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s-Cobordism Classification of 4-Manifolds Through the Group of Homotopy Self-equivalences

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Abstract. The aim of this paper is to give an s-cobordism classification of topological 4-manifolds in terms of the standard invariants using the group of homotopy self-equivalences. Hambleton and Kreck constructed a braid to study the group of homotopy self-equivalences of 4-manifolds. Using this braid together with the modified surgery theory of Kreck, we give an s-cobordism classification for certain 4-manifolds with fundamental group π, such that $\operatorname{cd}\pi \leq 2$.

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Keywords. s-Cobordism, 4-Manifold, cohomological dimension and Homotopy self-equivalence.

1. Introduction

The cohomological dimension of a group G, denoted $\operatorname{cd} G$, is the projective dimension of $\mathbb{Z}$ over $\mathbb{Z}G$. In other words, it is the smallest non-negative integer n such that $\mathbb{Z}$ admits a projective resolution $P = (P_i)_{i\leq 0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ of length n, satisfying $P_i = 0$ for $i > n$. If there exists no such n, then we set $\operatorname{cd} G = \infty$.

In this paper, we are going to deal with groups whose cohomological dimension is less than or equal to 2. This class of groups contains the free groups, knot groups and one-relator groups whose relator is not a proper power. Our aim here is to give an s-cobordism classification of topological 4-manifolds with fundamental group π such that $\operatorname{cd}\pi \leq 2$, in terms of the standard invariants such as the fundamental group, characteristic classes and the equivariant intersection form using the group of homotopy self-equivalences.

Let M be a closed, connected, oriented, 4-manifold with a fixed base point $x_0 \in M$. Throughout the paper, the fundamental group $\pi_1(M, x_0)$ will be denoted by π and the higher homotopy groups $\pi_i(M, x_0)$ by π_i. Let

$\Lambda = \mathbb{Z}[\pi]$ denote the integral group ring of π. The standard involution $\lambda \to \overline{\lambda}$ on Λ is induced by the formula

$$\sum n_g g \to \sum n_g g^{-1}$$

for $n_g \in \mathbb{Z}$ and $g \in \pi$. All modules considered in this paper will be right Λ-modules.

The first step in the classification of manifolds is the determination of their homotopy type. It is a well-known result of Milnor [13] and Whitehead [20] that a simply connected 4-dimensional manifold M is classified up to homotopy equivalence by its integral intersection form. In the non-simply connected case, one has to work with the equivariant intersection form s_M where

$$s_M\colon H_2(M;\Lambda) \times H_2(M;\Lambda) \to \Lambda;\ (a,b) \to s_M(a,b) = a^*(b).$$

This is a Hermitian pairing where $a^* \in H^2(M;\Lambda)$ is the Poincaré dual of a, such that $s_M(a,b) = \overline{s_M(b,a)} \in \Lambda$. This form does not detect the homotopy type and the missing invariant is the first k-invariant $k_M \in H^3(\pi;\pi_2)$, see [8, Remark 4.5] for an example.

Hambleton and Kreck [8] defined the quadratic 2-type as the quadruple $[\pi, \pi_2, k_M, s_M]$ and the group of isometries of the quadratic 2-type of M, $\mathrm{Isom}[\pi, \pi_2, k_M, s_M]$, consists of all pairs of isomorphisms

$$\chi\colon \pi \to \pi \quad \text{and} \quad \psi\colon \pi_2 \to \pi_2,$$

such that $\psi(gx) = \chi(g)\psi(x)$ for all $g \in \pi$ and $x \in \pi_2$, which preserve the k-invariant, $\psi_*(\chi^{-1})^* k_M = k_M$, and the equivariant intersection form, $s_M(\psi(x), \psi(y)) = \chi_* s_M(x,y)$. It was shown in [8] that the quadratic 2-type detects the homotopy type of an oriented 4-manifold M if π is a finite group with 4-periodic cohomology.

Throughout this paper $H^3(\pi;\pi_2) = 0$, so we have $k_M = 0$. For notational ease we will drop it from the notation and write $\mathrm{Isom}[\pi, \pi_2, s_M]$ for the group of isometries of the quadratic 2-type.

Let $\mathrm{Aut}_\bullet(M)$ denote the group of homotopy classes of homotopy self-equivalences of M, preserving both the given orientation on M and the base-point $x_0 \in M$. To study $\mathrm{Aut}_\bullet(M)$, Hambleton and Kreck [10] established a commutative braid of exact sequences, valid for any closed, oriented smooth or topological 4-manifold. To give an s-cobordism classification we use the above-mentioned braid together with the modified surgery theory of Kreck [12].

In Sect. 2, we briefly review some background material about the modified surgery theory and some of the terms of the braid. Throughout this paper we always refer to [10] for the details of the definitions concerning the braid. In Sect. 3, we are going to further assume that the the following three conditions are satisfied:

(A1) The assembly map $A_4\colon H_4(K(\pi,1);\mathbb{L}_0(\mathbb{Z})) \to L_4(\mathbb{Z}[\pi])$ is injective, where $\mathbb{L}_0(\mathbb{Z})$ stands for the connective cover of the periodic surgery spectrum;

(A2) Whitehead group $Wh(\pi)$ is trivial for π; and

(A3) The surgery obstruction map $\mathcal{T}(M \times I, \partial) \to L_5(\mathbb{Z}[\pi])$ is onto, where M is a closed, connected, oriented 4-manifold with $\pi_1(M) \cong \pi$.

Note that if the Farrell–Jones conjecture [6] is true for torsion-free groups, then π satisfies all the conditions above.

Now let $u_M : M \to K(\pi, 1)$ be a classifying map for the fundamental group π. Consider the homotopy fibration

$$\widetilde{M} \xrightarrow{p} M \xrightarrow{u_M} K(\pi, 1)$$

which induces a short exact sequence

$$0 \longrightarrow H^2(K(\pi,1);\mathbb{Z}/2) \xrightarrow{u_M^*} H^2(M;\mathbb{Z}/2) \xrightarrow{p^*} H^2(\widetilde{M};\mathbb{Z}/2).$$

Next we recall the following definition given in [9].

Definition 1.1. We say that a manifold M has w_2-type (I), (II), or (III) if one of the following holds:

(I) $w_2(\widetilde{M}) \neq 0$;
(II) $w_2(M) = 0$; or
(III) $w_2(M) \neq 0$ and $w_2(\widetilde{M}) = 0$.

Using the braid constructed in [10] together with the modified surgery theory of Kreck [12], we show that for topological 4-manifolds which have w_2-type (I) or (II), with $\operatorname{cd}\pi \leq 2$ and satisfying (A1), (A2) and (A3), Kirby–Siebenmann (ks) invariant and the quadratic 2-type give the s-cobordism classification. Our main result is the following:

Theorem 1.2. *Let M_1 and M_2 be closed, connected, oriented, topological 4-manifolds with fundamental group π such that $\operatorname{cd}\pi \leq 2$ and satisfying properties (A1), (A2) and (A3). Suppose also that they have the same Kirby–Siebenmann invariant and w_2-type (I) or (II). Then M_1 and M_2 are s-cobordant if and only if they have isometric quadratic 2-types.*

Let us finish this introductory section by pointing out the differences of methods used in this paper and the paper by Hambleton et al. [11] which classifies closed orientable 4-manifolds with fundamental groups of geometric dimension 2 subject to the same hypotheses of this paper.

The geometric dimension of a group G, denoted by $\operatorname{gd} G$, is the minimal dimension of a CW model for the classifying space BG. Eilenberg and Ganea [5] showed that for any group G we have $\operatorname{gd} G = \operatorname{cd} G$ for $\operatorname{cd} G > 2$ and if $\operatorname{cd} G = 2$ then $\operatorname{gd} G \leq 3$. Later Stallings [16] and Swan [17] showed that $\operatorname{cd} G = 1$ if and only if $\operatorname{gd} G = 1$. It follows that $\operatorname{gd} G = \operatorname{cd} G$, except possibly that there may exist a group G for which $\operatorname{cd} G = 2$ and $\operatorname{gd} G = 3$. The statement that $\operatorname{cd} G$ and $\operatorname{gd} G$ are always equal has become known as the Eilenberg–Ganea conjecture (see [4] for more details and potential counterexamples to Eilenberg–Ganea conjecture).

Although the Eilenberg–Ganea conjecture is still open, Bestvina and Brady [2] showed that at least one of the Eilenberg–Ganea and Whitehead conjectures has a negative answer, i.e., either there exists a group of cohomological dimension and geometric dimension a counterexample to the

Eilenberg–Ganea Conjecture or there exists a nonaspherical subcomplex of an aspherical complex a counterexample to the Whitehead Conjecture [19].

Therefore, our main result might be a slight generalization of Theorem C of [11]. Also our line of argument is different: we first work with the bordism group over the normal 1-type and then to use the braid constructed in [10], we work with the normal 2-type and the w_2-type, whereas in [11], the authors work with the *reduced* normal 2-type and a *refinement* of the w_2-type.

2. Background

The classical surgery theory, developed by Browder, Novikov, Sullivan and Wall in the 1960s, is a technique for classifying of high-dimensional manifolds. The theory starts with a normal cobordism $(F, f_1, f_2)\colon (W, N_1, N_2) \to X$ where f_1 and f_2 are homotopy equivalences, and then asks whether this cobordism is cobordant rel ∂ to an s-cobordism. There is an obstruction in a group $L_{n+1}(\mathbb{Z}[\pi_1(X)])$ which vanishes if and only if this is possible. Later in the 1980s Matthias Kreck [12] generalized this approach:

Definition 2.1. *([12])* Let $\xi\colon E \to \mathrm{BSO}$ be a fibration.

- (i) A normal (E, ξ) structure $\bar{\nu}\colon N \to E$ of an oriented manifold N in E is a normal k-smoothing, if it is a $(k+1)$-equivalence.
- (ii) We say that E is k-universal if the fibre of the map $E \to \mathrm{BSO}$ is connected and its homotopy groups vanish in dimension $\geq k+1$.

For each oriented manifold N, up to fibre homotopy equivalence, there is a unique k-universal fibration E over BSO admitting a normal k-smoothing of N. Thus, the fibre homotopy type of the fibration E over BSO is an invariant of the manifold N and we call it the *normal k-type* of N.

Instead of homotopy equivalences, Kreck started with cobordisms of normal smoothings $(F, f_1, f_2)\colon (W, N_1, N_2) \to X$ where f_1 and f_2 are only $[\frac{n+1}{2}]$-equivalences. There is an obstruction in a monoid $l_{n+1}(\mathbb{Z}[\pi_1(X)])$ which is elementary if and only if that cobordism is cobordant rel ∂ to an s-cobordism.

Let M be a closed oriented 4-manifold. We work with the normal 2-type of 4-manifolds. That is we need to construct a fibration $E \to \mathrm{BSO}$ whose finer has vanishing homotopy in dimensions $\geq k$ and there exists a 3-equivalence $M \to E$. Let B denote the 2-type of M (second stage of the Postnikov tower for M), i.e., there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "c"] \arrow[d, "u_M"'] & B \arrow[d, "u_B"] \\ B\pi \arrow[r, equal] & B\pi \end{tikzcd}$$

Here, u_M is unique up to homotopy and a classifying map for the universal covering $\widetilde{M}$ of M. We can attach cells of dimension ≥ 4 to obtain a CW-complex structure for B with the following properties:

- (i) The inclusion map $c : M \to B$ induces isomorphisms $\pi_k(M) \to \pi_k(B)$ for $k \leq 2$, and
- (ii) $\pi_k(B) = 0$ for $k \geq 3$.

Note that the universal covering space $\widetilde{B}$ of B is the Eilenberg–MacLane space $K(\pi_2, 2)$, and the inclusion $\widetilde{M} \to \widetilde{B}$ induces isomorphism on π_2.

The class $w_2 := w_2(M) \in H^2(M;\mathbb{Z}) \cong H^2(B;\mathbb{Z})$ gives a fibration and we can form the pullback

$$\begin{tikzcd} \mathrm{BSpin} \arrow[r] & B\langle w_2\rangle \arrow[r, "j"] \arrow[d, "\xi"'] & B \arrow[d, "w_2"] \\ \mathrm{BSpin} \arrow[r] & \mathrm{BSO} \arrow[r, "w"] & K(\mathbb{Z}/2, 2) \end{tikzcd}$$

where w pulls back the second Stiefel–Whitney class for the universal oriented vector bundle over BSO. Note that the fibration $B\langle w_2\rangle$ over BSO is the normal 2-type of M and if $w_2 = 0$, then $B\langle w_2\rangle = B \times \mathrm{BSpin}$.

We have a similar pullback diagram for M. Hambleton and Kreck [10] defined a thickening $\mathrm{Aut}_\bullet(M, w_2)$ of $\mathrm{Aut}_\bullet(M)$ and then established a commutative braid of exact sequences, valid for any closed, oriented smooth or topological 4-manifold.

Definition 2.2. ([10]) Let $\mathrm{Aut}_\bullet(M, w_2)$ denote the set of equivalence classes of maps $\widehat{f}\colon M \to M\langle w_2\rangle$ such that (i) $f := j \circ \widehat{f}$ is a base-point and orientation preserving homotopy equivalence, and (ii) $\xi \circ \widehat{f} = \nu_M$.

Given two maps $\widehat{f}, \widehat{g}\colon M \to M\langle w_2\rangle$ as above, we define

$$\widehat{f} \bullet \widehat{g}\colon M \to M\langle w_2\rangle$$

as the unique map from M into the pull-back $M\langle w_2\rangle$ defined by the pair $f \circ g\colon M \to M$ and $\nu_M\colon M \to \mathrm{BSO}$. It was proved in [10] that $\mathrm{Aut}_\bullet(M, w_2)$ is a group under this operation and there is a short exact sequence of groups

$$0 \longrightarrow H^1(M;\mathbb{Z}/2) \longrightarrow \mathrm{Aut}_\bullet(M, w_2) \longrightarrow \mathrm{Aut}_\bullet(M) \longrightarrow 1.$$

To define an analogous group $\mathrm{Aut}_\bullet(B, w_2)$ of self-equivalences, we must first state the following lemma.

Lemma 2.3. ([10]) *Given a base-point preserving map $f\colon M \to B$, there is a unique extension (up to base-point preserving homotopy) $\phi_f\colon B \to B$ such that $\phi_f \circ c = f$. If f is a 3-equivalence then ϕ_f is a homotopy equivalence. Moreover, if $w_2 \circ f = w_2$, then $w_2 \circ \phi_f = w_2$.*

Definition 2.4. ([10]) Let $\mathrm{Aut}_\bullet(B, w_2)$ denote the set of equivalence classes of maps $\widehat{f}\colon M \to B\langle w_2\rangle$ such that (i) $f := j \circ \widehat{f}$ is a base-point preserving 3-equivalence, and (ii) $\xi \circ \widehat{f} = \nu_M$.

Theorem 2.5 ([10]). *Let M be a closed, oriented 4-manifold. Then there is a sign-commutative diagram of exact sequences*

$$\begin{tikzcd}
\Omega_5(M\langle w_2\rangle) \arrow[dr] \arrow[rr, bend left] & & \widetilde{\mathcal{H}}(M,w_2) \arrow[dr] \arrow[rr, bend left] & & \mathrm{Aut}_\bullet(B,w_2) \arrow[dr, "\beta"] & \\
& \Omega_5(B\langle w_2\rangle) \arrow[ur] \arrow[dr] & & \mathrm{Aut}_\bullet(M,w_2) \arrow[ur] \arrow[dr, "\alpha"] & & \Omega_4(B\langle w_2\rangle) \\
\pi_1(\mathcal{E}_\bullet(B,w_2)) \arrow[ur] \arrow[rr, bend right] & & \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) \arrow[ur, "\gamma"] \arrow[rr, bend right] & & \widehat{\Omega}_4(M\langle w_2\rangle) \arrow[ur] &
\end{tikzcd}$$

such that the two composites ending in $\mathrm{Aut}_\bullet(M,w_2)$ *agree up to inversion, and the other sub-diagrams are strictly commutative.*

During the calculation of the terms on the above braid, we will be interested in certain subgroups of $\mathrm{Aut}_\bullet(B)$ and $\mathrm{Aut}_\bullet(B,w_2)$. Before we introduce these subgroups let us define a homomorphism

$$\widehat{j}\colon\ \mathrm{Aut}_\bullet(B,w_2)\to\mathrm{Aut}_\bullet(B)\quad\text{by}\quad \widehat{j}(\widehat{f})=\phi_f$$

where $\phi_f\colon B\to B$ is the unique homotopy equivalence with $\phi_f\circ c\simeq f$, and the following subgroup of $\mathrm{Aut}_\bullet(B,w_2)$

$$\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,c_*[M]] := \{\widehat{f}\in\mathrm{Aut}_\bullet(B,w_2)\mid \phi_f\in\mathrm{Isom}[\pi,\pi_2,c_*[M]]\}$$

where $\mathrm{Isom}[\pi,\pi_2,c_*[M]] := \{\phi\in\mathrm{Aut}_\bullet(B)\,|\,\phi_*(c_*[M])=c_*[M]\}$.

Lemma 2.6. *There is a short exact sequence of groups*

$$0\longrightarrow H^1(M;\mathbb{Z}/2)\longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,c_*[M]]\xrightarrow{\ \widehat{j}\ }\mathrm{Isom}[\pi,\pi_2,c_*[M]]\longrightarrow 1$$

Proof. For any $\phi\in\mathrm{Isom}[\pi,\pi_2,c_*[M]]$, we have an $f\in\mathrm{Aut}_\bullet(M)$ such that $c\circ f\simeq\phi\circ c$ (this is basically by [8, Lemma 1.3]). We may assume that the pair (f,ν_M) is an element of $\mathrm{Aut}_\bullet(M,w_2)$ ([10, Lemma 3.1]). The pair $(c\circ f,\nu_M)$ determines an element $\widehat{f}$ of $\mathrm{Aut}_\bullet(B,w_2)$ for which $\widehat{j}(\widehat{f})=\phi_f=\phi$.

Suppose now that $\widehat{f},\,\widehat{g}\in\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,c_*[M]]$ such that $h:\phi_f\simeq\phi_g$. We have the following diagram

$$\begin{tikzcd}
& K(\mathbb{Z}/2,1) \arrow[d] \\
\partial(M\times I) \arrow[r, "\widehat{f}\sqcup\widehat{g}"] \arrow[d, hook] & B\langle w_2\rangle \arrow[d, "{(j,\xi)}"] \\
M\times I \arrow[ur, dashed] \arrow[r, "{(h\circ c\times\mathrm{id},\nu_M\circ p_1)}"'] & B\times\mathrm{BSO}
\end{tikzcd}$$

The obstructions to lifting $(h\circ c\times\mathrm{id},\nu_M\circ p_1)$ lie in the groups

$$H^{i+1}(M\times I,\partial(M\times I);\pi_i(K(\mathbb{Z}/2,1)))\cong H^i(M;\pi_i(K(\mathbb{Z}/2,1))),$$

hence the only non-zero obstructions are in $H^1(M;\mathbb{Z}/2)$.

Let $\widehat{f} \in \mathrm{Aut}_\bullet(M, w_2)$, for any $\alpha \in H^1(M; \mathbb{Z}/2)$, we will construct a $\widehat{g} \in \mathrm{Aut}_\bullet(M, w_2)$ with the property that $f \simeq g$ and the obstruction to $\widehat{f}$ and $\widehat{g}$ being equivalent is α. Note that different maps $M \times I \to K(\mathbb{Z}/2, 2)$ relative to the given maps on the boundary are also classified by $H^1(M; \mathbb{Z}/2)$. So we may think $\alpha\colon M \times I \to K(\mathbb{Z}/2, 2)$ such that $\alpha|_{M\times\{0\}}$ and $\alpha|_{M\times\{1\}}$ are the constant maps to the base point $\{*\}$ of $K(\mathbb{Z}/2, 2)$. Consider the following diagram

$$\begin{tikzcd}
 & M\langle w_2\rangle \arrow[d, "{(j,\xi)}"] \\
M \times \{0\} \arrow[r, "{(f,\nu_M)}"] \arrow[d, hook] & M \times \mathrm{BSO} \arrow[d, "\rho"] \\
M \times I \arrow[r, "\alpha"] \arrow[ur, dashed, "\widehat{\alpha}"] & K(\mathbb{Z}/2, 2)
\end{tikzcd}$$

The fibration $\rho\colon M \times \mathrm{BSO} \to K(\mathbb{Z}/2, 2) = \Omega K(\mathbb{Z}/2, 3)$ is given by $(x, y) \to w_2(x) - w(y)$, for which the fiber over the base point is by definition $M\langle w_2\rangle$. By the homotopy lifting property we have $\widehat{\alpha}\colon M \times I \to M \times \mathrm{BSO}$ making the diagram commutative.

Let $\widehat{g} := \widehat{\alpha}|_{M\times\{1\}}$, then since $w_2(p_1 \circ \widehat{g}(x)) = w(p_2 \circ \widehat{g}(x))$, where p_1 and p_2 are projections to the first and second components, respectively, $\widehat{g}$ actually gives us a map $M \to M\langle w_2\rangle$. Observe that $p_1 \circ \widehat{\alpha}\colon M \times I \to M$ is a homotopy between f and g. To lift this homotopy to $M\langle w_2\rangle$, we should have $w_2((p_1 \circ \widehat{\alpha})(x, t)) = w((p_2 \circ \widehat{\alpha})(x, t))$ for all $x \in M$ and $t \in I$, which is possible if and only if α represents the trivial map. Hence α is the obstruction to $\widehat{f}$ and $\widehat{g}$ being equivalent. □

Lemma 2.7. *The kernel of* β, $\ker(\beta) := \beta^{-1}(0)$, *is equal to* $\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]]$.

Proof. The map $\beta\colon \mathrm{Aut}_\bullet(B, w_2) \to \Omega_4(B\langle w_2\rangle)$ is defined by $\beta(\widehat{f}) = [M, \widehat{f}\,] - [M, \widehat{c}]$. For the bordism group $\Omega_4(B\langle w_2\rangle)$, we use the Atiyah–Hirzebruch spectral sequence, whose E^2-term is $H_p(M; \Omega_q^{Spin}(*))$.

The non-zero terms on the E^2-page are $H_0(B; \Omega_4^{Spin}(*)) \cong \mathbb{Z}$ in the $(0, 4)$ position, $H_2(B; \mathbb{Z}/2)$ in the $(2, 2)$ position, $H_3(B; \mathbb{Z}/2)$ in the $(3, 1)$ position and $H_4(B)$ in the $(4, 0)$ position. To understand the kernel, we use the projection to $H_4(B)$.

Let $\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2)$ and suppose first that $\widehat{f} \in \ker\beta$, then $(j\circ\widehat{f})_*[M] = c_*[M]$. But since $(j \circ \widehat{f})$ is a 3-equivalence, there exists $\phi \in \mathrm{Aut}_\bullet(B)$ with $\phi \circ c = j \circ \widehat{f}$ (recall Lemma 2.3). So, $\phi_*(c_*[M]) = c_*[M]$ which means $\widehat{j}(\widehat{f}) = \phi \in \mathrm{Isom}[\pi, \pi_2, c_*[M]]$. Therefore, $\ker(\beta) \subseteq \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, c_*[M]]$. To see the other inclusion note that

$$\mathrm{coker}(d_2\colon H_4(B; \mathbb{Z}/2) \to H_2(B; \mathbb{Z}/2)) \cong \langle w_2\rangle$$

and the class w_2 is preserved by a self-homotopy equivalence. □

Definition 2.8. ([10]) Let $\widetilde{\mathcal{H}}(M,w_2)$ denote the bordism groups of pairs $(W,\widehat{F})$, where W is a compact, oriented 5-manifold with $\partial_1 W=-M$, $\partial_2 W=M$ and the map $\widehat{F}\colon W\to M\langle w_2\rangle$ restricts to $\widehat{id}_M$ on $\partial_1 W$, and on $\partial_2 W$ to a map $\hat{f}\colon M\to M\langle w_2\rangle$ satisfying properties (i) and (ii) of Definition 2.2 .

Corollary 2.9. *The images of* $\mathrm{Aut}_\bullet(M,w_2)$ *or* $\widetilde{\mathcal{H}}(M,w_2)$ *in* $\mathrm{Aut}_\bullet(B,w_2)$ *are precisely equal to* $\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,c_*[M]]$.

Proof. Let $\widehat{f}\in\mathrm{Aut}_\bullet(M,w_2)$ and $\phi_{\widehat{f}}$ denote the image of $\widehat{f}$ in $\mathrm{Aut}_\bullet(B,w_2)$. Then $\widehat{j}(\phi_{\widehat{f}})=\phi_f$ satisfies $\phi_f\circ c=c\circ f$ and ϕ_f preserves $c_*[M]$. Hence $\phi_f\in\mathrm{Isom}[\pi,\pi_2,c_*[M]]$. Now suppose that $\phi\in\mathrm{Isom}[\pi,\pi_2,c_*[M]]$, then by [8, Lemma 1.3] there exists $f\in\mathrm{Aut}_\bullet(M)$ such that $\phi\circ f\simeq c\circ f$. We may assume that $\widehat{f}=(f,\nu_M)\in\mathrm{Aut}_\bullet(M,w_2)$ [10, Lemma 3.1]. Let $\phi_{\widehat{f}}\in\mathrm{Aut}_\bullet(B,w_2)$ denote the image of $\widehat{f}$, we have $\widehat{j}(\phi_{\widehat{f}})=\phi$.

The result about the image of $\widetilde{\mathcal{H}}(M,w_2)$ follows from the exactness of the braid [10, Lemma 2.7] and the fact that $\ker(\beta)=\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,c_*[M]]$. □

Remark 2.10. By universal coefficient spectral sequence, we have an exact sequence

$$0\longrightarrow H^2(\pi;\Lambda)\longrightarrow H^2(M;\Lambda)\xrightarrow{\mathrm{ev}}\mathrm{Hom}_\Lambda(\pi_2,\Lambda)\longrightarrow 0$$

and the cohomology intersection pairing is defined by $s_M(u,v)=ev(v)(PD(u))$ for all $u,v\in H^2(M;\Lambda)$ where PD is the Poincaré duality isomorphism. Since $s_M(u,v)=0$ for all $u\in H^2(M;\Lambda)$ and $v\in H^2(\pi;\Lambda)$, the pairing s_M induces a nonsingular pairing

$$s'_M\colon H^2(M;\Lambda)/H^2(\pi;\Lambda)\times H^2(M;\Lambda)/H^2(\pi;\Lambda)\to\Lambda.$$

Before we finish this section, let us point out that for our purposes we need to look for a relation between the image of the fundamental class $c_*[M]\in H_4(B)$ and the equivariant intersection pairing s_M. Let $\mathrm{Her}(H^2(B;\Lambda))$ be the group of Hermitian pairings on $H^2(B;\Lambda)$. We can define a natural map $F\colon H_4(B)\to\mathrm{Her}(H^2(B;\Lambda))$ by

$$F(x)(u,v)=u(x\cap v)=(u\cup v)(x).$$

The construction of F applied to M yields s_M and by naturality $F(c_*[M])=s_M$. In other words, we have the following commutative diagram

$$\begin{tikzcd}
H^2(B;\Lambda)\times H^2(B;\Lambda) \arrow[r, "F(c_*[M])"] \arrow[d, "c^*\times c^*"', "\cong"] & \Lambda \\
H^2(M;\Lambda)\times H^2(M;\Lambda) \arrow[r, equal] & H^2(M;\Lambda)\times H^2(M;\Lambda) \arrow[u, "s_M"']
\end{tikzcd}$$

Therefore, any automorphism of B which preserves $c_*[M]$ also preserves the intersection form s_M. The converse of this statement is not necessarily true, i.e., $c_*[M]$ and s_M do not always uniquely determine each other.

3. s-Cobordism

In this section we are going to prove Theorem 1.2. Let M be a closed, connected, oriented, topological 4-manifold with fundamental group π such that $\operatorname{cd}\pi \leq 2$. We study bordism classes of such manifolds over the normal 1-type.

For type (I) manifolds, $w_2(\widetilde{M}) \neq 0$, oriented topological bordism group over the normal 1-type is

$$\Omega_4^{\mathrm{STOP}}(K(\pi,1)) \cong \Omega_4^{\mathrm{STOP}}(*) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

via the signature, $\sigma(M)$, and the ks-invariant. Recall that $\sigma(M)$ is determined via the integer valued intersection form $s_M^{\mathbb{Z}}$ on $H_2(M)$. Since the image

$$H^2(\pi;\mathbb{Z}) \xrightarrow{u_M} H^2(M;\mathbb{Z})$$

is the radical of $s_M^{\mathbb{Z}}$ $\sigma(M)$ that is equal to the signature of the form $s_M \otimes_\Lambda \mathbb{Z}$ [11, Remark 4.2]. Therefore, when $\operatorname{cd}\pi \leq 2$, the signature of M is determined by the formula

$$\sigma(M) = \sigma(s_M^{\mathbb{Z}}) = \sigma(s_M \otimes_\Lambda \mathbb{Z}).$$

On the other hand, in the type (II) case, $w_2(\widetilde{M}) = 0$, we have

$$\Omega_4^{\mathrm{TOPSPIN}}(K(\pi,1)) \cong \mathbb{Z} \oplus H_2(\pi;\mathbb{Z}/2).$$

In this case, the invariants are signature and an invariant in $H_2(\pi;\mathbb{Z}/2)$.

Now, let M_1 and M_2 be closed, connected, oriented, topological 4-manifolds with isomorphic fundamental groups. By fixing an isomorphism, we identify $\pi = \pi_1(M_1) = \pi_1(M_2)$. Suppose also that $\operatorname{cd}\pi \leq 2$. Suppose further that M_1 and M_2 have isometric quadratic 2-types. First we are going to show that M_1 and M_2 are homotopy equivalent using [1, Corollary 3.2]. Then, we are going to show that they are indeed bordant over the normal 1-type, if we further assume that π satisfies (A1).

Since M_1 and M_2 have isometric quadratic 2-types, we have

$$\chi\colon \pi_1(M_1) \to \pi_1(M_2) \quad \text{and} \quad \psi\colon \pi_2(M_1) \to \pi_2(M_2)$$

a pair of isomorphisms such that $\psi(gx) = \chi(g)\psi(x)$ for all $g \in \pi$, $x \in \pi_2(M_1)$ and preserving the intersection form i.e.,

$$s_{M_2}(\psi(x),\psi(y)) = \chi_*(s_{M_1}(x,y)).$$

Let $B(M_i)$ denote the 2-type of M_i and $c_i\colon M_i \to B(M_i)$ corresponding 3-equivalences for $i = 1,2$. We are going to construct a homotopy equivalence between $B(M_1)$ and $B(M_2)$. Note that we have isomorphisms $\pi_2(c_i)\colon \pi_2(M_i) \xrightarrow{\cong} \pi_2(B(M_i))$ for $i = 1,2$. Start with the composition

$$\pi_2(c_2) \circ \psi \circ \pi_2(c_1)^{-1}\colon \pi_2(B(M_1)) \xrightarrow{\cong} \pi_2(B(M_2)).$$

We can think of any Abelian group G as a topological group with discrete topology. Then we can define $K(G,1) = BG$, which is also an Abelian topological group, and $K(G,2) = BK(G,1) = B^2G$. This construction is functorial. Hence we have a homotopy equivalence

$$B^2(\pi_2(c_2) \circ \psi \circ \pi_2(c_1)^{-1})\colon K(\pi_2(B(M_2)),2) \to K(\pi_2(B(M_2)),2)$$

which is π_1-equivariant, since ψ is π_1-equivariant. We also have another π_1-equivariant homotopy equivalence, namely $E\chi\colon E\pi_1(M_1) \to E\pi_1(M_2)$, where the contractible space $E\pi_1(M_i)$ is the total space of the universal bundle over $B\pi_1(M_i)$ for $i = 1, 2$. Let

$$\tau := E(\chi) \times B^2(\pi_2(c_2) \circ \psi \circ \pi_2(c_1)^{-1})$$

and recall that $B(M_i) \simeq E\pi_1(M_i) \times_{\pi_1(M_i)} K(\pi_2(B(M_i)), 2)$. Then we have

$$\tau\colon B(M_1) \to B(M_2).$$

Also, since $B(M_i)$ is a fibration over $B\pi_1(M_i)$ with fiber $K(\pi_2(B(M_i)), 2)$ by five lemmas, we can see that τ is a homotopy equivalence. Summarizing, we have a homotopy equivalence τ with the following commutative diagram:

$$\begin{tikzcd}
\pi_2(M_1) \arrow[r, "\pi_2(c_1)"] \arrow[d, "\psi"'] & \pi_2(B(M_1)) \arrow[d, "\pi_2(\tau)"] \\
\pi_2(M_2) \arrow[r, "\pi_2(c_2)"'] & \pi_2(B(M_2))
\end{tikzcd}$$

Note that we have $\tau_\sharp(s_{M_2}) = s_{M_1}$. Since M_1 and M_2 have isometric quadratic 2-types, they have isomorphic intersection forms, which implies that $\tau_*((c_1)_*[M_1]) = (c_2)_*[M_2]$ (we may need to use the image of $(c_2)_*[M_2]$ under a self-equivalence of $B(M_2)$ if necessary, see [7, Lemma 3] and the proof of [7, Theorem 14]). Also see the discussion at the end of Section 2 for the relation between the image of the fundamental class and the equivariant intersection form. Therefore, M_1 and M_2 have isomorphic fundamental triples in the sense of [1] and hence they must be homotopy equivalent by [1, Corollary 3.2].

If we further assume that the assembly map

(A1) $A_4\colon H_4(K(\pi, 1); \mathbb{L}_0(\mathbb{Z})) \to L_4(\mathbb{Z}\pi)$ is injective,

then by [3, Corollary 3.11] M_1 and M_2 are bordant over the normal 1-type.

Therefore, if the fundamental group π satisfies (A1), then we have a cobordism W between M_1 and M_2 over the normal 1-type, which is a spin cobordism in the type (II) case.

Choose a handle decomposition of W. Since W is connected, we can cancel all 0- and 5-handles. Further, we may assume by low-dimensional surgery that the inclusion map $M_1 \hookrightarrow W$ is a 2 equivalence. So we can trade all 1-handles for 3-handles, and upside-down, all 4-handles for 2-handles. We end up with a handle decomposition of W that only contains 2- and 3-handles, and view W as

$$W = M_1 \times [0, 1] \cup \{2 - \text{handles}\} \cup \{3 - \text{handles}\} \cup M_2 \times [-1, 0].$$

Let $W_{3/2}$ be the ascending cobordism that contains just M_1 and all 2-handles and let $M_{3/2}$ be its 4-dimensional upper boundary. The inclusion map $M_1 \hookrightarrow W$ is a 2 equivalence, so attaching map $S^1 \times D^3 \to M_1$ of a 2-handle must be null-homotopic. Hence attaching a 2-handle is the same as connect summing with $S^2 \times S^2$ or the same as connect summing with $S^2 \widetilde{\times} S^2$. Since M_1 and

$\widetilde{M_1}$ are spin at the same time, we can assume that there are no $S^2\widetilde{\times}S^2$-terms present in $M_{3/2}$ (see for example [15, p. 80]).

From the lower half of W, we have $M_{3/2} \approx M_1\sharp m_1(S^2\times S^2)$, while from the upper half, we have $M_{3/2} \approx M_2\sharp m_2(S^2\times S^2)$. Since $\mathrm{rank}(H_2(M_1)) = \mathrm{rank}(H_2(M_2)))$, it follows that $m = m_1 = m_2$. We have a homeomorphism

$$\zeta\colon M_2\sharp m(S^2\times S^2) \xrightarrow{\approx} M_1\sharp m(S^2\times S^2)\,.$$

Next assume that:

(A2) Whitehead group $Wh(\pi)$ is trivial for π.

Hence being s-cobordant is equivalent to being h-cobordant. The strategy for the remainder of the proof is the following: we will cut W into two halves, then glue them back after sticking in an h-cobordism of $M_{3/2}$. This cut and reglue procedure will create a new cobordism from M_1 to M_2. If we choose the correct h-cobordism, then the 3-handles from the upper half will cancel the 2-handles from the lower half. This means that the newly created cobordism between M_1 and M_2 will have no homology relative to its boundaries, and so it will indeed be an h-cobordism from M_1 to M_2.

Note that we have $\tau_\sharp(s_{M_2}) = s_{M_1}$ and $s_{M_1}\cong s_{M_2}$ if and only if $s'_{M_1}\cong s'_{M_2}$. Hence we can immediately deduce that $\tau_\sharp s'_{M_1} = s'_{M_2}$. Now let $M := M_1\sharp m(S^2\times S^2)$ and $M' := M_2\sharp m(S^2\times S^2)$ with the following quadratic 2-types,

$$[\pi,\pi_2,s_M] := [\pi_1(M_1),\pi_2(M_1)\oplus\Lambda^{2m}, s_{M_1}\oplus H(\Lambda^m)]$$

and

$$[\pi_1(M_2),\pi_2(M_2)\oplus\Lambda^{2m}, s_{M_2}\oplus H(\Lambda^m)],$$

where $H(\Lambda^m)$ is the hyperbolic form on $\Lambda^m\oplus(\Lambda^m)^*$.

Since W is a cobordism over the normal 1-type,

$$(\pi_1(\zeta)\circ\chi, \pi_2(\zeta)\circ(\psi\oplus\mathrm{id})) = (\mathrm{id},\pi_2(\zeta)\circ(\psi\oplus\mathrm{id}))$$

is an element in $\mathrm{Isom}[\pi,\pi_2,s_M]$. Let $B = B(M)$ denote the 2-type of M. We have an exact sequence of the form [14]

$$0 \longrightarrow H^2(\pi;\pi_2) \longrightarrow \mathrm{Aut}_\bullet(B) \xrightarrow{(\pi_1,\pi_2)} \mathrm{Isom}[\pi,\pi_2] \longrightarrow 1\,. \qquad (1)$$

Therefore we can find a $\phi''\in\mathrm{Aut}_\bullet(B)$ such that

$$\pi_1(\phi'') = \mathrm{id} \quad \text{and} \quad \pi_2(\phi'') = \pi_2(\zeta)\circ(\psi\oplus\mathrm{id}).$$

The homotopy self-equivalence ϕ'' preserves the intersection form s_M but on the braid we see $\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,c_*[M]]$. So to use the braid, we need to construct a self-homotopy equivalence of B which preserves $c_*[M]$.

Hillman [7] showed that for $\mathrm{cd}\,\pi\le 2$, we have $\pi_2(M)\cong P\oplus H^2(\pi;\Lambda)$ where P is a projective Λ-module. He also showed that there exists a 2-connected degree-1 map $g_M\colon M\to Z$ where Z is a PD_4 complex with $\pi_2(Z)\cong H^2(\pi;\Lambda)$ and $\ker(\pi_2(g_M)) = P$. He called Z as the strongly minimal model for M.

We may assume that $\pi_2(g_M)$ is projection to the second factor and $c_Z\circ g_M = g\circ c$ for some 2-connected map $g\colon B\to B(Z)$, where $B(Z)$ denotes the 2-type of Z . The map g is a fibration with fibre $K(P,2)$, and the inclusion of $H^2(\pi;\Lambda)$ into $\pi_2(M_2)$ determines a section s for g. Summarizing we have the diagram below with a commutative square

$$\begin{tikzcd} & M \arrow[r, "g_M"] \arrow[d, "c"'] & Z \arrow[d, "c_Z"] \\ K(P,2) \arrow[r] & B \arrow[r, "g"] & B(Z) \arrow[l, bend left, "s"] \end{tikzcd}$$

Note that since ϕ'' preserves the intersection form and identity on π,

$$\pi_2(\phi'')\colon P \oplus H^2(\pi;\Lambda) \to P \oplus H^2(\pi;\Lambda)$$

has a matrix representation of the form

$$\pi_2(\phi'') = \begin{bmatrix} * & * \\ 0 & \mathrm{id} \end{bmatrix}$$

where the first $*$ represents an π-module isomorphism $P \to P$ and the second $*$ represents an Λ-module homomorphism $P \to H^2(\pi;\Lambda)$. We modify ϕ'', first to $\phi' \in \mathrm{Aut}_\bullet(B)$ so that $\pi_2(\phi')$ has a matrix representation of the form

$$\pi_2(\phi') = \begin{bmatrix} * & 0 \\ 0 & \mathrm{id} \end{bmatrix}$$

i.e., it induces the zero homomorphism from P to $H^2(\pi;\Lambda)$. To achieve this first define

$$\theta\colon P \to H^2(\pi;\Lambda) \qquad \text{by} \qquad \theta(p) = \mathrm{pr}_2(\pi_2(\phi'')(p,0)).$$

Then define

$$\alpha_\theta\colon P \oplus H^2(\pi;\Lambda) \to P \oplus H^2(\pi;\Lambda) \qquad \text{by} \qquad \alpha_\theta(p,e) = (p, e-\theta(p)).$$

This newly defined map α_θ is a Λ-module isomorphism of π_2 by [7, Lemma 3]. Now the pair $(\mathrm{id}, \alpha_\theta)$ gives us an isomorphism ϕ''_θ of B by the sequence (1) on the previous page. Define $\phi' := \phi''_\theta \circ \phi''$, and observe that $g \circ \phi' = g$.

Let $L := L_\pi(P,2)$ be the space with algebraic 2-type $[\pi, P, 0]$ and universal covering space $\widetilde{L} \simeq K(P,2)$. We may construct L by adjoining 3-cells to M to kill the kernel of the projection from π_2 to P and then adjoining higher dimensional cells to kill the higher homotopy groups. The splitting $\pi_2 \cong P \oplus H^2(\pi;\Lambda)$ also determines a projection $q\colon B \to L$.

To begin with we have the following isomorphisms where Γ denotes the Whitehead quadratic functor [21].

$$\begin{aligned} H_4(B) &\cong \Gamma(\pi_2) \otimes_\Lambda \mathbb{Z} \oplus H_2(\pi;\pi_2) \\ &\cong \Gamma(H^2(\pi;\Lambda) \oplus P) \otimes_\Lambda \mathbb{Z} \oplus H_2(\pi; H^2(\pi,\Lambda)) \\ &\cong (\Gamma(H^2(\pi,\Lambda)) \oplus \Gamma(P) \oplus H^2(\pi,\Lambda) \otimes P) \otimes_\Lambda \mathbb{Z} \oplus H_2(\pi; H^2(\pi,\Lambda)) \\ &\cong \Gamma(P) \otimes_\Lambda \mathbb{Z} \oplus \Gamma(H^2(\pi,\Lambda)) \otimes_\Lambda \mathbb{Z} \oplus H_2(\pi; H^2(\pi,\Lambda)) \oplus (H^2(\pi,\Lambda) \otimes P) \otimes_\Lambda \mathbb{Z} \\ &\cong H_4(L) \oplus H_4(B(Z)) \oplus (H^2(\pi,\Lambda) \otimes P) \otimes_\Lambda \mathbb{Z}. \end{aligned}$$

We are going to consider the difference $\phi'_*(c_*[M]) - c_*[M] \in H_4(B)$. We start by projecting $\phi'_*(c_*[M])$ and $c_*[M]$ to $H_4(L) \cong \Gamma(P) \otimes_\Lambda \mathbb{Z}$. Recall that we have a nonsingular pairing

$$s'_M \colon H^2(M;\Lambda)/H^2(\pi;\Lambda) \times H^2(M;\Lambda)/H^2(\pi;\Lambda) \to \Lambda.$$

If we further restrict s'_M to $\mathrm{Hom}_\Lambda(P,\Lambda) \cong H^2(L;\Lambda)/H^2(\pi;\Lambda)$, we get a Hermitian pairing $s''_M \in \mathrm{Her}(P)$. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
H_4(B) \arrow[r, "F"] \arrow[d, "q_*"'] & \mathrm{Her}(H^2(B;\Lambda)) \arrow[d, "q_\sharp"] \\
\Gamma(P) \otimes_\Lambda \mathbb{Z} \arrow[r, "\cong"] & \mathrm{Her}(P)
\end{tikzcd}$$

The bottom row is an isomorphism [7, Theorem 2]. Both $q_*(c_*[M])$ and $q_*(\phi'_*(c_*[M]))$ map to s''_M, hence $q_*(c_*[M]) = q_*(\phi'_*(c_*[M]))$. Since $g \circ \phi' = g$, we have

$$\phi'_*(c_*[M]) - c_*[M] \in (H^2(\pi;\Lambda) \otimes P) \otimes_\Lambda \mathbb{Z}\,.$$

As a final modification, as in [7, Lemma 3], we can choose a self-equivalence ϕ'_θ of B so that $(\phi'_\theta \circ \phi')_*(c_*[M]) = c_*[M] \bmod \Gamma(H^2(\pi,\Lambda)) \otimes_\Lambda \mathbb{Z}$. Hence $(\phi'_\theta \circ \phi')_*(c_*[M]) = c_*[M]$ in $H_4(B)$, see also the proof of [7, Theorem 14]. Let $\phi := \phi'_\theta \circ \phi'$.

We have $\phi \in \mathrm{Isom}[\pi, \pi_2, c_*[M]]$. Recall that we have the following short exact sequence by Lemma 2.6

$$0 \longrightarrow H^1(M;\mathbb{Z}/2) \longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,c_*[M]] \xrightarrow{\widehat{j}} \mathrm{Isom}[\pi,\pi_2,c_*[M]] \longrightarrow 1.$$

Choose $\widehat{f} \in \mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,c_*[M]]$ such that $\widehat{j}(\widehat{f}) = \phi$. There exists $(W,\widehat{F}) \in \widetilde{\mathcal{H}}(M,w_2)$ which maps to $\widehat{f}$, i.e., $\widehat{F}\colon W \to B\langle w_2\rangle$ and $F|_{\partial_2 W} = \widehat{f}$.

Comparison of Wall's[18] surgery program with Kreck's modified surgery program gives a commutative diagram of exact sequences (see [10], Lemma 4. 1)

$$\begin{tikzcd}
\tilde{L}_6(\mathbb{Z}[\pi]) \arrow[r, equal] \arrow[d] & \tilde{L}_6(\mathbb{Z}[\pi]) \arrow[d] & \\
\mathcal{S}(M \times I, \partial) \arrow[r] \arrow[d] & \mathcal{H}(M) \arrow[r] \arrow[d] & \mathrm{Aut}_\bullet(M) \\
\mathcal{T}(M \times I, \partial) \arrow[r] \arrow[d] & \widetilde{\mathcal{H}}(M,w_2) \arrow[r, two heads] \arrow[d] & \mathrm{Isom}[\pi,\pi_2,c_*[M],w_2] \\
L_5(\mathbb{Z}[\pi]) \arrow[r, equal] & L_5(\mathbb{Z}[\pi]) &
\end{tikzcd}$$

The group $\mathcal{H}(M)$ consists of oriented h-cobordisms W^5 from M to M, under the equivalence relation induced by h-cobordism relative to the boundary.

The tangential structures $\mathcal{T}(M \times I, \partial)$, is the set of degree 1 normal maps $F\colon (W, \partial W) \to (M \times I, \partial)$, inducing the identity on the boundary. The group structure on $\mathcal{T}(M \times I, \partial)$ is defined as for $\widetilde{\mathcal{H}}(M, w_2)$. The map $\mathcal{T}(M \times I, \partial) \to \widetilde{\mathcal{H}}(M, w_2)$ takes $F\colon (W, \partial W) \to (M \times I, \partial)$ to $(W, \widehat{F}) \in \widetilde{\mathcal{H}}(M, w_2)$, where $\widehat{F} = \widehat{p_1} \circ F$ (see [18] for further details). Let $\sigma_5 \in L_5(\mathbb{Z}[\pi])$ be the image of $(W, \widehat{F})$. We further assume that

(A3) The map $\mathcal{T}(M \times I, \partial) \to L_5(\mathbb{Z}[\pi])$ is onto.

Let $(W', F') \in \mathcal{T}(M \times I, \partial)$ map to σ_5 and let $(W', \widehat{F'}) \in \widetilde{\mathcal{H}}(M, w_2)$ be the image of (W', F'). Consider the difference of these elements in $\widetilde{\mathcal{H}}(M, w_2)$,

$$(W'', \widehat{F''}) := (W', \widehat{F'}) \bullet (-W, \hat{f}^{-1} \bullet \widehat{F}) \in \widetilde{\mathcal{H}}(M, w_2).$$

Note that $\hat{f}^{-1} = \widehat{id}_M\colon M \to M\langle w_2\rangle$ denotes the map defined by the pair $(id_M\colon M \to M, \nu_M\colon M \to \mathrm{BSO})$. The element $(W'', \widehat{F''}) \in \widetilde{\mathcal{H}}(M, w_2)$ maps to $0 \in L_5(\mathbb{Z}[\pi_1])$. By the exactness of the right-hand vertical sequence there exists an h-cobordism T of M which maps to $(W'', \widehat{F''})$. Let f denote the induced homotopy self equivalence of M. By construction we have $c \circ f \simeq \phi \circ c$ where $c \circ f = j \circ \widehat{f}$. Note that $\pi_2(\zeta^{-1} \circ f) = \psi \oplus \mathrm{id}$ and also $\zeta^{-1} \circ f$ gives us a self-equivalence of $M_{3/2}$. Now, if we put the s-cobordism T in between the two halves of W, then the 3-handles from the upper half cancel the 2-handles from the lower half. This finishes the proof of Theorem 1.2.

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