda) \longleftarrow C^{\#}(W \cap V, \Lambda) \longrightarrow C^{\#}(\partial V, \Lambda)$$

$$C_{n-1-\#}(\partial W, \Lambda) \longrightarrow C_{n-1-\#}(W \cap V, \Lambda) \longleftarrow C_{n-1-\#}(\partial V, \Lambda)$$

$$\mathcal{S}_{\#}$$

defining two simple Λ-PD structures on $W \cap V$.

Now, $\mathcal{S}$ is an ε-equivalence, hence it follows by [10, Theorem 1'] that $\mathcal{S}_{\#}$, and $\mathcal{S}^{\#}$ are simple chain equivalences for sufficiently small $\varepsilon > 0$. It then follows by [30, Proposition 2.7], that X_{ε} is a simple Λ-PD complex. The induced chain equivalence of $X_{\varepsilon} \to X$ determines a simple Λ-PD structure on X:

$$S_c^{\#}(\widetilde{X}) \longrightarrow C^{\#}(X_{\varepsilon}, \Lambda)$$

$$\bullet \cap [X] \qquad \qquad \bullet \cap [X_{\varepsilon}]$$

$$S_{n-\#}(\widetilde{X}) \longleftarrow C_{n-\#}(X_{\varepsilon}, \Lambda)$$

3.4. The simple Λ-Poincaré duality type

The simple Λ-PD structure on a generalized manifold X, given by $X_{\varepsilon} \to X$, can be improved by requiring that it is a simple homotopy equivalence. However, this requires the extension of the Whitehead torsion theory to ENR spaces as done in [9]: To any homotopy equivalence $f : X \to Y$ between ANR's one can assign an element $\overline{\tau}(f) \in Wh(\pi_1(M))$ such that:

(i) If X, Y are finite CW complexes then $\overline{\tau}(f) = \tau(f)$, where τ denotes the classical torsion.
(ii) $\overline{\tau}(f) = 0$ if and only if there exists an ANR Z and cell-like-maps $Z \xrightarrow{\alpha} X$, $Z \xrightarrow{\beta} Y$ such that

$$Z$$
$$\alpha \swarrow \qquad \searrow \beta$$
$$X \xrightarrow{\ f\ } Y$$
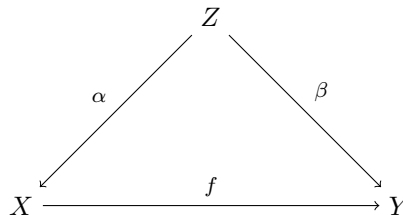
commutes up to homotopy. Any ANR X has a simple homotopy type given by $Id : X \to X$.

Applying this to our situations we obtain that $X_\varepsilon \to X$ is a simple homotopy equivalence because it is an ε-homotopy equivalence, assuming that $\varepsilon > 0$ is sufficiently small [10, Theorem 1'].

If N is the mapping cylinder neighborhood of $r : \partial N \to X$, then $r : N \to X$ is a simple homotopy equivalence with inverse the inclusion $i : X \hookrightarrow N$. One has

$$\overline{\tau}(X_\varepsilon \to X \xrightarrow{i} N) = \overline{\tau}(X_\varepsilon \to X) + \overline{\tau}(i) = 0.$$

Corollary 3.9. *The map*

$$\bullet \cap [\Sigma] : C^{\#}(N, \Lambda) \to C_{n-\#}(N, \Lambda)$$

is a simple chain equivalence.

Proof. This follows from the diagram

$$
\begin{array}{ccc}
C^{\#}(N, \Lambda) & \xrightarrow{\;\bullet \cap [\Sigma]\;} & C_{n-\#}(N, \Lambda) \\[2pt]
\big\uparrow & & \big\downarrow \\[2pt]
C^{\#}(X_\varepsilon, \Lambda) & \xrightarrow{\;\bullet \cap [X_\varepsilon]\;} & C_{n-\#}(X_\varepsilon)
\end{array}
$$
$\qquad\square$

Definition 3.10. A *simple Λ-PD type* on X is defined as a simple symmetric Λ-PD structure $(D_{\#}, \Phi)$

$$
\begin{array}{ccc}
S_c^{\#}(\widetilde{X}) & \xrightarrow{\;\alpha^{\#}\;} & D^{\#} \\[2pt]
\Big\downarrow{\scriptstyle \bullet \cap [X]} & & \Big\downarrow{\scriptstyle \Phi_0} \\[2pt]
S_{n-\#}(\widetilde{X}) & \xleftarrow{\;\;\alpha\;\;} & D_{n-\#}
\end{array}
$$

with α a simple chain equivalence. Two types are said to be *equivalent* if the chain equivalence $D_{\#} \xrightarrow{\gamma} D'_{\#}$ is simple.

In this sense we have the following:

Summary 3.11. Any compact oriented generalized n-manifold X has a simple Λ-PD type, unique up to equivalence determined by a mapping cylinder neighborhood N of $X \subset \mathbb{R}^m$, where $n \geq 6$ and m is sufficiently large.

Remark 3.12. Strictly speaking, it does not make sense to state that $\alpha : D_{\#} \to S_{\#}(\widetilde{X})$ is a simple chain equivalence. However, the extension of Whitehead torsion to compact ANR spaces is built on the fact that there is a cell-like map $K \times Q \to X$, where Q is a Hilbert cube manifold and K is a finite complex. Then "simple" refers to the finite complex K.

4. Recognizing generalized manifolds among ENR spaces with simple Λ-PD type

4.1. Controlled Poincaré duality complexes and approximate fibrations

Controlled Poincaré duality complexes are a bridge between simple Λ-PD complexes and generalized manifolds. There is a controlled geometric aspect linked to [4, Proposition 4.5] (see also [24, Example 2.3]). The geometric aspect leads to approximate fibrations introduced in [12].

Definition 4.1. An *oriented n-dimensional ε-PD complex* K (over K) is a finite complex K, an n-cycle $[K] \in C_n(K)$ such that cap product with it

$$\bullet \cap [K] : C^{\#}(K, \Lambda) \to C_{n-\#}(K, \Lambda)$$

is an ε-chain equivalence over K.

To define ε-chain maps, resp. ε-chain equivalences, one needs the notion of geometric chain complexes. We refer to the literature, in particular to [32] and [33, Remark 2 on p. 120] and [16, Definition 2.1]. One observes that the ε-Poincaré-duality is a much stronger condition than the simple Poincaré duality, assuming that ε is sufficiently small (again [10, Theorem 1']).

Let $N \subset \mathbb{R}^m$ be a regular neighborhood of $K \subset \mathbb{R}^m$, m sufficiently large, $r : N \to K$ the restriction. Then by [4, Proposition 4.5], the restrictions of r to ∂N (also denoted by r), $r : \partial N \to K$, has the δ-controlled homotopy lifting property. Here ε is sufficiently small and δ depends on ε and K.

Definition 4.2. The *δ-controlled homotopy lifting property* is defined as follows: Given a separable metric space Z, and a commutative diagram

$$
\begin{array}{ccc}
Z \times \{0\} & \xrightarrow{\ h_0\ } & \partial N \\
\downarrow & {\Large\nearrow}^{H} & \downarrow {\scriptstyle r} \\
Z \times I & \xrightarrow{\ h_t\ } & K
\end{array}
$$

there is a homotopy $H : Z \times I \to \partial N$, such that $\mathrm{dist}(r \circ H_t, h_t) < \delta$. (Note the relationship with the $UV^k(\varepsilon)$-property in Definition 2.9. For details we refer to [3], in particular Theorem 6 therein.)

We emphasize that this holds for some $\delta = T \cdot \varepsilon$, where $T > 0$ is a factor depending on K [4, Proposition 4.5]. However, if this holds for all $\delta > 0$ then one obtains approximate fibrations, more precisely (see [12]):

Definition 4.3. An *approximate fibration* $p : E \to B$ is a surjective map between locally compact separable metric ANR's E and B such that the above δ-homotopy lifting property holds for all separable metric spaces Z and all $\delta > 0$.

It is remarkable and useful to note that it is sufficient to require the lifting property only for cells [12, Theorem 2.2]. The last building stone of the bridge is the following:

Theorem 4.4 *([13]). Suppose that $p : M \to B$ is a proper map, where M is a closed connected manifold and B is an ANR. Then, if p is an approximate fibration, B is a generalized manifold.*

Putting this together, one can phrase the recognition principle as follows: A Λ-PD complex is a generalized manifold if it satisfies the ε-Poincaré duality for all $\varepsilon > 0$. However, generalized manifolds in general do not have CW or simplicial structures. It is therefore appropriate to define a class of spaces to which the above theorems can be applied to characterize generalized manifolds in this class. This will be done in the next section.

4.2. The class of simple Λ-PD types and approximations by controlled PD complexes

In this section we consider the following class $\mathcal{B}$ of PD spaces B characterized by

(i) B is a compact separable metric ENR which satisfies $\mathbb{Z}$-Poincaré duality of dimension n.
(ii) B has a simple Λ-PD type, i.e. there is a diagram

$$
\begin{array}{ccc}
S_c^\#(\widetilde{B}) & \xrightarrow{\ \alpha^\#\ } & D^\# \\
\downarrow {\scriptstyle \bullet\,\cap\,[B]} & & \downarrow {\scriptstyle \Phi_0} \\
S_{n-\#}(\widetilde{B}) & \xleftarrow{\ \alpha\ } & D_{n-\#}
\end{array}
$$

where $(D_\#, \Phi)$ is a simple symmetric algebraic Poincaré duality chain complex and α is a simple chain equivalence ($\Lambda = \mathbb{Z}[\pi_1 B]$), see Remark 3.12). We denote this class by $\mathcal{B}$.

Let $N \cong \partial N \times I \underset{r}{\cup} B$ be a mapping cylinder neighborhood of $B \subset \mathbb{R}^m$. It is equivalent to the Spivak fibration by [1], and it gives rise to the following diagram (see Section 3)

$$
\begin{array}{ccccc}
C^\#(N, \Lambda) & \xleftarrow{\ r^\#\ } & S_c^\#(\widetilde{B}) & \xrightarrow{\ \alpha^\#\ } & D^\# \\
\downarrow {\scriptstyle \bullet\,\cap\,[\Sigma]} & & \downarrow {\scriptstyle \bullet\,\cap\,[B]} & & \downarrow {\scriptstyle \Phi_0} \\
C_{n-\#}(N, \Lambda) & \xrightarrow{\ r_\#\ } & S_{n-\#}(\widetilde{B}) & \xleftarrow{\ \alpha\ } & D_{n-\#}
\end{array}
$$

Since $r : N \to B$ is a simple equivalence, $\bullet \cap [\Sigma]$ is also a simple chain equivalence. This proves Lemma 3.5 from Section 3.2.

As explained in Section 3, generalized manifolds belong to this class, and of course, also finite simple Λ-PD complexes. To recognize generalized manifolds within the above defined class $\mathcal{B}$, we define ε-Λ-structures:

Definition 4.5. A *(simple)* Λ-PD *structure* $(D_\#, \Phi)$ on B is a (simple) ε-Λ-PD structure on B if it makes $\bullet \cap [\Sigma] : C^\#(N) \to C_{n-\#}(N)$ a (simple) ε-chain equivalence by means of the diagram

$$
\begin{array}{ccccc}
C^\#(N, \Lambda) & \xleftarrow{\ r^\#\ } & S_c^\#(\widetilde{B}) & \xrightarrow{\ \alpha^\#\ } & D^\# \\
\downarrow {\scriptstyle \bullet\,\cap\,[\Sigma]} & & \downarrow {\scriptstyle \bullet\,\cap\,[B]} & & \downarrow {\scriptstyle \Phi_0} \\
C_{n-\#}(N, \Lambda) & \xrightarrow{\ r_\#\ } & S_{n-\#}(\widetilde{B}) & \xleftarrow{\ \alpha\ } & D_{n-\#}
\end{array}
$$

A result of Daverman and Hush (see Theorem 4.4 above) leads to the following:

Theorem 4.6 *(Recognition Criterion). Suppose that B belongs to $\mathcal{B}$. Then B is a generalized manifold if for every $\varepsilon > 0$, B admits an ε-Λ-PD structure.*

Remark 4.7. Clearly, B in Theorem 4.6 is simple, if the ε-Λ-PD structures are simple.

Proof. The hypothesis implies that B is an ε-PD space for all $\varepsilon > 0$. By [4, Proposition 4.5], it follows that $\partial N \to B$ has the $T \cdot \varepsilon$-lifting property for all $\varepsilon > 0$, where T is a factor depending on B. Hence $\partial N \to B$ is an approximate fibration, i.e. B is a generalized manifold (see Theorem 4.4 above). □

4.3. Generalized manifolds as limits in the Gromov–Hausdorff metric space

The Bryant–Ferry–Mio–Weinberger lemma [5, Lemma 7.2] also allows a characterization of generalized manifolds in terms of the Gromov–Hausdorff metric, denoted here by d_G. It is defined on the set of isometry classes of compact metric spaces, and d_G can be proved to be a complete metric (for details see [15], or [16, Section 1.V]).

We consider elements of the class $\mathcal{B}$ as being points of the Gromov–Hausdorff space and we prove the following result.

Lemma 4.8. *Let X be a generalized compact n-manifold and let $\delta > 0$ be given. Then there exists a 2-patch space X' such that $d_G(X, X') < \delta$.*

Remark 4.9. Note that X' in Lemma 4.8 is in general not a generalized manifold.

Proof. Let N be a cylindrical (regular) neighborhood of an embedding $X \subset \mathbb{R}^m$, $m \geq 2n + 1$, $m - n \geq 3$. If $C \subset \mathbb{R}^m$ is compact, we denote by $\varrho(x, C)$ the distance of x from C in the metric of $\mathbb{R}^m$, and $C_\varepsilon = \{x \in \mathbb{R}^m | \varrho(x, C) < \varepsilon\}$.

By Lemma 3.8 we can choose an ε-homotopy equivalence $f : X' \to X$. We can choose $\varepsilon > 0$ small enough to get $X_\varepsilon \subset N$. Inside X_ε we can find a cylindrical (regular) neighborhood $Z \xrightarrow{p} X$. By Proposition [4, Proposition 4.10], there is an embedding $X' \xrightarrow{j} Z$ and a retraction $r : Z \to X'$ such that $\varrho(p(z), r(z)) < 2\varepsilon$ for $z \in Z$.

For clarity we write down the maps in the diagram

$$
\begin{array}{ccccc}
Z & = & Z & \subset & X_\varepsilon \\
r \big\downarrow \uparrow j & & p \big\downarrow \uparrow i & & \\
X' & \xrightarrow{\;\;f\;\;} & X & &
\end{array}
$$

For $z \in Z$ we therefore get

$$\varrho(z, r(z)) \leq \varrho(z, p(z)) + \varrho(p(z), r(z)) < \varrho(z, p(z)) + 2\varepsilon < 3\varepsilon,$$

since $Z \subset X_\varepsilon$. Hence $\varrho(z, X') < 3\varepsilon$, for all $z \in Z$, i.e. $X \subset Z \subset X'_{3\varepsilon}$. Since $X' \subset Z \subset X_\varepsilon$ we get by definition of d_G that $d_G(X', X) < 3\varepsilon$. This proves the lemma, since we can choose ε arbitrarily small. □

Caveat 4.10. The notion X_ε used in the above proof differs from the notion used in Section 3.3.

Corollary 4.11. *Generalized manifolds are isolated limits of the subspace of 2-patch spaces in the Gromov–Hausdorff space.*

Remark 4.12. Other examples of limits in the Gromov–Hausdorff space were considered in [16, Section 1.V].

Acknowledgements

Appendix A. Appendix

The relations between $\mathbb{Z}$-PD and Λ-PD of a space K were always behind the discussion in the previous sections. Since it is of general interest and is not explicitly presented in the literature, we state them in the following lemma:

Lemma A.1. *Let K be a finitely dominated CW complex. If K satisfies $\mathbb{Z}$-PD, then it satisfies Λ-PD, where $\Lambda = \mathbb{Z}[\pi_1(K)]$.*

For the proof we note the following key facts:

- By Browder's Theorem [1, Theorem A], K has a Spivak normal fibration $\pi : E\nu_K \to K$.
- A result of Ranicki [26, Proposition 3.10] shows that the cap product with the Thom class $[\mathcal{U}]$,

$$\cap : H_*(T\nu_K, \Lambda) \to H_{*-k}(K, \Lambda)$$

  is an isomorphism, where k is the fiber dimension of $E\nu_K$. The proof of this consists of applying Browder's lemma [2, Lemma I.4.3] to the universal cover $\widetilde{K}$ of K.
- Using the homotopy equivalence $(N, \partial N) \to (E\nu_K, \partial E\nu_K)$, where $N \subset \mathbb{R}^{n+k}$ is a regular neighborhood of $K \subset \mathbb{R}^{n+k}$, one obtains Λ-PD as the following composite map

$$H^*(K, \Lambda) \cong H^*(N, \Lambda) \to H_{n+k-*}(N, \partial N, \Lambda) \xrightarrow{\cap[\mathcal{U}]} H_{n-*}(K, \Lambda). \quad \Box$$

We emphasize that this lemma is not helpful for proving the results of this paper.

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