CONTROLLED HOMOTOPY EQUIVALENCES AND STRUCTURE SETS OF MANIFOLDS

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ABSTRACT. For a closed topological n–manifold K and a map $p : K \to B$ inducing an isomorphism $\pi_1(K) \to \pi_1(B)$, there is a canonically defined morphism $b : H_{n+1}(B, K, \mathbb{L}) \to \mathcal{S}(K)$, where $\mathbb{L}$ is the periodic simply connected surgery spectrum and $\mathcal{S}(K)$ is the topological structure set. We construct a refinement $a : H_{n+1}^+(B, K, \mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(K)$ in the case when p is UV^1, and we show that a is bijective if B is a finite-dimensional compact metric ANR. Here, $H_{n+1}^+(B, K, \mathbb{L}) \subset H_{n+1}(B, K, \mathbb{L})$, and $\mathcal{S}_{\varepsilon,\delta}(K)$ is the controlled structure set. We show that the Pedersen-Quinn-Ranicki controlled surgery sequence is equivalent to the exact $\mathbb{L}$-homology sequence of the map $p : K \to B$, i.e. that

$$H_{n+1}(B, \mathbb{L}) \to H_{n+1}^+(B, K, \mathbb{L}) \to H_n(K, \mathbb{L}^+) \to H_n(B, \mathbb{L}), \ \mathbb{L}^+ \to \mathbb{L},$$

is the connected covering spectrum of $\mathbb{L}$. By taking for B various stages of the Postnikov tower of K, one obtains an interesting filtration of the controlled structure set.

1. INTRODUCTION

Let K denote a closed topological manifold of dimension n. As usual, $\mathcal{S}(K)$ denotes the topological structure set of K. Elements of $\mathcal{S}(K)$ are equivalence classes of pairs (M, h), where M is a closed topological n-manifold and $h : M \to K$ is a simple homotopy equivalence. The pairs $(M, h), (M_1, h_1)$ are equivalent if there is a homeomorphism $\varphi : M_1 \to M$ such that $h \circ \varphi$ is homotopic to h_1. $\mathcal{S}(K)$ has a group structure. If B is a finite-dimensional compact metric ANR and $p : K \to B$ a continuous UV^1 map, the controlled structure set $\mathcal{S}_{\varepsilon,\delta}(K \to B)$ (cf. [PQR]) of δ-homotopy equivalences can be defined (see precise definitions below). There is an obvious forgetful map $\mathcal{S}_{\varepsilon,\delta}(K \to B) \to \mathcal{S}(K)$. An element of the image will be called a controlled homotopy equivalence $h : M \to K$ with respect to p. Varying $p : K \to B$, we get the subset of controlled homotopy equivalences in $\mathcal{S}(K)$.

Dranishnikov and Ferry studied the subset (indeed a subgroup) $\mathcal{S}^{CE}(K) \subset \mathcal{S}(K)$ of elements realized by cell-like maps (cf. [DrFe]). They observed that these elements belong to the image of a natural map

$$b : H_{n+1}(E_2(K), K; \mathbb{L}) \to \mathcal{S}(K)$$

localized away from 2.

Here $\mathbb{L}$ is the simply connected 4-periodic surgery spectrum, and $E_2(K)$ is the second stage of the Postnikov tower of K. The map b can be defined using the diagram on p. 207 of [Ran]. The main ingredient is the $\pi - \pi$–theorem. The morphism b can be defined for any map $p : K \to B$ satisfying $p_* : \pi_1(K) \xrightarrow{\cong} \pi_1(B)$. We shall generically call it b.

One may ask of what type are homotopy equivalences of images of other Postnikov stages. In this paper we shall give an answer in the case of the first stage, i.e. $B\pi$, where $\pi = \pi_1(K)$. As a consequence of the main theorem we shall get the following result:

Theorem 1. *Let $n \geq 5$. Then the image of the map*

$$b : H_{n+1}(B\pi, K, \mathbb{L}) \to \mathcal{S}(K)$$

contains the controlled homotopy equivalences.

In order to be able to state the main result we shall need some more notations. Recall that $\mathbb{L}_0 = \mathbb{Z} \times G/TOP$. Let $\mathbb{L}^+ \to \mathbb{L}$ be the simply connected spectrum covering $\mathbb{L}$. In particular, $\mathbb{L}_0^+ = G/TOP$. For any pair (Z, Y) there is the following exact sequence (cf. [Ran, p. 153]):

$(*)$ $\cdots \to H_{m+2}(Z, Y, \mathbb{Z}) \to H_{m+1}(Z, Y, \mathbb{L}^+) \to H_{m+1}(Z, Y, \mathbb{L}) \to H_{m+1}(Z, Y, \mathbb{Z}) \to \cdots$.

In particular, for $(K, \emptyset)$, where K is an oriented closed topological n-manifold, we have an injection

$$H_n(K, \mathbb{L}^+) \to H_n(K, \mathbb{L}).$$

Let us denote the inverse image of $H_n(K, \mathbb{L}^+)$ under $\partial_* : H_{n+1}(B, K, \mathbb{L}) \to H_n(K, \mathbb{L})$ by $H_{n+1}^+(B, K, \mathbb{L}) \subset H_{n+1}(B, K, \mathbb{L})$. Here B is a space containing K, or more generally, a map $K \to B$.

Theorem 2. *Suppose that $p : K \to B$ is a UV^1-map into a compact metric ANR-space, with K as above and $n \geq 5$. Then there is a map*

$$a : H_{n+1}^+(B, K, \mathbb{L}) \to \mathcal{S}_{\varepsilon, \delta}(K \to B),$$

where ε, δ are appropriately choosen (see below). The map a fits into the following diagram:

$(**)$
$$
\begin{array}{ccc}
H_{n+1}^+(B, K, \mathbb{L}) & \xrightarrow{\ a\ } & \mathcal{S}_{\varepsilon, \delta}(K \to B) \\
\downarrow & & \downarrow \\
H_{n+1}(B, K, \mathbb{L}) & \xrightarrow{\ b\ } & \mathcal{S}(K).
\end{array}
$$

The natural map $H_{n+1}(B, \mathbb{L}) \to H_{n+1}(B, K, \mathbb{L})$ factors over $H_{n+1}^+(B, K, \mathbb{L})$, and one can easily see that the sequence

$$\cdots \to H_{n+1}(B, \mathbb{L}) \to H_{n+1}^+(B, K, \mathbb{L}) \to H_n(K, \mathbb{L}^+) \to H_n(B, \mathbb{L}) \to \cdots$$

is exact.

Recall that by the Poincaré duality $H_n(K, \mathbb{L}^+) \cong [K, G/TOP]$.

Theorem 3. *Let K be a closed oriented topological n-manifold, $n \geq 5$, and let $p : K \to B$ be a UV^1-map into a finite-dimensional compact metric ANR. Then*

there exists a commutative diagram of exact sequences

$$H_{n+1}(K, \mathbb{L}^+) \rightarrow H_{n+1}(B, \mathbb{L}) \rightarrow H_{n+1}^+(B, K, \mathbb{L}) \rightarrow H_n(K, \mathbb{L}^+) \rightarrow H_n(B, \mathbb{L})$$

$$\downarrow{\cong} \qquad\qquad \| \qquad\qquad \downarrow{a} \qquad\qquad \downarrow{\cong} \qquad\qquad \|$$

$$[SX, G/TOP] \rightarrow H_n(B, \mathbb{L}) \rightarrow \mathcal{S}_{\varepsilon,\delta}(K \rightarrow B) \rightarrow [K, G/TOP] \rightarrow H_n(B, \mathbb{L}).$$

In particular, a is bijective.

Here the lower exact sequence is the controlled surgery sequence from [PQR] and ε and δ are to be appropriately chosen: There is an $\varepsilon_0 > 0$, depending on B and n, such that for any $0 < \varepsilon < \varepsilon_0$ there is $\delta > 0$ such that the lower sequence is exact.

Theorem 2 and Theorem 3 will be proved in §5 and §6, respectively. Theorem 1 follows from Theorem 2, since any UV^1-map $K \rightarrow B$ composed with the canonical map $B \rightarrow B\pi$ determines a map $(B, K) \rightarrow (B\pi, K)$. Since K is a manifold of dimension ≥ 5 we can assume that $K \rightarrow B\pi$ is UV^1 (cf. [BFMW]). Hence we get an induced morphism

$$H_{n+1}(B, K, \mathbb{L}) \rightarrow H_{n+1}(B\pi, K, \mathbb{L}).$$

On the other hand, any $[y] \in H_{n+1}(B\pi, K, \mathbb{L})$ has a compact support $(B, K) \subset (B\pi, K)$ (cf. [Ran, §12]), and we may assume $K \rightarrow B$ is UV^1.

To define the map a we recall the geometric $\mathbb{L}$-spectrum, and represent elements $[x] \in H_n(K, \mathbb{L})$ and $[y] \in H_{n+1}(B, K, \mathbb{L})$ in terms of normal degree 1 maps. We shall follow [Qu1], [Nic] and [Ran]. One reason to recall these is to stabilize the notations used in the proofs. Another reason is that [Ran] used the algebraic $\mathbb{L}$-spectrum defined by quadratic Poincaré chain complexes which led to representing $[x]$ by a normal degree 1 map between closed manifolds (cf. Corollary 18.6(I) in [Ran]). The gluing construction using the geometric $\mathbb{L}$-spectrum gives surgery problems with homotopy equivalences on boundaries. Explicit examples have recently been given (cf. [Ham, Example 5.4]).

2. THE PERIODIC SIMPLY CONNECTED $\mathbb{L}$-SPECTRUM

We briefly recall the semi-simplicial surgery spectrum introduced in [Qu1] (cf. also [Nic]), denoted by $\mathbb{L} = \{\mathbb{L}_g | g \in \mathbb{Z}\}$. It is an Ω-spectrum, and each $\mathbb{L}_g$ satisfies the Kan condition. Its homotopy groups are the Wall groups of the trivial fundamental group, more precisely, $\pi_j(\mathbb{L}_g) = L_{j+g}(\{1\})$, and $\mathbb{L}_0 = \mathbb{Z} \times G/TOP$.

Let $\mathbb{L}_g(j)$ denote the j-simplices of $\mathbb{L}_g$. An element x in $\mathbb{L}_g(j)$ is a degree 1 normal map of $(j+3)$-ads, which we shall shortly denote by

$$x = \{(f, b) : M \rightarrow X\},$$

where $(M, \partial_0 M, \ldots, \partial_j M, \partial_{j+1} M)$ (resp. $(X, \partial_0 X, \ldots, \partial_j X, \partial_{j+1} X)$) are the $(j+3)$-ads M (resp. X). It is required that $f|_{\partial_{j+1} M}$ is a (simple) homotopy equivalence. M and X are topological manifolds of dimension $j+g$ with boundaries $\partial X = \cup_{i=0}^{j+1} \partial_i X$ and $\partial M = \cup_{i=0}^{j+1} \partial_i M$. To x belongs a reference map from X to the $(j+3)$-ad Δ^j, i.e. a map $(X, \partial_0 X, \ldots, \partial_j X, \partial_{j+1} X) \rightarrow (\Delta^j, \partial_0 \Delta^j, \ldots, \partial_j \Delta^j, \Delta^j)$. We emphasize that the condition that $f|_{\partial_{j+1} M} : \partial_{j+1} M \rightarrow \partial_{j+1} X$ is a homotopy equivalence is necessary to indentify the homotopy groups of $\mathbb{L}$ with the Wall groups.

The Cartesian product with $\mathbb{C}P^2$ defines a semi-simplicial map $\mathbb{L}_g \rightarrow \mathbb{L}_{g+4}$ which is a homotopy equivalence, called the periodicity map.

3. $\mathbb{L}$-HOMOLOGY CLASSES

We shall follow [Ran, §12], to describe elements $[x] \in H_n(K, \mathbb{L})$, where K is a finite ordered simplicial complex. The relative case will be given in §4.

The class $[x]$ is represented by a "cycle" x, which roughly is a semi-simplicial map from K to $\mathbb{L}_g$ for some g. More precisely, one embeds K into $\partial\Delta^{m+1}$, Δ^{m+1} the standard $(m+1)$-simplex with vertices denoted $\{0, 1, \ldots, m+1\}$. The embedding is order-preserving and one may assume $m+1 = $ number of vertices of K. For any $\sigma \in \partial\Delta^{m+1}$, let $D(\sigma, \partial\Delta^{m+1})$ be the dual cell, and let Σ^m be the dual cell-complex decomposition of $\partial\Delta^{m+1}$. Note that $D(\sigma, \partial\Delta^{m+1})$ is an $(m - |\sigma|)$-ball with boundaries $\partial_i D(\sigma, \partial\Delta^{m+1}) = D(\delta_i\sigma, \partial\Delta^{m+1})$, where $\sigma = \{j_0, \ldots, j_k\} \subset \{0, \ldots, m+1\}$, and $\delta_i\sigma = \sigma \cup \{\lambda_i\}$ with $\{0, \ldots, m+1\} \setminus \{j_0, \ldots, j_h\} = \{\lambda_0, \ldots, \lambda_{m-k}\}$ (written in that order). Hence $\{D(\sigma, \partial\Delta^{m+1}), \partial_0 D(\sigma, \partial\Delta^{m+1}), \ldots, \partial_{m-|\sigma|} D(\sigma, \partial\Delta^{m+1})\}$ is an $(m - |\sigma| + 2)$-ball-ad.

Define $\overline{K} = \{D(\sigma, \partial\Delta^{m+1}) | \sigma \in \partial\Delta^{m+1} \setminus K\}$ and call it the supplement of K. The cycle x can be considered as a semi-simplicial map

$$x : (\Sigma^m, \overline{K}) \to (\mathbb{L}_{n-m}, \emptyset);$$

i.e. $x(D(\sigma, \partial\Delta^{m+1})) \in \mathbb{L}_{n-m}(m - |\sigma|)$ is a surgery problem of $(m - |\sigma| + 3)$-ads denoted simply

$$x = \{x_\sigma = (f_\sigma, b_\sigma) : M_\sigma \to X_\sigma | \sigma \in \partial\Delta^{m+1}\}$$

between manifold $(m - |\sigma| + 3)$-ads of dimension $n - |\sigma|$ together with a reference map of $(n - |\sigma| + 3)$-ads

$$(X_\sigma, \partial_0 X_\sigma, \ldots, \partial_{m-|\sigma|} X_\sigma, \partial_{m-|\sigma|+1} X_\sigma)$$

$$\to (D(\sigma, \partial\Delta^{m+1}), \partial_0 D(\sigma, \partial\Delta^{m+1}), \ldots, \partial_{m-|\sigma|} D(\sigma, \partial\Delta^{m+1}), D(\sigma, \partial\Delta^{m+1})).$$

Moreover, f_σ restricted to $\partial_{m-|\sigma|+1} M_\sigma \to \partial_{m-|\sigma|+1} X_\sigma$ is a homotopy equivalence. Note that $X_\sigma \neq \emptyset \Rightarrow \sigma \in K$.

If $x' = \{x'_\sigma | \sigma \in \partial\Delta^{m+1}\}$ is a cycle defining the same class as x, there is a semi-simplicial map

$$y = \Delta^1 \times (\Sigma^m, \overline{K}) \to (\mathbb{L}_{n-m}, \emptyset)$$

such that $y(\partial_0\Delta^1 \times \sigma) = x(\sigma)$, $y(\partial_1\Delta^1 \times \sigma) = x'(\sigma)$.

Next we assemble the pieces $\{X_\sigma\}$, resp. $\{M_\sigma\}$; i.e. we build colimits. For this we use results from [LaMc] which are elegant reformulation of [BRS].

Let A_{STOP} be the graded category with objects n-dimensional oriented topological manifolds with boundaries. The morphisms are orientation preserving inclusions (if dimension preserving) or inclusions with the image in boundaries (if dimension increasing). We refer to Example 3.5 from [LaMc].

Given the cycle x as above we obtain two functors $(\Sigma^m, \overline{K}) \to (A_{STOP}, \emptyset)$ of degree $m - n$, namely we associate to $x(\sigma)$ either the target or the domain: i.e.

$$D(\sigma, \partial\Delta^{m+1}) \to X_\sigma \text{ and } D(\sigma, \partial\Delta^{m+1}) \to M_\sigma.$$

Let X, resp. M, denote its colimits, and denote by $(f, b) : M \to X$ the resulting map. By Proposition 6.6 of [LaMc], M and X are n-manifolds with boundaries ∂M, ∂X, respectively. In fact, ∂M and ∂X are the colimits of the restricted functors to the boundary components $\partial_{m-|\sigma|+1} M_\sigma$, resp. $\partial_{m-|\sigma|+1} X_\sigma$.

There are no more boundary components. To see this we may consider a (simplicial) neighborhood $N(K)$ of $K \subset \partial\Delta^{m+1}$. Then $X_\sigma = \emptyset$ for all $\sigma \notin K$; in particular $X_\sigma = \emptyset$ if $\sigma \in \partial N(K)$.

Together with the colimit of the reference maps one has obtained a normal degree 1 map $(f, b) : M \to X$ with a reference map $\varphi : X \to N(K) \to K$, where $N(K) \to K$ is a retraction.

Remarks.

(I) The restriction of f to ∂M is a homotopy equivalence. Indeed, it is a δ-homotopy equivalence. This can be seen by applying the gluing construction (cf. e.g. [Hat, Ch. 4.G]) inductively to the family $\{f_\sigma| : \partial_{m-|\sigma|+1}M_\sigma \to \partial_{m-|\sigma|+1}X_\sigma | \sigma \in K\}$. Here δ can be chosen as small as necessary by taking a sufficiently fine subdivision.

(II) A careful construction shows that φ is transverse to all $D(\sigma, K)$, the duals of σ in K, and that

$$\varphi^{-1}(D(\sigma, K)) = X_\sigma.$$

Conversely, any simplicial map from a manifold $\varphi : X \to K$ is transversal to dual cells ([Coh]). Hence any normal degree 1 map $M \to X$ defines a class $[x] \in H_n(K, \mathbb{L})$.

(III) If K is not a finite complex, then one can choose for any $[x] \in H_n(K, \mathbb{L})$ a finite complex $J \subset K$, so that $[x]$ belongs to the image of $H_n(J, \mathbb{L}) \to H_n(K, \mathbb{L})$.

(IV) Instead of Proposition 6.6 from [LaMc], one can invoke the gluing procedure applied in [Nic, Ch. 3]. However, the notion of degree decreasing functors from [LaMc] is more appropriate here.

4. Relative $\mathbb{L}$-homology classes

Here we consider ordered simplicial complexes B, K and a simplicial map $p : K \to B$. We assume B, K to be finite complexes, and as in §3 let $B \subset \partial\Delta^{m+1}$ be an order preserving embedding. By the simplicial mapping cylinder construction we substitute p by an inclusion $K \subset B$. An element $[y] \in H_{n+1}(B, K, \mathbb{L})$ is given by a semi-simplicial map

$$y : (\overline{K}, \overline{B}) \to (\mathbb{L}_{n+1-m}, \emptyset),$$

i.e. by a family $\{y(\tau) \in \mathbb{L}_{n+1-m}(m-|\tau|)\}$.

Each $y(\tau)$ is a surgery problem $(f_\tau, b_\tau) : W_\tau \to V_\tau$ of manifold $(m-|\tau|+3)$-ads with $dim V_\tau = dim W_\tau = n+1-|\tau|$, such that f_τ restricts to a homotopy equivalence

$$\partial_{m-|\tau|+1}W_\tau \to \partial_{m-|\tau|+1}V_\tau.$$

It is convenient to give the family $\{y(\tau)\}$ a "cobordism" interpretation: y can be written as a semi-simplicial map:

$$y : (\Delta^1 \times \Sigma^m, \Delta^1 \times \overline{B} \cup \partial_0\Delta^1 \times \overline{K} \cup \partial_1\Delta^1 \times \Sigma^m) \to (\mathbb{L}_{n-m}, \emptyset)$$

(cf. [Ran, p. 128]).

The connecting homomorphism $\partial_* : H_{n+1}(B, K, \mathbb{L}) \to H_n(K, \mathbb{L})$ sends $[y]$ to $\partial_*[y] = [x]$, where

$$x : (\partial_0\Delta^1 \times \Sigma^m, \partial_0\Delta^1 \times \overline{K}) \to (\mathbb{L}_{n-m}, \emptyset)$$

is the restriction of y.

If we write $\Delta^1 \times \sigma$ for $\Delta^1 \times D(\sigma, \partial\Delta^{m+1}) \subset \Delta^1 \times \Sigma^m$, considered as $(m-|\sigma|+1) - ball$, we have $y(\Delta^1 \times \sigma) \in \mathbb{L}_{n-m}(m-|\sigma|+1)$; i.e. it is a surgery problem

$$(f_{\Delta^1 \times \sigma}, b_{\Delta^1 \times \sigma}) : W_{\Delta^1 \times \sigma}^{n+1-|\sigma|} \to V_{\Delta^1 \times \sigma}^{n+1-|\sigma|}$$

between $(m - |\sigma| + 5)$-ads, such that $W_{\Delta^1 \times \sigma}$, $V_{\Delta^1 \times \sigma}$ are $(n + 1 - |\sigma|)$-dimensional manifolds and $f_{\Delta^1 \times \sigma}$ restricts to a homotopy equivalence

$$\partial_{m-|\sigma|+3} W_{\Delta^1 \times \sigma} \to \partial_{m-|\sigma|+3} V_{\Delta^1 \times \sigma}.$$

Note that $\partial_0 V_{\Delta^1 \times \sigma} = V_{\partial_0 \Delta^1 \times \sigma}$, $\partial_0 W_{\Delta^1 \times \sigma} = W_{\partial_0 \Delta^1 \times \sigma}$. Furthermore, we have

(I) $V_{\Delta^1 \times \sigma} \neq \emptyset \Rightarrow \sigma \in B$;

(II) $\partial_0 V_{\Delta^1 \times \sigma} \neq \emptyset \Rightarrow \sigma \in K$; and

(III) $\partial_1 V_{\Delta^1 \times \sigma} = V_{\partial_1 \Delta^1 \times \sigma} = \emptyset$ for all $\sigma \in \partial \Delta^{m+1}$.

Hence $\{W_{\Delta^1 \times \sigma} \to V_{\Delta^1 \times \sigma}\}$ can be considered as the adic-surgery problem bounding the adic-surgery problem $\{\partial_0 W_{\Delta^1 \times \sigma} \to \partial_0 V_{\Delta^1 \times \sigma}\}$. The latter represents $\partial_*[y] = [x] \in H_n(K, \mathbb{L})$.

Let $(f, b) : W \to V$ be the colimit understood as in §3. From Proposition 6.6 in [LaMc] we obtain a degree 1 normal map (f, b) between $(n + 1)$-manifolds with boundaries

$$\partial V = \partial_0 V \cup \partial' V, \; \partial W = \partial_0 W \cup \partial' W,$$

where $\partial_0 V$ is the colimit of x, $\partial' V$ is the colimit of the restrictions to all $\partial_{m-|\sigma|+3} V_{\Delta^1 \times \sigma}$, and similarly for ∂W. Clearly, f restricted to $\partial' W$ is a homotopy equivalence. The colimit of the reference maps gives a reference map $(V, \partial_0 V) \xrightarrow{S} (B, K)$.

We now describe the map $p_* : H_n(K, \mathbb{L}) \to H_n(B, \mathbb{L})$. Suppose that $[x]$ is represented by the normal degree 1 map $(f, b) : M \to X$ with the reference map $S : X \to K$. Then the family $M_\sigma \xrightarrow{f_\sigma, b_\sigma} X_\sigma$, $\sigma \in K$, with $X_\sigma = S^{-1}(D(\sigma, K))$, $M_\sigma = f^{-1}(X_\sigma)$, $f_\sigma = f|_{M_\sigma}$, defines a cycle x. Note that $p_*[x]$ is represented by the family $\widehat{M}_\tau \xrightarrow{\widehat{f}_\tau, \widehat{b}_\tau} \widehat{X}_\tau$, $\tau \in B$, with $\widehat{X}_\tau = \widehat{S}^{-1}(D(\tau, B))$, $\widehat{M}_\tau = f^{-1}(\widehat{X}_\tau)$, $\widehat{f}_\tau = f|_{\widehat{M}_\tau}$, where we have put $\widehat{S} = S \circ p$.

We also have to understand in this context the meaning of "homologies" between cycles x, x' on K, i.e. $[x] = [x'] \in H_n(K, \mathbb{L})$. There is a semi-simplicial map

$$y : (\Delta^1 \times \Sigma^m, \Delta^1 \times \overline{K}) \to (\mathbb{L}_{n-m}, \emptyset)$$

such that $y(\partial_0 \Delta^1 \times \sigma) = x(\sigma)$, $y(\partial_1 \Delta^1 \times \sigma) = x'(\sigma)$, i.e.

$$y(\Delta^1 \times \sigma) \in \mathbb{L}_{n-m}(m - |\sigma| + 1),$$

defining surgery problems

$$(F_{\Delta^1 \times \sigma}, B_{\Delta^1 \times \sigma}) : W_{\Delta^1 \times \sigma} \to V_{\Delta^1 \times \sigma}$$

of $(m - |\sigma| + 5)$-ads as above, such that it restricts to

$$(f_\sigma, b_\sigma) : \partial_0 W_{\Delta^1 \times \sigma} = M_\sigma \to \partial_0 V_{\Delta^1 \times \sigma} = X_\sigma$$

and

$$f'_\sigma, b'_\sigma : \partial_1 W_{\Delta^1 \times \sigma} = M'_\sigma \to \partial_1 V_{\Delta^1 \times \sigma} = X'_\sigma$$

being $x(\sigma)$ and $x'(\sigma)$, respectively. Moreover, $F_{\Delta^1 \times \sigma}$ restricts to a homotopy equivalence

$$\partial_{m-|\sigma|+3} W_{\Delta^1 \times \sigma} \to \partial_{m-|\sigma|+3} V_{\Delta^1 \times \sigma}.$$

The colimit construction gives a normal cobordism $(F, B) : W^{n+1} \to V^{n+1}$ such that $\partial W = \partial_0 W \cup \partial_1 W \cup \partial' W \to \partial_0 V \cup \partial_1 V \cup \partial' V$ with $\partial' W \to \partial' V$ as a controlled homotopy equivalence extending the homotopy equivalences $\partial M \to \partial X$ and $\partial M' \to \partial X'$. Moreover, $\partial_0 W \to \partial_0 V$ and $\partial_1 W \to \partial_1 V$ represent x (resp. x') by normal degree 1 maps.

We shall now consider the case of a closed topological n-manifold K. By the Poincaré duality (cf. [Ran]), we have

$$H_n(K, \mathbb{L}) = H^0(K, \mathbb{L}) = [K, \mathbb{Z} \times G/TOP].$$

The $\mathbb{Z}$-sector is related to Quinn's index invariant. The simply connected cover $\mathbb{L}^+ \to \mathbb{L}$ is characterized by $\mathbb{L}_0^+ = G/TOP$, and induces an injection $[K, G/TOP] \to [K, \mathbb{Z} \times G/TOP]$ onto the 1-sector. By duality it is the image of $H_n(K, \mathbb{L}^+) \to H_n(K, \mathbb{L})$ (cf. [Qu2], [Ran, §25], and [BFMW] for more details).

Given an element $[x] \in H_n(K, \mathbb{L}^+)$ it defines a normal degree 1 map between closed manifolds

$$(g, c) : Z^n \to K^n \text{ giving a cycle representation}$$

$$\{z_\sigma\} = \{g_\sigma, c_\sigma\} : Z_\sigma \to K_\sigma = D(\sigma, K) \subset D(\sigma, \partial \Delta^{m+1}).$$

Obviously, $\partial_{m-|\sigma|+1} K_\sigma = \emptyset$, and the colimit of the family $\{K_\sigma | \sigma \in K\}$ gives K. The cycle $\{z_\sigma\} = z$ represents $[z] = [x] \in H_n(K, \mathbb{L}^+) \subset H_n(K, \mathbb{L})$. If $(f, b) : M^n \to X^n$ is the normal degree 1 map obtained from the colimit of the cycle x, there is a normal cobordism $C^{n+1} \to Y^{n+1}$ with $\partial_0 C^{n+1} \to \partial_0 Y^{n+1}$ equal to (f, b) and $\partial_1 C^{n+1} \to \partial_1 Y^{n+1}$ equal to (g, c). $\partial_0 Y \cup \partial_1 Y \subset \partial Y$, $\partial_0 C \cup \partial_1 C \subset \partial C$, and $\partial Y \setminus (\partial_0 Y \cup \partial_1 Y)$ have boundaries $\partial' X = \partial X$ defined above. Similarly, $\partial C \setminus (\partial_0 C \cup \partial_1 C)$ have boundaries $\partial' M = \partial M$.

We define $H_{n+1}^+(B, K, \mathbb{L}) \subset H_{n+1}(B, K, \mathbb{L})$ as the inverse image of $H_n(K, \mathbb{L}^+) \subset H_n(K, \mathbb{L})$ under $\partial_* : H_{n+1}(B, K, \mathbb{L}) \to H_n(K, \mathbb{L})$. Obviously, $H_{n+1}(B, K, \mathbb{L}^+) \subset H_{n+1}^+(B, K, \mathbb{L})$.

Finally, let a class $[y] \in H_{n+1}^+(B, K, \mathbb{L})$ be given with boundary cycle x, i.e. $[x] = \partial_*[y]$. Let

$$W \to V, \partial_0 W = M \xrightarrow{(f,b)} \partial_0 V = X$$

be the colimits of y and x. We glue $C \to Y$ to $W \to V$ along $M \xrightarrow{(f,b)} X$ and obtain a normal degree 1 map:

$$(F^+, B^+) : W^+ \to V^+ \text{ with restriction}$$

$$(g, c) : \partial_0 W^+ = Z \to \partial_0 V^+ = K, \partial' W^+ = \partial' W, \partial' V^+ = \partial' V,$$

$$(F^+, B^+) \text{ defines an element } [y^+] \in H_{n+1}^+(B, K, \mathbb{L}).$$

By the normal cobordism extension property (cf. [Med, p. 45] or [Wal, p. 93] applied in the proof of Theorem 9.6), $W^+ \to V^+$ and $W \to V$ are normally cobordant by a cobordism being a product outside a small neighborhood of $C \to Y$.

In summary, we got for any closed oriented n-manifold $p : K \to B$ a continuous map into a simplicial complex and any $[y] \in H_{n+1}^+(B, K, \mathbb{L})$ a cycle representation y with colimit a degree 1 normal map

$$(f, b) : W^{n+1} \to V^{n+1}$$

between $(n+1)$-dimensional manifolds such that $K = \partial_0 V \subset \partial V$, f restricted to $\partial_0 W = M \to K$ is the colimit of a cycle of $\partial_*[y] \in H_n(K, \mathbb{L})$. Moreover, f restricted to $\partial W \setminus \partial_0 W \to \partial V \setminus \partial_0 V$ is a homotopy equivalence. It is a δ-homotopy equivalence with respect to the reference map $\partial V \setminus \partial_0 V \to B$.

A special case is given when $[y] \in H_{n+1}^+(B, K, \mathbb{L})$ is in the image of $i_* : H_{n+1}(B, \mathbb{L}) \to H_{n+1}^+(B, K, \mathbb{L})$, i.e. $i_*([z]) = [y]$. In this case one gets a normal degree 1 map $W \to V$ with $\partial_0 W \to \partial_0 V = \emptyset$ and $\partial W = \partial' W \to \partial' V = \partial V$ a controlled homotopy equivalence.

For our later use we substitute $W \to V$ by adding $C \to Y$ representing the trivial element in $H_{n+1}^+(B, K, \mathbb{L})$. We can take $C = K \times I = Y$, with $\partial_0 C = K \times 0 = \partial_0 Y$, $\partial_1 C = \partial_1 Y = \emptyset$ and $\partial' C = K \times 1 = \partial' Y$. We shall later refer to this case as the "absolute case" and write it as

$$W \dot{\cup} C \to V \dot{\cup} Y.$$

Moreover, we shall ignore the boundaries $\partial' W$, $\partial' V$, $\partial' C$ and $\partial' Y$.

5. The map $a : H_{n+1}^+(B, K, \mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(K)$

We assume K to be a closed oriented n-manifold, $p : K \to B$ a UV^1-map, and B a compact finite-dimensional metric ANR. To define the map a it suffices to take B a finite ordered simplicial complex contained in $\partial \Delta^{m+1}$ as before.

We must first introduce the controlled structure set. Its definition comes with the proof of the controlled surgery sequence of [PQR] (cf. also [Fer]):

$$H_{n+1}(B, \mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(K \xrightarrow{p} B) \to [K, G_{/TOP}] \to H_n(B, \mathbb{L}).$$

The precise statement is: There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there is $\delta > 0$ such that the sequence is exact provided p is a UV^1-map ($UV^1(\delta)$ would be sufficient). With these ε and δ, the controlled structure set $\mathcal{S}_{\varepsilon,\delta}(K \to B)$ can be defined: Elements in $\mathcal{S}_{\varepsilon,\delta}(K \to B)$ are represented by pairs (M, f), where M is a closed n-manifold and $f : M \to K$ is a δ-homotopy equivalence with respect to the control map $p : K \to B$. Two pairs (M_1, f_1) and (M_2, f_2) are "ε-related" if there is a homeomorphism $h : M_1 \to M_2$ such that $f_2 \circ h$ and f_1 are ε-homotopic over $p : K \to B$ (i.e. the homotopy has "tracks" of size $< \varepsilon$ in B). Now, with this ε and f, "ε-related" is an equivalence relation. We shall shortly write $\mathcal{S}_{\varepsilon,\delta}(K \xrightarrow{p} B) = \mathcal{S}_{\varepsilon,\delta}(K)$.

Further remarks. In [PQR], $\mathbb{L}$ denotes the algebraic $\mathbb{L}$-spectrum defined by adic quadratic Poincaré complexes (see [Ran]). To a normal degree 1-map $(f, b) : M \to K$ there is an obstruction in a controlled Wall group $L_n(B, \varepsilon, \delta)$. It consists of quadratic n-dimensional Poincaré complexes of radius $\delta > 0$ over B, modulo ε-cobordism. If this obstruction vanishes, controlled surgery can be completed in the middle dimension. The controlled Hurewicz and Whitehead theorems give an element in $\mathcal{S}_{\varepsilon,\delta}(K)$. All this involves ε-δ-estimates. There is an assembly map $A : H_n(B, \mathbb{L}) \to L_n(B, \varepsilon, \delta)$ defined by gluing the adic parts of an element in $H_n(B, \mathbb{L})$, similarly to how it was done in §3. This map is proved to be an isomorphism for suitable ε, δ. It is here, among other places, where the "stability threshold" $\varepsilon_0 > 0$ drops in (see also [RanYa]).

Let $[y] \in H_{n+1}^+(B, K, \mathbb{L})$ be given, and let y be a cycle with $y(\sigma)$, given by normal degree 1 $(m - |\sigma| + 5)$-ads $(f_\sigma, b_\sigma) : W_\sigma \to V_\sigma$. Let $(f, b) : W \to V$ be the colimit obtained in §4; i.e. f restricts to a normal degree 1 map between closed manifolds $\partial_0 W \to K$, and to a δ'-homotopy equivalence $\partial' W \to \partial' V$. We claim that δ' can be chosen smaller than δ. First note that the above $\varepsilon_0 > 0$ depends on B (and on n), and δ' becomes smaller and smaller if we subdivide B finer and finer.

Let $\mathcal{S} : V \to B$ be the reference map. Restricting $\mathcal{S}$ to $\partial_0 V = K$ gives $p : K \to B$, and by assumption it is UV^1.

We can assume that $S : V \to B$ is also UV^1: First, change $W \to V$ as in the proof of Theorem 9.4 in [Wal] to obtain an isomorphism $S_* : \pi_1(V) \to \pi_1(B)$.

Then since $\dim V \geq 5$ one can approximate S (rel. ∂V) to become a UV^1-map (cf. [BFMW, Theorem 4.4]).

We can invoke the controlled simply connected $\pi - \pi$-theorem (cf. e.g. [Fer]): We have a δ'-equivalence $\partial_0 W \to \partial_0 V = K$, and we can assume that $\partial' W \to \partial' V$ is a δ'-equivalence too, and δ' can be as small as we need it to be. By the $\pi - \pi$-theorem there are $\delta_0 > 0$, $k > 0$, such that for $\delta' < \delta_0$ there is a normal cobordism between $(W, \partial_0 W) \to (V, \partial_0 V)$ relative $\partial' V$ to a $k\delta'$-homotopy equivalence

$$f' : (W', \partial_0 W') \to (V, \partial_0 V = K).$$

Choosing further $k\delta' < \delta$, we get a class in $\mathcal{S}_{\varepsilon, \delta}(K)$ represented by

$$f'_0 : \partial_0 W' \to K \ , \ f'_0 = f'|_{\partial_0 W'}.$$

We shall define $a : H^+_{n+1}(B, K, \mathbb{L}) \to \mathcal{S}_{\varepsilon, \delta}(K)$ by $a([y]) = [f'_0, \partial_0 W']$.

It remains to show that a is well-defined. Suppose y, y_1 are cycles with $[y_1] = [y]$; let

$$(W, \partial_0 W) \xrightarrow{(f, b)} (V, \partial_0 V = K)$$
$$(W_1, \partial_0 W_1) \xrightarrow{(f_1, b_1)} (V_1, \partial_0 V_1 = K)$$

be colimits and let

$$(W', \partial_0 W') \xrightarrow{f'} (V, \partial_0 V = K)$$
$$(W'_1, \partial_0 W'_1) \xrightarrow{f'_1} (V_1, \partial_0 V_1 = K)$$

be $k\delta'$-homotopy (resp. $k\delta'_1$-homotopy) equivalences obtained from the controlled $\pi - \pi$-theorem. We may assume $k\delta' < \delta$, $k\delta'_1 < \delta$.

Let $Q \to \Omega$ be a normal cobordism between (f, b) and (f_1, b_1). It restricts to a normal cobordism $P \to K \times I$ with $\partial P = M \cup M_1 \xrightarrow{f \cup f_1} K \times \{0, 1\}$, and with boundary $\partial Q = W \cup P \cup W_1 \to \partial \Omega = V \cup K \times I \cup V_1$.

To proceed we briefly have to recall the proof of the $\pi - \pi$-theorem: The first step is to make $(W, M = \partial_0 W) \to (V, \partial_0 V = K)$ highly connected, which we can assume.

The second step consists of handle subtraction in the pairs (W, M) and (W_1, M_1), i.e. $(W', \partial_0 W')$ is obtained from W by subtracting handles. We add these handles to P along M. Similarly, we add the handles subtracted from (W_1, M_1) along M_1 to P_1. One obtains P' with $\partial P' = M' \cup M'_1$. In other words, we obtain a new decomposition of ∂Q and $\partial \Omega$ and $\partial Q \to \partial \Omega$ as

$$\partial Q = W' \cup P' \cup W'_1 \to \partial \Omega = V \cup K \times I \cup V_1.$$

Hence $Q \to \Omega$ displays a normal bordism between

$$(P', \partial P' = M' \cup M'_1) \to (K \times I, K \times \{0\} \cup K \times \{1\})$$

and the controlled homotopy equivalence

$$(W', M') \cup (W'_1, M'_1) \to (V, K) \cup (V_1, K).$$

We denote by $\phi' : P' \to K \times I$ the first map. Then ϕ' restricted to M', M'_1 is a $k\delta'$ (resp. $k\delta'_1$) equivalence, with $k\delta'$, $k\delta'_1 < \delta$.

Let us assume $\delta'_1 = \delta'$. Controlled surgery theory then implies (cf. [PQR], [Qu1], [Fer]) that ϕ' can be surgered to a simple $\overline{k} k_1 \delta'$-homotopy equivalence

$$H : (P'', M', M'_1) \to (K \times I, K \times \{0\}, K \times \{1\})$$

(for some $\overline{k} > 0$) with $H|_{M'} = f'$, $H|_{M'_1} = f'_1$. In particular P'' is an s-cobordism.

Let $G : P'' \to M' \times [0,1]$ be a homeomorphism with

$$G|_{M'} = Id \, , \, g = G|_{M'_1} : M'_1 \to M.$$

The composition

$$M'_1 \times I \xrightarrow{g \times Id} M' \times I \xrightarrow{G^{-1}} P'' \xrightarrow{H} K \times I$$

restricts to $f' \circ g : M'_1 \times 0 \to K \times 0$ and $f'_1 : M'_1 \times \{1\} \to K \times \{1\}$. Therefore $f' \circ g$ and f'_1 are $\overline{k}k\delta'$-homotopic over B.

However, we can choose δ' such that $\overline{k}k\delta' < \varepsilon$, showing $f' : M' \to K$, $f'_1 : M'_1 \to K$ define the same element in $\mathcal{S}_{\varepsilon,\delta}(K)$; hence the map a is well-defined.

This finishes the generic case, and we shall now come to the absolute case. Now let $[y] \in H^+_{n+1}(B, K, \mathbb{L})$ be the image of $[z] \in H^+_{n+1}(B, \mathbb{L})$, i.e. $i_*[z] = [y]$. Then we get a normal degree 1 map $W \dot\cup C \to V \dot\cup Y$ such that $\partial_0 V = \emptyset$, $\partial_0 W = \emptyset$, $C \to Y$ is a controlled homotopy equivalence with $\partial C = K \to \partial Y = K$ a homeomorphism. The $\pi - \pi$-theorem cannot be applied to $(W, \partial_0 W)$ since $\partial_0 W = \emptyset$. We define $a([y])$ as the result of the controlled Wall action of $[z]$ on $Id : K \to K$; i.e. $a[y]$ is the image of $[z]$ under

$$H_{n+1}(B, \mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(K)$$

(cf. [PQR]). We have to show that the map a is well-defined. It suffices to consider the following two types of cycle representations of $i_*([z]) = [y]$:

(I) $W \dot\cup C \to V \dot\cup Y$ is an absolute representation;

(II) $W_1 \to V_1$ is a generic representation;

with $\partial_0 W_1 \to \partial_0 V_1 \neq \emptyset$.

Let $Q \to \Omega$ be a normal cobordism between them. Then $\partial Q = W \dot\cup C \underset{K}{\cup} P \cup W_1 \to \partial\Omega = V \dot\cup Y \underset{K}{\cup} K \times I \cup V_1$, where $(P, \partial_0 P = K, \partial_1 P = \partial_0 W_1) \to (K \times I, K \times 0, K \times 1)$.

The restriction to $W_2 = C \cup P \cup W_1 \to V_2 = Y \cup K \times I \cup V_1$ is a normal degree 1 map with controlled homotopy equivalence on the boundary. It defines the same element $i_*[z] = [y] \in H^+_{n+1}(B, K, \mathbb{L})$. We may also cancel $C \to Y$ thus obtaining

$$W_3 = P \cup W_1 \to V_3 \cong V_1, \text{ and identity on}$$

$$\partial_0 W_3 = K \to \partial_0 V_3 = K.$$

It still defines the same element $[y]$. We can apply the $\pi - \pi$-theorem to $(W_3, \partial_0 W_3 = K) \to (V_3, \partial_0 V_3 = K)$ and obtain a controlled homotopy equivalence

$$(W'_3, \partial_0 W'_3) \to (V_3, \partial_0 V_3 = K).$$

We have to show that the δ-homotopy equivalence $\partial_0 W'_3 \to K$ defines the same element as the image of

$$[z] \in H_{n+1}(B, \mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(K).$$

First, one makes $W_3 \to V_3$ highly connected by surgeries in the interior of W_3. We shall assume this for $W_3 \to V_3$. Assume $n + 1 = 2k$. The middle-dimensional classes defining the surgery kernel can be represented by framed immersed $(k+1)$-handles

$$(H', \partial H') \to (W_3, K).$$

Let $U = K \times [0, \rho] \cup H' \subset W_3$, where $K \times [0, \rho]$ denotes a small collar of K. Note that U can be identified with the controlled Wall action of $[z]$ on K. On the other hand one can remove the intersection of the immersed handles across K (here $\pi_1(K) \underset{\cong}{\to} \pi_1 W_3$ is used) to get embedded handles $(H, \partial H) \subset (W_3, K)$.

We may assume that $U = K \times [o, \rho] \cup H$. Then $(W_3', \partial_0 W_3') = (W_3 \setminus \overset{\circ}{U}, \partial_1 U) \to$ (V_3, K) is the result from the $\pi - \pi$-theorem. This shows that both definitions of the map a on $i_*[z] = [y]$ give the same result.

The case when $n + 1$ is odd goes as follows: One has to consider neighbourhoods U of K in W_3 given by attached embedded handles of the type $S^k \times D^{k+1}$, $2k+1 = n + 1$. They have to be subtracted. On the other hand, in order to realize an element of $H_{n+1}(B, \mathbb{L})$ one attaches trivial handles and then applies the controlled obstruction to the transverse spheres $1 \times S^k$. On this result one then performs the surgeries. The resulting homotopy equivalence is the same.

6. Proof of Theorem 3

The map b was defined in [DrFe] as the composite map

$$H_{n+1}(B, K, \mathbb{L}) \cong S_{n+1}(B, K, \mathbb{L}) \xrightarrow{\partial_*} S_n(K, \mathbb{L}) \to \mathcal{S}(K).$$

Here, $S_{n+1}(B, K, \mathbb{L})$, $S_n(K, \mathbb{L})$ are the structure sets defined in [Ran] (however, denoted differently as $S_{n+2}(B, K)$, $S_{n+1}(K)$).

An element in $S_n(K, \mathbb{L})$ is represented by a homotopy equivalence of n-manifolds $P^n \to Q^n$ with the reference map $Q^n \to K$. There is a natural inclusion $\mathcal{S}(K) \subset S_n(K, \mathbb{L})$. The first isomorphism is due to the $\pi - \pi$-theorem. The map $S_n(K, \mathbb{L}) \to \mathcal{S}(K)$ is such that the composite $\mathcal{S}(K) \subset S_n(K, \mathbb{L}) \to \mathcal{S}(K)$ is the identity.

Let $[y] \in H^+_{n+1}(B, K, \mathbb{L})$. Our construction of $a[y]$ produces a homotopy equivalence $(W', \partial_0 W') \to (V, \partial_0 V = K)$ with the reference map $(V, \partial_0 V = K) \to (B, K)$.

This is the image of $[y]$ in $S_{n+1}(B, K, \mathbb{L})$; hence $\partial_0 W' \to \partial_0 V$ is the image in $S_n(K, \mathbb{L})$. However, it belongs already to $\mathcal{S}(K) \subset S_n(K, \mathbb{L})$, and $\mathcal{S}(K) \subset S_n(K, \mathbb{L}) \to \mathcal{S}(K)$ is the identity. This shows that the diagram

$$
\begin{array}{ccccccc}
H_{n+1}(B, K, \mathbb{L}) & \cong & S_{n+1}(B, K, \mathbb{L}) & \to & S_n(K, \mathbb{L}) & \to & \mathcal{S}(K) \\
\cup & & & & & & \uparrow \\
H^+_{n+1}(B, K, \mathbb{L}) & & & \xrightarrow{\quad a \quad} & & & \mathcal{S}_{\varepsilon, \delta}(K \to B)
\end{array}
$$

commutes.

To complete the proof of Theorem 3 we have to verify the commutativity of the diagrams

$$
\text{(I)} \qquad
\begin{array}{ccc}
H_{n+1}(B, \mathbb{L}) & \longrightarrow & H^+_{n+1}(B, K, \mathbb{L}) \\
\| & & \downarrow a \\
H_{n+1}(B, \mathbb{L}) & \longrightarrow & \mathcal{S}_{\varepsilon, \delta}(K \to B)
\end{array}
$$

and

$$
\text{(II)} \qquad
\begin{array}{ccc}
H^+_{n+1}(B, K, \mathbb{L}) & \longrightarrow & H_n(K, \mathbb{L}^+) \\
\downarrow a & & \downarrow \cong \\
\mathcal{S}_{\varepsilon, \delta}(K \to B) & \longrightarrow & [K, G/TOP].
\end{array}
$$

Now, (I) follows from the definition. To prove commutativity of (II), we consider the big diagram

$$
\begin{array}{ccc}
S_{n+1}(B,K,\mathbb{L}) & \xrightarrow{\;\;\partial_*\;\;} & S_n(K,\mathbb{L}) \\
\cong\downarrow & & \downarrow \\
H_{n+1}(B,K,\mathbb{L}) & \xrightarrow{\;\;\partial_*\;\;} & H_n(K,\mathbb{L}) \\
\| & & \| \\
H_{n+1}(B,K,\mathbb{L}) \supset H_{n+1}^+(B,K,\mathbb{L}) \xrightarrow{\partial_*} H_n(K,\mathbb{L}^+) \longrightarrow & & H_n(K,\mathbb{L}) \\
b\downarrow \qquad\qquad a\downarrow \qquad\qquad \cong\downarrow & & \cong\downarrow \\
S(K) \longleftarrow S_{\varepsilon,\delta}(K) \longrightarrow [K,G_{/TOP}] \xleftarrow{pr} & & [K,Z\times G_{/TOP}] \\
\| \qquad\qquad\qquad\qquad \| & & \\
S(K) \xrightarrow{\quad\eta\quad} [K,G_{/TOP}] & &
\end{array}
$$

The "inner" diagram is (II). The upper 3 rows are just definitions of various ∂_* (see also [Ran], p. 207). Commutativity of the middle left diagram is explained in the above diagram. The middle right diagram commutes because of compatibility of $\mathbb{L}$ and $\mathbb{L}^+$-Poincaré duality. The lower diagram commutes; here η associates to a homotopy equivalence $f : M \to K$ its normal invariant, and pr denotes the projection map. The composed maps of the "outer" diagram give the diagram

$$
\begin{array}{ccc}
S_{n+1}(B,K,\mathbb{L}) & \xrightarrow{\;\partial_*\;} & S_n(K,\mathbb{L}) \\
\downarrow & & \downarrow \\
S(K) & \xrightarrow{\;\eta\;} & [K,G_{/TOP}].
\end{array}
$$

This commutes because of commutativity of

$$
\begin{array}{ccc}
H_n(K,\mathbb{L}) & \xrightarrow{\;\cong\;} & [K,G_{/TOP}] \\
\uparrow & & \downarrow{pr} \\
H_n(K,\mathbb{L}^+) & \xrightarrow{\;\cong\;} & [K,G_{/TOP}].
\end{array}
$$

Commutativity of (II) follows from commutativity of all these diagrams.

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