

Homotopy classification of PD_4 -complexes relative an order relation

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Abstract We define an order relation among oriented PD_4 -complexes. We show that with respect to this relation, two PD_4 -complexes over the same complex are homotopy equivalent if and only if there is an isometry between the second homology groups. We also consider minimal objects of this relation.

Keywords Minimal PD_4 -complex · Order relation · Symmetric L -group

Mathematics Subject Classification Primary 57P10; Secondary 55N22 · 55S35 · 55S45

1 Introduction

Let X, P be compact oriented PD_4 -complexes and $[X] \in H_4(X; \mathbb{Z}), [P] \in H_4(P; \mathbb{Z})$ be their fundamental classes, respectively. We are going to use the notation $X > P$ if there is a continuous map $f: X \rightarrow P$ such that

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1. $f_*[X] = [P]$, i.e., f has degree 1,
2. $f_* : \pi_1(X) \rightarrow \pi_1(P)$ is an isomorphism.

In this case, we shall say that f realizes $X \succ P$. Note that f is not unique with respect to the properties (1) and (2), i.e., there could exist a map $g : X \rightarrow P$ satisfying (1) and (2), but not homotopic to f .

Let X, X' and P be PD_4 -complexes such that $X \succ P$ and $X' \succ P$ are realized by $f : X \rightarrow P$ and $f' : X' \rightarrow P$. One of the main questions that we want to address in this paper is, when are X and X' homotopy equivalent over P ? We show that X and X' are homotopy equivalent over P if and only if there is an isometry between the second homology groups (Theorem 5.1).

Remark 1.1 Throughout the paper π will denote the fundamental group $\pi_1(X)$. Also note that for a PD_4 -complex X , the integral group ring $\Lambda := \mathbb{Z}\pi$ has an involution defined on it. Every right(left) Λ -module can be considered as a left(right) Λ -module with the conjugate structure given by this involution. Throughout this paper the functors \otimes_Λ and Hom_Λ are defined using this fact.

Starting with a PD_4 -complex X , we also define a minimal PD_4 -complex P for X , called X -minimal, which is minimal with respect to the order relation \succ (see Definition 3.1). Minimal PD_4 -complexes are also considered by Hillman [6–8] with special emphasis on a particular type of minimal PD_4 -complex, called a strongly minimal PD_4 -complex. Recall that for a PD_4 -complex X , the radical of the intersection form λ_X , denoted by $\text{Rad}(\lambda_X)$, is isomorphic to the module $H^2(\pi; \Lambda)$.

Definition 1.2 A PD_4 -complex P is said to be strongly minimal if

$$H_2(P; \mathbb{Z}[\pi_1(P)]) / \text{Rad}(\lambda_P) = 0.$$

Remark 1.3 Obviously, if P is strongly minimal and $X \succ P$, then P is X -minimal.

These two notions of minimality coincide whenever the cohomological dimension of the fundamental group is less than or equal to 2 (see for example [8, Theorem 25]). All known examples of strongly minimal models are PD_4 -complexes with such fundamental groups ([6–8]). Therefore one might consider the following natural question:

Problem 1.4 Find examples of (strongly) minimal PD_4 -complexes whose fundamental group has cohomological dimension greater than 2.

Hillman [6, 8] gives a homotopy classification for PD_4 -complexes over the strongly minimal models subject to a k -invariant constraint. He considers the same obstruction as in the proof of our main result Theorem 5.1. However, we point out that our method in this paper is different: to see that the obstruction vanishes Hillman realizes it by a self-equivalence, whereas we use a map A' to relate the obstruction to intersection forms and cap products. We also remove the hypothesis on the k -invariant.

The outline of the paper is as follows: In Sect. 2 we list some of the immediate properties of the order relation \succ . In Sect. 3, for a PD_4 -complex X , we define X -minimal PD_4 complexes. We show that if $H_2(X; \Lambda)$ is finitely generated, then such

minimal complexes exist (Theorem 3.5). Section 4 is about Postnikov decomposition of the map $f: X \rightarrow P$. In Sect. 5, we prove our main result: two PD_4 -complexes X and X' over the same minimal complex P are homotopy equivalent if and only if there is an isometry $\Phi: H_2(X; \Lambda) \rightarrow H_2(X'; \Lambda)$ (Theorem 5.1).

2 Some remarks and preliminary results

In this section we will list some of the immediate properties of the above definition of the order relation \succ .

1. The relation \succ is transitive and since $\text{Id}: X \rightarrow X$ realizes $X \succ X$ it is clear that \succ is also reflexive.
2. The relation \succ is symmetric in the sense of the following theorem:

Theorem 2.1 *If $X \succ P$ and $P \succ X$ then X is homotopy equivalent to P .*

Proof Let f and g realize $X \succ P$ and $P \succ X$, respectively. Then $g \circ f$ and $f \circ g$ realize $X \succ X$ and $P \succ P$, respectively. Then by [5, Theorem 3, page 15] $f \circ g$ and $g \circ f$ are homotopy equivalences, hence f and g are homotopy equivalences.

3. If f realizes $X \succ P$, then

$$K_2(f, \Lambda) := \text{Ker}(f_*: H_2(X; \Lambda) \rightarrow H_2(P; \Lambda))$$

is stably Λ -free. Here $\Lambda = \mathbb{Z}[\pi_1(P)]$ is the integral group ring. Moreover, the restriction of the intersection form

$$\lambda_X: H_2(X; \Lambda) \times H_2(X; \Lambda) \rightarrow \Lambda$$

to $K_2(f, \Lambda)$ is non-singular. Also note that the module $K_2(f, \Lambda)$ is finitely generated. See [11, Lemmas 2.3, 2.6, 5.1] for these arguments.

- (4) The converse of (3) is also true, as witnessed by the following theorem:

Theorem 2.2 *Let X be a PD_4 -complex and $G \subset H_2(X; \Lambda)$ a stably free Λ -submodule such that λ_X restricted to G is non-singular. Then there is a PD_4 -complex P such that $X \succ P$ is realized by $f: X \rightarrow P$ with $K_2(f, \Lambda) = G$.*

Proof (see [4]) If G is Λ -free with Λ -basis $e_1, \dots, e_r \in G \subset H_2(X; \Lambda) \cong \pi_2(X)$, then we can take

$$P = X \cup_{\varphi_i} \cup_1^r D^3,$$

where $[\varphi_i] = e_i$, for $i = 1, \dots, r$ and we have the inclusion map $f: X \hookrightarrow P$ realizing $X \succ P$.

If G is stably Λ -free, i.e., $G \oplus \Lambda^{2a}$ is free with basis e_1, \dots, e_q , we can consider the PD_4 -complex

$$Y = X \# (\#_1^a S^2 \times S^2)$$

so that $G \oplus \Lambda^{2a} \subset H_2(Y; \Lambda)$. Then we can construct

$$P = Y \cup_{\varphi_i} \cup_1^q D^3,$$

where $[\varphi_i] = e_i$, for $i = 1, \dots, q$, and $f : Y \hookrightarrow P$ is the inclusion as above, realizing $Y \succ P$.

We claim that f is homotopic to a map $g : Y \rightarrow P$ such that g factors over the collapsing map $c : Y \rightarrow X$, i.e., there exists a map $f' : X \rightarrow P$ such that $g = f' \circ c$. Since c realizes $Y \succ X$, f' realizes $X \succ P$.

To see that the claim is true, we write $T = \sharp_1^a S^2 \times S^2$ and hence $Y = X \sharp T$. The connected sum is formed by deleting a 4-disc \mathring{D}^4 from T , letting $\mathring{T} = T \setminus \mathring{D}^4$, and attaching it to $X \setminus \{\text{interior of the } 4\text{-cell}\}$ along S^3 . Note that, forming connected sums of PD_4 -complexes can be done by using representations of $X = K \cup_\varphi D^4$ where K is a 3-complex [11, Lemma 2.9]. By construction, $f|_{\mathring{T}}$ is homotopic to the constant map by a homotopy $h_t : \mathring{T} \rightarrow P$. Applying the homotopy extension property, h_t can be extended to a homotopy $H_t : Y \rightarrow P$ with $H_0 = f$, $H_1(\mathring{T}) = \{\text{pt}\}$. Let $H_1 = g$, which factors over $Y/\mathring{T} = X$. □

(5) Any degree 1-map $f : X \rightarrow P$ defines, by Poincaré duality, a split short exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_2(f, \Lambda) & \longrightarrow & H_2(X; \Lambda) & \xrightarrow{f_*} & H_2(P; \Lambda) & \longrightarrow & 0 \\
 & & & & \uparrow \cong & \swarrow s_f & \uparrow \cong & & \\
 & & & & \cap[X] & & \cap[P] & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & H^2(X; \Lambda) & \xleftarrow{f^*} & H^2(P; \Lambda) & &
 \end{array}$$

such that $\text{Im } s_f$ and $K_2(f, \Lambda)$ are orthogonal with respect to λ_X (see [11, Theorem 5.2]).

(6) Assume that we are given $f' : X \rightarrow P'$ realizing $X \succ P'$ with $K_2(f', \Lambda) = G$ which is stably free and λ_X restricted to G is non-singular. The above construction (see the proof of Theorem 2.2) also provides $X \succ P$ realized by $f : X \rightarrow P$ with $K_2(f, \Lambda) = G$. For this situation we shall need the following lemma:

Lemma 2.3 *There is a homotopy equivalence $h : P \rightarrow P'$ such that the diagram*

$$\begin{array}{ccc}
 P & \xrightarrow{h} & P' \\
 & \swarrow f & \nearrow f' \\
 & X &
 \end{array}$$

commutes up to homotopy.

Proof Assume first that G is Λ -free, with base e_1, \dots, e_r . Hence $P = X \cup_{\varphi_i} \cup_1^r D^3$, where $[\varphi_i] = e_i$, for $i = 1, \dots, r$. Since $S^2 \xrightarrow{\varphi_i} X \xrightarrow{f'} P'$ is null homotopic, the map $f' : X \rightarrow P'$ extends to $h : P \rightarrow P'$. Obviously, $h_* : \pi_q(P) \xrightarrow{\cong} \pi_q(P')$ for $q = 1, 2$. But h is of degree 1, hence by duality we get

$h_* : H_*(P; \Lambda) \xrightarrow{\cong} H_*(P'; \Lambda)$, so h is a homotopy equivalence by the Hurewicz-Whitehead theorem.

If G is stably Λ -free, i.e., $G \oplus \Lambda^{2a}$ is free, we first stabilize $X \sharp(\sharp_1^a S^2 \times S^2)$. Then as in the proof of Theorem 2.2, the map $X \sharp(\sharp_1^a S^2 \times S^2) \rightarrow P$ factors over X :

$$\begin{array}{ccccc}
 X \sharp(\sharp_1^a S^2 \times S^2) & \longrightarrow & X & \longrightarrow & P' \\
 & & & \searrow & \downarrow \cong \\
 & & & & P
 \end{array}$$

The above facts can be summarized as follows:

Corollary 2.4 *Given a PD_4 -complex X , there is a bijective correspondence between the following sets:*

$$\begin{array}{c}
 \{(G, \lambda_X|_G) \mid G \subset H_2(X; \Lambda) \text{ stably free } \Lambda\text{-module, } \lambda_X|_G \text{ non-singular}\} \\
 \updownarrow \\
 \{P \mid P : PD_4\text{-complex, } X \succ P\} / \text{homotopy equivalence.}
 \end{array}$$

3 Minimal PD_4 -complexes

We start this section by fixing a PD_4 -complex X .

Definition 3.1 We say that the PD_4 -complex P is a minimal PD_4 -complex for X (or X -minimal for short) if

1. We have $X \succ P$, and
2. Whenever $P \succ Q$ for some PD_4 -complex Q , then P is homotopy equivalent to Q .

The goal of this section is to show that X -minimal PD_4 -complexes exist. The following observation follows easily from the previous section.

Lemma 3.2 *Let $X \succ P_1 \succ P_2$ be realized by $f_0 : X \rightarrow P_1$ and $f_1 : P_1 \rightarrow P_2$. Then*

- (a) $K_2(f_1 \circ f_0, \Lambda) \cong K_2(f_0, \Lambda) \oplus K_2(f_1, \Lambda)$.
- (b) *Let s_{f_i} denote the splitting defined by f_i for $i = 0, 1$ (see Section 1, Remark (5) to recall the definition of s_{f_i}). Then we have*

$$s_{f_1 \circ f_0} = s_{f_0} \circ s_{f_1}.$$

Proof (a) The first assertion follows from the following isomorphisms

$$\begin{aligned}
 H_2(X; \Lambda) &\cong K_2(f_1 \circ f_0, \Lambda) \oplus H_2(P_2; \Lambda) \\
 &\cong K_2(f_0, \Lambda) \oplus K_2(f_1, \Lambda) \oplus H_2(P_2; \Lambda).
 \end{aligned}$$

(b) The second assertion follows from the degree 1-property of the maps f_0 and f_1 , and also by well-known formulas of the cup-(respectively cap) products. □

Suppose we are given an infinite sequence of PD_4 -complexes

$$P_0 = X \succ P_1 \succ P_2 \succ \dots \succ P_i \succ P_{i+1} \succ \dots$$

which are realized by

$$f_0: X \rightarrow P_1, \quad f_i: P_i \rightarrow P_{i+1} \quad i = 1, 2, \dots$$

Let Q be the direct limit of $\{P_i, f_i\}$, and let $f: X \rightarrow Q$ be the limit of the maps f_i . Note that in general, we cannot assume that Q is a PD_4 -complex. By Lemma 3.2 (a), we have

$$\begin{aligned} K_2(f, \Lambda) &\cong K_2(f_0, \Lambda) \oplus K_2(f_1, \Lambda) \oplus \dots \\ &= \bigoplus_0^\infty K_2(f_i, \Lambda). \end{aligned}$$

Lemma 3.3 *For f as above, we have $K_2(f, \Lambda) \subset H_2(X; \Lambda)$ as a direct summand.*

Proof Compatibility of direct limits with homology and exact sequences gives the following exact sequence

$$0 \rightarrow K_2(f, \Lambda) \rightarrow H_2(X; \Lambda) \rightarrow \varinjlim H_2(P_i; \Lambda) \rightarrow 0.$$

For the next argument it is convenient to write explicitly the following ladder:

$$\begin{array}{ccccccccccc} H_2(X; \Lambda) & \longrightarrow & H_2(P_1; \Lambda) & \longrightarrow & H_2(P_2; \Lambda) & \longrightarrow & \dots & \longrightarrow & H_2(P_i; \Lambda) & \longrightarrow & H_2(P_{i+1}; \Lambda) & \longrightarrow & \dots \\ \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & & & \uparrow \cong & & \uparrow \cong & & \\ H^2(X; \Lambda) & \longleftarrow & H^2(P_1; \Lambda) & \longleftarrow & H^2(P_2; \Lambda) & \longleftarrow & \dots & \longleftarrow & H^2(P_i; \Lambda) & \longleftarrow & H^2(P_{i+1}; \Lambda) & \longleftarrow & \dots \end{array}$$

with the obvious maps and isomorphisms. Property (b) of Lemma 3.2 gives inclusions

$$H_2(X; \Lambda) \supseteq H_2(P_1; \Lambda) \supseteq H_2(P_2; \Lambda) \supseteq \dots \supseteq H_2(P_i; \Lambda) \supseteq \dots$$

and

$$\varinjlim H_2(P_i; \Lambda) \cong \bigcap_0^\infty H_2(P_i; \Lambda).$$

Moreover,

$$s = \varinjlim s_i : \varinjlim H_2(P_i; \Lambda) \rightarrow H_2(X; \Lambda)$$

is a splitting of the above exact sequence.

Alternatively,¹ to obtain the above short exact sequence, one can use the fact that the homology group of the colimit is the colimit of the homology groups. Then the universal property of the colimit of the homology groups yields the splitting s .

Remark 3.4 In the proof above the direct limit is identified with the inverse limit

$$\varinjlim H_2(P_i; \Lambda) = \varprojlim H^2(P_i; \Lambda)$$

which in general is not equal to

$$H^2(\varinjlim P_i; \Lambda) = H^2(Q; \Lambda).$$

Theorem 3.5 *If $H_2(X; \Lambda)$ is a finitely generated Λ -module, then there are X -minimal PD_4 -complexes.*

Proof Note first that X itself can be X -minimal. This occurs if the following set

$$\{(G, \lambda_X|_G) \mid G \subset H_2(X; \Lambda) \text{ stably free } \Lambda - \text{ module, } \lambda_X|_G \text{ non-singular}\}$$

contains only the trivial submodule 0. If the above set contains non-trivial submodules, then one can choose an arbitrary G in it and construct the PD_4 -complex P_1 with $X \succ P_1$ as in Theorem 2.2. If P_1 is not X -minimal, then one takes an element G_1 from

$$\{(G, \lambda_{P_1}|_G) \mid G \subset H_2(P_1; \Lambda) \text{ stably free } \Lambda - \text{ module, } \lambda_{P_1}|_G \text{ non-singular}\}$$

giving $X \succ P_1 \succ P_2$. Continuing in this way, one obtains a sequence

$$X \succ P_1 \succ P_2 \succ \dots$$

realized by

$$f_0, f_1, f_2, \dots$$

By Lemma 3.3 we have the following splitting of $H_2(X; \Lambda)$

$$H_2(X; \Lambda) \cong \bigoplus_i K_2(f_i, \Lambda) \oplus H_2(Q; \Lambda).$$

Because $H_2(X; \Lambda)$ is finitely generated, the direct sum $\bigoplus_i K_2(f_i, \Lambda)$ is a finite direct sum, hence the sequence

$$X \succ P_1 \succ P_2 \succ \dots$$

¹ We thank the referee for providing us with this argument.

is finite of type

$$X \succ P_1 \succ P_2 \succ \dots \succ P_k.$$

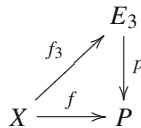
Hence P_k is a X -minimal PD_4 -complex. □

Note that the proof above indicates that in general presumably there might be more than one X -minimal PD_4 -complexes. One might consider the following question:

Problem 3.6 Give examples of several X -minimal PD_4 -complexes.

4 Postnikov decomposition of $X \succ P$

Let X, P be PD_4 -complexes such that $X \succ P$, realized by $f: X \rightarrow P$, which we may assume to be a fibration, and let $G = K_2(f, \Lambda)$. Then we have a decomposition [1, pp. 141–142],



where $p: E_3 \rightarrow P$ is a fibration with fiber $K(G, 2)$. The above diagram satisfies the following:

1. The map $p: E_3 \rightarrow P$ is 3-coconnected, i.e.,

$$\begin{aligned}
 p_*: \pi_q(E_3) &\rightarrow \pi_q(P) \text{ is an isomorphism for } q > 3, \\
 p_*: \pi_3(E_3) &\rightarrow \pi_3(P) \text{ is a monomorphism.}
 \end{aligned}$$

2. The map $f_3: X \rightarrow E_3$ is 3-connected, i.e.,

$$\begin{aligned}
 (f_3)_*: \pi_q(X) &\rightarrow \pi_q(E_3) \text{ is an isomorphism for } q = 1, 2, \\
 (f_3)_*: \pi_3(X) &\rightarrow \pi_3(E_3) \text{ is an epimorphism.}
 \end{aligned}$$

Taking mapping cylinders of f_3, p and f we get the following inclusions $X \subset E_3 \subset P$, and now properties (1) and (2) above become

- 1'. $\pi_q(P, E_3) = 0$ for $q \geq 4$.
- 2'. $\pi_q(E_3, X) = 0$ for $q \leq 3$.

Hence, up to homotopy equivalence E_3 can be constructed from X by attaching cells of dimension ≥ 4 , so $X^{(3)} = (E_3)^{(3)}$. In fact, this is the way E_3 is constructed. Moreover, we have $f|_{X^{(3)}} = p|_{(E_3)^{(3)}}$.

Now, note that $E_3 \xrightarrow{p} P$ is a $K(G, 2)$ fibration which is not necessarily simple, that is $\pi_1(P) \cong \pi$ does not have to act trivially on the homotopy group G of the fiber. We refer the reader to [9] for the details of the theory of non-simple fibrations.

There is a classifying space for $K(G, 2)$ -fibrations denoted by $\widehat{K}(G, 3)$ as described in [9]. Let $Q = K(\text{aut } G, 1)$ where $\text{aut } G$ is the group of isomorphisms of the Abelian group G . The universal covering space \widetilde{Q} is contractible and $\text{aut } G$ acts freely on it. Then

$$\widehat{K}(G, 3) = (K(G, 3) \times \widetilde{Q}) / \text{aut } G.$$

Here $K(G, 3)$ is interpreted as a topological(Abelian) group on which $\text{aut } G$ acts from the left. There is a universal $K(G, 2)$ -fibration over $\widehat{K}(G, 3)$ as described in [9, Section 2] which classifies $K(G, 2)$ -fibrations. Hence there is a classifying map $\widehat{k}_3: P \rightarrow \widehat{K}(G, 3)$ for $p: E_3 \rightarrow P$. Moreover there is an obvious fibration

$$K(G, 3) \longrightarrow \widehat{K}(G, 3) \xrightarrow{q} Q,$$

for which the null-element in $K(G, 3)$ gives a section $s: Q \rightarrow \widehat{K}(G, 3)$.

There is a $\pi_1(P)$ action on $G = K_2(f, \Lambda)$ and hence there is a homomorphism

$$\pi = \pi_1(X) \cong \pi_1(P) \xrightarrow{\rho} \text{aut } G$$

inducing $B\rho: B\pi_1 \rightarrow Q$, such that

$$\begin{array}{ccc} P & \xrightarrow{\widehat{k}_3} & \widehat{K}(G, 3) \\ \chi \downarrow & & \downarrow q \\ B\pi_1 & \xrightarrow{B\rho} & Q \end{array} \tag{4.1}$$

commutes. Here χ classifies the universal covering $\widetilde{P} \rightarrow P$.

Suppose $X' \succ P$ is realized by $f': X' \rightarrow P$. Set $G' = K_2(f', \Lambda)$ and $Q' = K(\text{aut } G', 1)$. We obtain a similar diagram

$$\begin{array}{ccc} P & \xrightarrow{\widehat{k}'_3} & \widehat{K}(G', 3) \\ \chi \downarrow & & \downarrow q' \\ B\pi_1 & \xrightarrow{B\rho'} & Q' \end{array} \tag{4.2}$$

Recall the non-degenerate hermitian forms

$$\lambda = \lambda_X|_G: G \times G \rightarrow \Lambda \quad \text{and} \quad \lambda' = \lambda_{X'}|_{G'}: G' \times G' \rightarrow \Lambda,$$

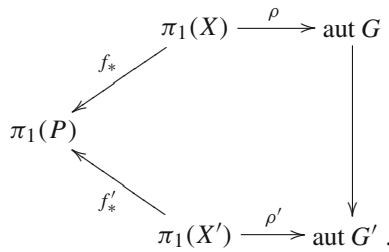
and let $p': E'_3 \rightarrow P$ be the $K(G', 2)$ fibration constructed from X' .

Proposition 4.1 *If $\Phi: G' \rightarrow G$ is an isometry, then $p': E'_3 \rightarrow P$ and $p: E_3 \rightarrow P$ are fiber homotopy equivalent.*

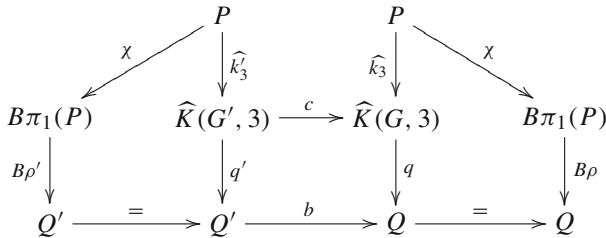
Proof The isometry Φ induces the following equivalences

$$\text{aut } G' \xrightarrow[\cong]{a} \text{aut } G, \quad Q' \xrightarrow[\cong]{b} Q \quad \text{and} \quad \widehat{K}(G', 3) \xrightarrow[\cong]{c} \widehat{K}(G, 3).$$

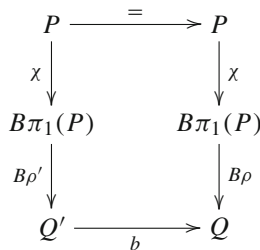
Note also that the definition of isometry includes commutativity of the following diagram:



All these maps induce maps between the diagrams (4.1) and (4.2) when \widehat{k}_3 and \widehat{k}'_3 are deleted. Therefore, we have $b \circ q' \circ \widehat{k}'_3 = q \circ \widehat{k}_3$ which can be seen from the diagram below.



All subdiagrams commute by the commutativity of the diagrams (4.1) and (4.2), by the hypothesis and construction of the maps b and c . Moreover, the following diagram



is commutative by hypothesis.

Hence we have

$$q \circ c \circ \widehat{k}'_3 = b \circ q' \circ \widehat{k}'_3 = b \circ B\rho' \circ \chi = B\rho \circ \chi = q \circ \widehat{k}_3.$$

Recall that we have the following fibration: $K(G, 3) \longrightarrow \widehat{K}(G, 3) \xrightarrow{q} Q$. Using obstruction theory as in [1, Chapter 4], particularly 4.29 (with local coefficients), the only obstruction for $c \circ \widehat{k}'_3 - \widehat{k}_3$ to be homotopic to the constant map belongs to

$$H^3(P; \pi_3(G, 3)) = H^3(P; G) \cong H_1(P; G).$$

The group on the right is trivial as $- \otimes_{\Lambda} G$ is right exact. Hence $c \circ \widehat{k}'_3, \widehat{k}_3: P \rightarrow \widehat{K}(G, 3)$ are homotopic maps. The result follows, since c is a homotopy equivalence. \square

Example 4.2 Suppose $G = \oplus_1^m \Lambda$, then $[K(G, 2)]^{(2)} = \vee_1^m (\vee_{g \in \pi_1} S_g^2)$ and π_1 acts on $\vee_{g \in \pi_1} S_g^2$ by permutation. This is the case when π_1 is the free group on l generators, and in this case $P = \sharp_1^l S^1 \times S^3$.

Next, we are going to show that $p_*: \pi_3(E_3) \rightarrow \pi_3(P)$ is an isomorphism. For this consider the following diagram of Whitehead sequences:

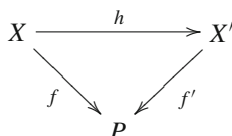
$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\pi_2(X)) & \longrightarrow & \pi_3(X) & \longrightarrow & H_3(X; \Lambda) \longrightarrow 0 \\ & & \cong \downarrow & & \downarrow & & \\ & & \Gamma(\pi_2(E_3)) & \longrightarrow & \pi_3(E_3) & \longrightarrow & H_3(E_3; \Lambda) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(\pi_2(P)) & \longrightarrow & \pi_3(P) & \longrightarrow & H_3(P; \Lambda) \longrightarrow 0. \end{array}$$

By Poincare duality, we have $p_* \circ (f_3)_* = f_*: H_3(X; \Lambda) \xrightarrow{\cong} H_3(P; \Lambda)$ an isomorphism, hence the map $H_3(E_3; \Lambda) \xrightarrow{P_*} H_3(P; \Lambda)$ is surjective. Since $\Gamma(\pi_2(X)) = \Gamma(\pi_2(P) \oplus G) \rightarrow \Gamma(\pi_2(P))$ is surjective too, note that $\pi_3(E_3) \xrightarrow{P_*} \pi_3(P)$ is also surjective. By Property (1), it is also injective, hence it must be an isomorphism.

5 Classification relative order

Let X, X' and P be PD_4 -complexes such that $X > P$ and $X' > P$ are realized by $f: X \rightarrow P$ and $f': X' \rightarrow P$. As before we set $G = K_2(f, \Lambda)$ and $G' = K_2(f', \Lambda)$.

The question we want to consider in this section is, when are X and X' homotopy equivalent over P ? In other words, does there exist a homotopy equivalence $h: X \rightarrow X'$ such that $f' \circ h$ is homotopic to f ?



Suppose such an h exists, then the following sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_2(f, \Lambda) & \longrightarrow & H_2(X; \Lambda) & \longrightarrow & H_2(P; \Lambda) \longrightarrow 0 \\
 & & \Phi \downarrow & & \Phi \downarrow & & = \downarrow \\
 0 & \longrightarrow & K_2(f', \Lambda) & \longrightarrow & H_2(X'; \Lambda) & \longrightarrow & H_2(P; \Lambda) \longrightarrow 0
 \end{array}
 \tag{5.1}$$

are isomorphic, namely $\Phi := h_*$ is an isometry. We are going to prove that this condition is also sufficient.

Theorem 5.1 *Suppose there is an isometry $\Phi: H_2(X; \Lambda) \rightarrow H_2(X'; \Lambda)$ satisfying the diagram (5.1). Then there is a homotopy equivalence $h: X \rightarrow X'$ over P inducing Φ .*

Remark 5.2 We should point out that our result gives a classification over the complex P , whereas Baues and Bleile [2] give classification result over $B\pi$ and Hillman [6, 8] gives a classification result over the strongly minimal model.

Proof Since there is an isometry $\Phi: K_2(f; \Lambda) \rightarrow K_2(f'; \Lambda)$, we have a homotopy equivalence g between the Postnikov systems

$$\begin{array}{ccc}
 E_3 & \xrightarrow{g} & E'_3 \\
 & \searrow p & \swarrow p' \\
 & & P
 \end{array}$$

We are going to denote $g|_{E_3^{(3)}}$ by \bar{h} , i.e., $\bar{h}: E_3^{(3)} = X^{(3)} \rightarrow X'^{(3)} = E_3'^{(3)}$ such that $\bar{h} = g|_{E_3^{(3)}}$. Let $X = X^{(3)} \cup_{\varphi} D^4$ and $X' = X'^{(3)} \cup_{\varphi'} D^4$, where $\varphi: S^3 \rightarrow X^{(3)}$ and $\varphi': S^3 \rightarrow X'^{(3)}$ are the attaching maps of the 4-cells [10, Theorem 2.4]. For simplicity, we denote $\varphi(S^3) = \partial D^4$ and $\varphi'(S^3) = \partial D^4$. The obstruction to extend \bar{h} over X belongs to

$$H^4(X; \pi_3(X')) \cong H_0(X; \pi_3(X')) = \pi_3(X') \otimes_{\Lambda} \mathbb{Z}.$$

This obstruction is given by $\bar{w} = w \otimes_{\Lambda} 1 = (\bar{h}_*[\partial D^4] - [\partial D^4]) \otimes_{\Lambda} 1$. We first consider the difference $w := \bar{h}_*[\partial D^4] - [\partial D^4] \in \pi_3(X'^{(3)})$.

Lemma 5.3 *The class $w \in \pi_3(X'^{(3)})$ maps to zero under the Hurewicz homomorphism $\pi_3(X'^{(3)}) \rightarrow H_3(X'^{(3)}; \Lambda)$.*

Proof We write $P = P^{(3)} \cup_{\psi} D^4$ and we have the following isomorphisms

$$f_*: H_4(X, X^{(3)}; \Lambda) \xrightarrow{\cong} H_4(P, P^{(3)}; \Lambda),$$

$$f'_*: H_4(X', X'^{(3)}; \Lambda) \xrightarrow{\cong} H_4(P, P^{(3)}; \Lambda)$$

by the degree-1 property of f and f' , respectively. Consider the diagram

$$\begin{array}{ccccccc}
 \pi_4(X, X^{(3)}) = H_4(X, X^{(3)}; \Lambda) & \longrightarrow & H_3(X^{(3)}; \Lambda) & \longrightarrow & H_3(X; \Lambda) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \cong \downarrow f_* & & \\
 \pi_4(P, P^{(3)}) = H_4(P, P^{(3)}; \Lambda) & \longrightarrow & H_3(P^{(3)}; \Lambda) & \longrightarrow & H_3(P; \Lambda) & \longrightarrow & 0 \\
 \uparrow & & \uparrow & & \cong \uparrow f'_* & & \\
 \pi_4(X', X'^{(3)}) = H_4(X', X'^{(3)}; \Lambda) & \longrightarrow & H_3(X'^{(3)}; \Lambda) & \longrightarrow & H_3(X'; \Lambda) & \longrightarrow & 0.
 \end{array}$$

The vertical maps are induced by f and f' , respectively. The rightmost and leftmost vertical maps are isomorphisms because of Poincaré duality and the degree-1 properties. Hence we have the following isomorphisms

$$f_* : H_3(X^{(3)}; \Lambda) \xrightarrow{\cong} H_3(P^{(3)}; \Lambda),$$

$$f'_* : H_3(X'^{(3)}; \Lambda) \xrightarrow{\cong} H_3(P^{(3)}; \Lambda).$$

It follows that $f_*[\partial D^4] = f'_*[\partial D^4]$. Also the diagram below commutes.

$$\begin{array}{ccc}
 H_3(X^{(3)}; \Lambda) & \xrightarrow{\bar{h}_*} & H_3(X'^{(3)}; \Lambda) \\
 \searrow f_* & & \swarrow f'_* \\
 & \cong & H_3(P^{(3)}; \Lambda)
 \end{array}$$

Hence $f'_* \circ \bar{h}_*[\partial D^4] = f_*[\partial D^4] = f'_*[\partial D^4]$ which implies $\bar{h}_*[\partial D^4] - [\partial D^4] = 0 \in H_3(X'^{(3)}; \Lambda)$. □

Note that if $w \in \pi_3(X'^{(3)})$ is zero, then \bar{h} extends to a map $h : X \rightarrow X'$ making the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{h} & X' \\
 \searrow f & & \swarrow f' \\
 & P &
 \end{array}$$

commutative up to homotopy. The above arguments show that the map h is then of degree 1, and it follows from this that h is a homotopy equivalence.

From the Whitehead sequence [12]

$$0 \longrightarrow \Gamma(\pi_2(X'^{(3)})) \longrightarrow \pi_3(X'^{(3)}) \longrightarrow H_3(X'^{(3)}; \Lambda) \longrightarrow 0,$$

it follows that $w \in \Gamma(\pi_2(X'^{(3)})) = \Gamma(\pi_2(X'))$ which has a decomposition

$$\Gamma(\pi_2(X')) \cong \Gamma(G' \oplus \pi_2) \cong \Gamma(\pi_2(P)) \oplus \pi_2(P) \otimes G' \oplus \Gamma(G').$$

We write $w = w_1 + w_2 + w_3$ according to this decomposition and since it suffices to show that $w \otimes_{\Lambda} 1 \in \pi_3(X'^{(3)}) \otimes_{\Lambda} \mathbb{Z}$ is zero, we have to prove:

1. $\bar{w}_1 = w_1 \otimes_{\Lambda} 1 = 0 \in \Gamma(\pi_2(P)) \otimes_{\Lambda} \mathbb{Z}$,
2. $\bar{w}_2 = w_2 \otimes_{\Lambda} 1 = 0 \in G' \otimes_{\Lambda} \pi_2(P) = G' \otimes_{\Lambda} H_2(P; \Lambda)$,
3. $\bar{w}_3 = w_3 \otimes_{\Lambda} 1 = 0 \in \Gamma(G') \otimes_{\Lambda} \mathbb{Z}$.

We are going to consider the above components one by one.

Lemma 5.4 *The element $\bar{w} \in \pi_3(X'^{(3)}) \otimes_{\Lambda} \mathbb{Z}$ maps to zero under the map induced by f' ,*

$$f'_* : \pi_3(X'^{(3)}) \otimes_{\Lambda} \mathbb{Z} \rightarrow \pi_3(P^{(3)}) \otimes_{\Lambda} \mathbb{Z}.$$

Proof Recall that $\bar{w} = w \otimes_{\Lambda} 1 = (\bar{h}_*[\partial D^4] - [\partial D^4]) \otimes_{\Lambda} 1$. Now, for every oriented PD_4 -complex Y with $Y = Y^{(3)} \cup_{\alpha} D^4$ the composite map $\bar{\partial}_Y$,

$$\begin{array}{c}
 \xrightarrow{\quad \bar{\partial}_Y \quad} \\
 H_4(Y; \mathbb{Z}) \longrightarrow H_4(Y, Y^{(3)}) \otimes_{\Lambda} \mathbb{Z} = \pi_4(Y, Y^{(3)}) \otimes_{\Lambda} \mathbb{Z} \longrightarrow \pi_3(Y^{(3)}) \otimes_{\Lambda} \mathbb{Z},
 \end{array}$$

sends $[Y]$ to $[\alpha] \otimes_{\Lambda} 1$. Consider the commutative diagram

$$\begin{array}{ccccc}
 H_4(X; \mathbb{Z}) & \xrightarrow{f_*} & H_4(P; \mathbb{Z}) & \xleftarrow{f'_*} & H_4(X'; \mathbb{Z}) \\
 \bar{\partial}_X \downarrow & & \downarrow & & \downarrow \bar{\partial}_{X'} \\
 \pi_3(X^{(3)}) \otimes_{\Lambda} \mathbb{Z} & \xrightarrow{f_*} & \pi_3(P^{(3)}) \otimes_{\Lambda} \mathbb{Z} & \xleftarrow{f'_*} & \pi_3(X'^{(3)}) \otimes_{\Lambda} \mathbb{Z}.
 \end{array} \tag{5.2}$$

Since the diagram

$$\begin{array}{ccc}
 X^{(3)} & \xrightarrow{\bar{h}} & X'^{(3)} \\
 & \searrow f & \swarrow f' \\
 & P &
 \end{array}$$

is homotopy commutative, we have $f'_* \circ \bar{h}_* = f_*$ also in the lower line of (5.2). The result then follows because we have $f_*[X] = [P] = f'_*[X']$. □

We have the following diagram of Whitehead sequences

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \Gamma(G') \oplus \pi_2(P^{(3)}) \otimes G' & & \Omega & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \Gamma(\pi_2(X^{(3)})) & \longrightarrow & \pi_3(X'^{(3)}) & \longrightarrow & H_3(X'^{(3)}; \Lambda) \longrightarrow 0 \\
 & & \downarrow \Gamma(f'_*) & & \downarrow f'_* & & \cong \downarrow f'^* \\
 & & \Gamma(\pi_2(P^{(3)})) & \longrightarrow & \pi_3(P^{(3)}) & \longrightarrow & H_3(P'^{(3)}; \Lambda) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Note that $\Gamma(f'_*)$ is induced from the split surjective homomorphism

$$\pi_2(X^{(3)}) \cong H_2(X'; \Lambda) \xrightarrow{f'_*} H_2(P; \Lambda) \cong \pi_2(P^{(3)}),$$

so $\Gamma(f'_*)$ is split surjective, too. Therefore $f'_*: \pi_3(X'^{(3)}) \rightarrow \pi_3(P'^{(3)})$ is surjective with

$$\text{Kernel} = \Omega \cong \Gamma(G') \oplus \pi_2(P^{(3)}) \otimes G'.$$

Now $- \otimes_{\Lambda} \mathbb{Z}$ is right exact, so we have exactness of

$$\Omega \otimes_{\Lambda} \mathbb{Z} \rightarrow \pi_3(X'^{(3)}) \otimes_{\Lambda} \mathbb{Z} \rightarrow \pi_3(P'^{(3)}) \otimes_{\Lambda} \mathbb{Z} \rightarrow 0.$$

It follows from Lemma 5.4 that the obstruction

$$(\bar{h}_*[\partial D^4] - [\partial D'^4]) \otimes_{\Lambda} 1 \in \text{Im}(\Omega \otimes_{\Lambda} \mathbb{Z} \rightarrow \pi_3(X'^{(3)}) \otimes_{\Lambda} \mathbb{Z}),$$

that is, it comes from $\Gamma(G') \otimes_{\Lambda} \mathbb{Z} \oplus \pi_2(P^{(3)}) \otimes_{\Lambda} G'$. This immediately implies that the component $\bar{w}_1 \in \Gamma(\pi_2(P)) \otimes_{\Lambda} \mathbb{Z}$ should vanish.

Corollary 5.5 *The component $\bar{w}_1 = w_1 \otimes_{\Lambda} 1 = 0$.*

To further analyze the obstruction we will use a map A' which relates our obstruction \bar{w} to intersection forms and cap products. For convenience we shall give the details in the following remark.

Remark 5.6 Let $Y = Y^{(3)} \cup_{\alpha} D^4$ be an oriented PD_4 -complex with $\alpha: S^3 \rightarrow Y^{(3)} \in \pi_3(Y^{(3)})$. Given $\beta: S^3 \rightarrow Y^{(3)} \in \pi_3(Y^{(3)})$, we denote the complexes $Y_{\beta} = Y^{(3)} \cup_{\beta} D^4$ and $Y_{\alpha+\beta} = Y^{(3)} \cup_{\alpha+\beta} D^4$ (in this notation $Y = Y_{\alpha}$). Then we have the following:

1. If $\beta \in \Gamma(\pi_2(Y^{(3)})) \subset \pi_3(Y^{(3)})$, then $H_4(Y_\beta; \mathbb{Z}) \cong \mathbb{Z}$ with generator $[Y_\beta]$ given by the top cell.
2. If $\beta \in \Gamma(\pi_2(Y^{(3)}))$, this also implies that $H_4(Y_{\alpha+\beta}; \mathbb{Z}) \cong \mathbb{Z}$ with a generator given by the top cell (see [3, Lemma 4.3]).

We define the map

$$\tilde{A}: \Gamma(\pi_2(Y^{(3)})) \rightarrow \text{Hom}_\Lambda(H^2(Y^{(3)}; \Lambda), H_2(Y^{(3)}; \Lambda))$$

by

$$\begin{array}{ccc} \beta & \longrightarrow & \cdot \cap [Y_\beta]: H^2(Y_\beta; \Lambda) \longrightarrow H_2(Y_\beta; \Lambda) \\ & & \parallel \qquad \qquad \qquad \parallel \\ & & H^2(Y^{(3)}; \Lambda) \qquad \qquad H_2(Y^{(3)}; \Lambda) \end{array}$$

Observe that we also have

$$\begin{array}{ccc} \cdot \cap [Y_{\alpha+\beta}]: H^2(Y_{\alpha+\beta}; \Lambda) & \longrightarrow & H_2(Y_{\alpha+\beta}; \Lambda) \\ \parallel & & \parallel \\ H^2(Y^{(3)}; \Lambda) & & H_2(Y^{(3)}; \Lambda) \end{array}$$

It is obvious that $\cdot \cap [Y_\beta] = \cdot \cap [Y_{\alpha+\beta}] - \cdot \cap [Y]$.

- (3) \tilde{A} induces a map $A: \Gamma(\pi_2(Y^{(3)})) \otimes_\Lambda \mathbb{Z} \rightarrow \text{Hom}_\Lambda(H^2(Y^{(3)}; \Lambda), H_2(Y^{(3)}; \Lambda))$. The map A can be seen to be the composite map

$$\Gamma(\pi_2(Y^{(3)})) \otimes_\Lambda \mathbb{Z} \longrightarrow H_2(Y^{(3)}; \Lambda) \otimes_\Lambda H_2(Y^{(3)}; \Lambda) \longrightarrow \text{Hom}_\Lambda(H^2(Y^{(3)}; \Lambda), H_2(Y^{(3)}; \Lambda)).$$

where the first map is induced from $\Gamma(\pi_2) \rightarrow \pi_2 \otimes \pi_2 = H_2(Y^{(3)}; \Lambda) \otimes H_2(Y^{(3)}; \Lambda)$ and the second one maps $u \otimes_\Lambda v \rightarrow \{\xi \rightarrow (\xi \cap u)v\}$.

Take now $X = X^{(3)} \cup_\varphi D^4$ giving

$$A: \Gamma(\pi_2(X^{(3)})) \otimes_\Lambda \mathbb{Z} \rightarrow \text{Hom}_\Lambda(H^2(X^{(3)}; \Lambda), H_2(X^{(3)}; \Lambda))$$

and similarly for $X' = X'^{(3)} \cup_{\varphi'} D^4$ one obtains the map A'

$$A': \Gamma(\pi_2(X'^{(3)})) \otimes_\Lambda \mathbb{Z} \rightarrow \text{Hom}_\Lambda(H^2(X'^{(3)}; \Lambda), H_2(X'^{(3)}; \Lambda)).$$

Recall that $\bar{h}: X^{(3)} \rightarrow X'^{(3)}$, and $w = \bar{h}_*[\partial D^4] - [\partial D^4] \in \pi_3(X')$ gives our obstruction $\bar{w} = w \otimes_\Lambda 1 \in \Gamma(\pi_2(X'^{(3)})) \otimes_\Lambda \mathbb{Z}$. We have $\bar{h}_*[\partial D^4] = [\partial D^4] + w$, so by Remark 5.6 (2)

$$[X'^{(3)} \cup_{\bar{h}\varphi} D^4] \in H_4(X'^{(3)} \cup_{\bar{h}\varphi} D^4; \mathbb{Z}) \cong \mathbb{Z}$$

is the canonical generator and $\bar{h}: X^{(3)} \rightarrow X'^{(3)}$ extends to the map

$$\tilde{h}: X = X^{(3)} \cup_{\varphi} D^4 \rightarrow X'^{(3)} \cup_{\bar{h}\varphi} D^4$$

in the obvious way. Hence $\tilde{h}_*[X] = [X'^{(3)} \cup_{\bar{h}\varphi} D^4]$.

Now by Remark 5.6 (2), we have

$$\begin{aligned} A'(\bar{w}) &= \cdot \cap [X'^{(3)} \cup_w D^4] = \cdot \cap [X'^{(3)} \cup_{\bar{h}\varphi} D^4] - \cdot \cap [X'^{(3)}] \\ &= \bar{h}_* \circ (\cap[X]) \circ \bar{h}^* - \cdot \cap [X']. \end{aligned}$$

We shall now prove that $\bar{w}_2 = 0$, and $\bar{w}_3 = 0$. According to the splittings

$$H_2(X'^{(3)}; \Lambda) = H_2(P^{(3)}; \Lambda) \oplus G' \text{ and } H^2(X'^{(3)}; \Lambda) = H^2(P^{(3)}; \Lambda) \oplus G'^*,$$

the map A' has components

$$\begin{aligned} A'_2: G' \otimes_{\Lambda} H_2(P^{(3)}) &\rightarrow \text{Hom}_{\Lambda}(G'^*, H_2(P^{(3)}; \Lambda)) \\ A'_3: \Gamma(G') \otimes_{\Lambda} \mathbb{Z} &\rightarrow \text{Hom}_{\Lambda}(G'^*, G'). \end{aligned}$$

Note that both maps are injective because G' is stably free.

We consider first $A'_3(\bar{w}_3)$. By our hypothesis the restriction of \bar{h}^* to G'^* is equal to Φ^* , similarly the map Φ is the restriction of \bar{h}_* . Moreover, the restriction of the cap product map $\cdot \cap [X]$ to G^* is equal to the inverse of the adjoint

$$\hat{\lambda}_X: H_2(X^{(3)}; \Lambda) \rightarrow \text{Hom}_{\Lambda}(H_2(X^{(3)}; \Lambda), \Lambda)$$

restricted to G , i.e., to $(\hat{\lambda}_{X|G})^{-1}$. Hence

$$A'_3(\bar{w}_3) = \Phi(\hat{\lambda}_{X|G})^{-1}\Phi^* - (\hat{\lambda}_{X'|G'})^{-1}.$$

But

$$\begin{array}{ccc} G & \xrightarrow{\hat{\lambda}_{X|G}} & G^* \\ \Phi \downarrow & & \uparrow \Phi^* \\ G' & \xrightarrow{\hat{\lambda}_{X'|G'}} & G'^* \end{array}$$

commutes because Φ is an isometry. This shows that $A'_3(\bar{w}_3) = 0$, hence $\bar{w}_3 = 0$.

We now come to

$$A'_2(\bar{w}_2): G'^* \rightarrow H_2(P^{(3)}; \Lambda).$$

For this we identify

$$G'^* = \text{coker}(H^2(P^{(3)}; \Lambda) \rightarrow H^2(X'^{(3)}; \Lambda)).$$

We have the following diagram

$$\begin{array}{ccccccc}
 & & & & & & H_2(P^{(3)}; \Lambda) \\
 & & & & & & \uparrow f_* \\
 & & & & & & \uparrow \\
 & & & & & & H_2(X'^{(3)}; \Lambda) \\
 H^2(X'^{(3)}; \Lambda) & \xrightarrow{\bar{h}^*} & H^2(X^{(3)}; \Lambda) & \xrightarrow{\cap[X]} & H_2(X^{(3)}; \Lambda) & \xrightarrow{\bar{h}_*} & H_2(X'^{(3)}; \Lambda) \\
 \downarrow & & \downarrow & & \uparrow & & \uparrow \\
 G'^* & \xrightarrow{\Phi^*} & G^* & \xrightarrow{\quad} & G & \xrightarrow{\Phi} & G'
 \end{array}
 \tag{5.3}$$

Commutativity of the right square follows from the hypothesis (5.1). Commutativity of the left square is a consequence of $\bar{h}: X^{(3)} \rightarrow X'^{(3)}$ being a map over $P^{(3)}$. Consider the composition $f_* \circ \bar{h}_* \circ (\cap[X]) \circ \bar{h}^*$. Its restriction to G'^* is the lower row of (5.3) followed by

$$G' \rightarrow H_2(X'^{(3)}; \Lambda) \rightarrow H_2(P^{(3)}; \Lambda),$$

hence is zero. Moreover, note also that the following

$$\begin{array}{ccc}
 H^2(X'^{(3)}; \Lambda) & \xrightarrow{\cdot \cap[X']} & H_2(X'^{(3)}; \Lambda) \longrightarrow H_2(P^{(3)}; \Lambda) \\
 \downarrow & & \uparrow \\
 G'^* & \xrightarrow{\quad} & G'
 \end{array}$$

implies that $\cdot \cap [X']$ restricted to G'^* has a vanishing component in $H_2(P^{(3)}; \Lambda)$.

Therefore $A'_2(\bar{w}_2) = 0$, implying $\bar{w}_2 = 0$. □

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