Browder-Livesay Filtrations and the Example of Cappell and Shaneson

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Abstract. Let M^3 be a 3-dimensional manifold with fundamental group $\pi_1(M)$ which contains a quaternion subgroup Q of order 8. In 1979 Cappell and Shaneson constructed a nontrivial normal map $f\colon M^3 \times T^2 \to M^3 \times S^2$ which cannot be detected by simply connected surgery obstructions along submanifolds of codimension 0, 1, or 2, but it can be detected by the codimension 3 Kervaire-Arf invariant. The proof of non-triviality of $\sigma(f) \in L_5(\pi_1(M))$ is based on consideration of a Browder-Livesay filtration of a manifold X with $\pi_1(X) \cong \pi_1(M)$. For a Browder-Livesay pair $Y^{n-1} \subset X^n$, the restriction of a normal map to the submanifold Y is given by a partial multivalued map $\Gamma\colon L_n(\pi_1(X)) \to L_{n-1}(\pi_1(Y))$, and the Browder-Livesay filtration provides an iteration Γ^n. This map is a basic step in the definition of the iterated Browder-Livesay invariants which give obstructions to realization of surgery obstructions by normal maps of closed manifolds. In the present paper we prove that $\Gamma^3(\sigma(f)) = 0$ for any Browder-Livesay filtration of a manifold X^{4k+1} with $\pi_1(X) \cong Q$. We compute splitting obstruction groups for various inclusions $\rho \to Q$ of index 2, describe natural maps in the braids of exact sequences, and make more precise several results about surgery obstruction groups of the group Q.

Mathematics Subject Classification (2010). Primary 57R67, 19J25; Secondary 55T99, 58A35, 18F25.

Keywords. Cappell and Shaneson example, surgery obstruction groups, closed manifolds surgery problem, splitting obstruction groups, Browder–Livesay invariants, Browder-Livesay groups, manifolds with filtration.

1. Introduction

Let M^3 be a 3-dimensional manifold with fundamental group $\pi_1(M)$ which contains the quaternion subgroup

$$Q = \{x, y | x^2 = y^2, yxy^{-1} = x^{-1}\} \tag{1.1}$$

of order 8. Consider a map

$$f\colon M^3 \times T^2 \longrightarrow M^3 \times S^2 \tag{1.2}$$

where $T^2 = S^1 \times S^1$ is the torus equipped with the Lie group invariant framing [1], and $T^2 \longrightarrow S^2$ is a normal map with a nontrivial Kervaire-Arf invariant. In 1979 Cappell and Shaneson [1] proved that surgery obstruction $\sigma(f)$ is nontrivial. The nontriviality of $\sigma(f)$ cannot be detected by simply connected surgery obstructions along submanifolds of codimension 0, 1, and 2, but it can be detected by the codimension 3 Arf-invariant (see [1] and [5]).

Recall that a pair of manifolds

$$Y^{n-1} \subset X^n \tag{1.3}$$

is called *a Browder-Livesay pair* if Y is a codimension one locally-flat closed submanifold and the natural inclusion $\pi_1(Y) \to \pi_1(X)$ induces an isomorphism of fundamental groups. In what follows we shall consider only the case when $\dim Y = n - 1 \geq 5$. For a Browder-Livesy pair, we can write down a push-out square of fundamental groups

$$\begin{array}{ccccccc}
\pi_1(\partial U) & \overset{\cong}{\Rightarrow} & \pi_1(X \setminus Y) & \rho & \to & \rho \\
\downarrow & & \downarrow & = \downarrow i_- & & \downarrow i_+ \\
\pi_1(Y) & \to & \pi_1(X) & G^- & \to & G^+
\end{array} \tag{1.4}$$

where ∂U is the boundary of a tubular neighborhood of Y in X, and $X \setminus Y$ is the closure of the complement of a tubular neighborhood. The groups in square (1.4) are equipped with orientation. The upper horizontal map and the vertical maps agree with orientations. The bottom horizontal map preserves the orientation on the images of the vertical maps which are inclusions of index 2 and reverses orientation outside of these images. We shall denote this fact by superscript "$+$" or "$-$". We shall omit this superscript if the orientation is clear from the context.

For a Browder-Livesay pair (1.3) we have a braid of exact sequences (see [3] and [8]):

$$\begin{array}{ccccccc}
\to L_n(\rho) & \xrightarrow{i_*} & L_n(G) & \xrightarrow{\partial} & LN_{n-2} \to \\
\nearrow \quad \searrow & & s \nearrow \quad \searrow & & \nearrow \quad \searrow \\
& LP_{n-1} & & \Gamma \downarrow \quad L_n(i_*) & & \\
\searrow \quad \nearrow & & q \searrow \quad \nearrow & & \searrow \quad \nearrow \\
\to LN_{n-1} & \xrightarrow{c} & L_{n-1}(G^-) & \xrightarrow{i^!_-} & L_{n-1}(\rho) \to
\end{array} \tag{1.5}$$

where $LP_{n-1} = LP_{n-1}(F) \cong L_n(i^!_-)$ are surgery obstruction groups for the manifold pair (X, Y), $LN_n = LN_n(\rho \to G)$ are the splitting obstruction groups, and $L_n(i_*)$ are the relative surgery obstruction groups for the inclusion map i (see [3], [8] and [9]). The horizontal rows in (1.5) are chain complexes and Γ is an isomorphism of the corresponding homology groups. The maps s and q are the natural forgetful maps, and the map c denotes passing from surgery problem inside the manifold X^n to an abstract surgery problem. For a normal map $f\colon M \to X$, the restriction to a transversal preimage $f|_{f^{-1}(Y)}\colon f^{-1}(Y) \to Y^{n-1}$ is a normal map to Y^{n-1}, and the

corresponding map of surgery obstruction groups is given by a partial multivalued map $\Gamma\colon L_n(\pi_1(X)) \to L_{n-1}(\pi_1(Y))$.

Let $\mathcal{X}$ be a filtration

$$X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X \tag{1.6}$$

of a closed n-dimensional manifold X by means of locally flat closed submanifolds. A filtration in (1.6) is called a *Browder-Livesay filtration*, if every pair of submanifolds (X_i, X_{i+1}) $(0 \le i \le k-1)$ is a Browder-Livesay pair (see [2], [5] and [7]). In what follows we shall assume that $\dim X_k = n - k \ge 5$.

Let F_i $(0 \le i \le k-1)$ be a square of fundamental groups in the splitting problem for a manifold pair (X_i, X_{i+1}) of a filtration in (1.6), $G_i = \pi_1(X_i)$, and $\rho_i = \pi_1(\partial U_{i+1}) \cong \pi_1(X_i \setminus X_{i+1})$, where ∂U_{i+1} is a boundary of a tubular neighborhood of X_{i+1} in X_i. Then $LN_*(\rho_i \to G_i)$ are the splitting obstruction groups for the manifold pair (X_i, X_{i+1}).

Every inclusion $\rho_i \to G_i$ of index 2 of oriented groups gives a commutative braid of exact sequences (1.5). Putting together central squares from these diagrams (see [4], [5] and [7]) we obtain a commutative diagram

$$
\begin{array}{cccc}
\longrightarrow & L_n(G_0) & \xrightarrow{\partial_0} & LN_{n-2}(\rho_0 \to G_0) \\
 & {}_s\nearrow \quad \searrow^p & & \\
LP_{n-1}(F_0) & \Gamma\downarrow \quad L_n(\rho_0 \to G_0) & & \\
 & {}_q\searrow \quad \nearrow_r & & \\
\longrightarrow & L_{n-1}(G_1) & \xrightarrow{\partial_1} & LN_{n-3}(\rho_1 \to G_1) \\
 & {}_s\nearrow \quad \searrow^p & & \\
LP_{n-2}(F_1) & \Gamma\downarrow \quad L_{n-1}(\rho_1 \to G_1) & & \\
 & {}_q\searrow \quad \nearrow_r & & \\
\longrightarrow & L_{n-2}(G_2) & \xrightarrow{\partial_2} & LN_n(\rho_2 \to G_2) \\
 & {}_s\nearrow \quad \searrow^p & & \\
LP_{n-3}(F_2) & \Gamma\downarrow \quad L_{n-2}(\rho_2 \to G_2) & & \\
 & {}_q\searrow \quad \nearrow_r & & \\
 & L_{n-3}(G_3) & & \\
 & \vdots & & \\
\longrightarrow & L_{n-m}(G_m) & \xrightarrow{\partial_m} & LN_{n-l}(\rho_m \to G_m) \\
 & {}_s\nearrow \quad \searrow^p & & \\
LP_{n-k}(F_m) & \Gamma\downarrow \quad L_{n-m}(\rho_m \to G_m) & & \\
 & {}_q\searrow \quad \nearrow_r & & \\
 & L_{n-k}(G_k) & &
\end{array}
\tag{1.7}
$$

where $m = k - 1$, $l = k + 1$.

In this diagram we denote by s, q, p, and r similar maps from different diagrams. However, in what follows, it will be clear from the context which map is under consideration. Note that the groups and the maps in diagram (1.7) are defined by the subscripts taken mod 4.

Now we recall an inductive definition of the sets

$$\Gamma^j(x) \subset L_{n-j}(G_j) \text{ for } (0 \le j \le k)$$

and the iterated Browder-Livesay j-invariants $(1 \le j \le k)$ with respect to filtration (1.6) (see [5] and [7]).

Definition 1.1. Let $x \in L_n(G_0)$. By definition,

$$\Gamma^0(x) = \{x\} \subset L_n(G_0).$$

The set $\Gamma^0(x)$ is said to be trivial if $x \in \text{Image}\{L_n(\rho_0) \to L_n(G_0)\}$. Let a set

$$\Gamma^j(x) \subset L_{n-j}(G_j) \quad (0 \le j \le k-1)$$

be defined. For $j \ge 1$, it is called trivial if $0 \in \Gamma^j(x)$.

If $\Gamma^j(x)(0 \le j \le k-1)$ is defined and nontrivial, then the $(j+1)$-th Browder-Livesay invariant with respect filtration (1.6) is the set

$$\partial_j(\Gamma^j(x)) \subset LN_{n-j-2}(\rho_{j-1} \to G_{j-1}).$$

The $(j+1)$-th invariant is nontrivial if $0 \notin \partial_j(\Gamma^j(x))$.

If the $(j+1) - th$ $(1 \le j \le k-1)$ Browder-Livesay invariant is defined and trivial then the set $\Gamma^{j+1}(x)$ is defined as

$$\Gamma^{j+1}(x) \stackrel{def}{=} \Gamma(\Gamma^j(x)) \stackrel{def}{=} \{qs^{-1}(z) | z \in \Gamma^j(x), \partial_j(z) = 0\} \subset L_{n-j-1}(G_{j+1}).$$

Theorem 1.2. (See [5].) *Let $x \in L_n(G_0)$ be an element with a nontrivial j-th Browder-Livesay invariant for some $j \ge 1$ relative to a Browder-Livesay filtration $\mathcal{X}$ of the manifold X. Then the element x cannot be realized by a normal map of closed manifolds.*

Note that the necessary condition for nontriviality of the j-th Browder-Livesay invariant of $x \in L_n(G_0)$ $(1 \le j)$ is nontriviality of $\Gamma^{j-1}(x)$.

Taking in (1.2) a 3-dimensional manifold M with $\pi_1(M) \cong Q$ we obtain a surgery obstruction $\sigma(f) \in L_1(Q)$ (see [1]). The proof of nontriviality $\sigma(f)$ in [1] is based on a consideration of the Browder-Livesay filtration

$$X_3 \subset X_2 \subset X_1 \subset X_0 = X, \quad (\dim X_3 \ge 5) \tag{1.8}$$

with $\pi_1(X) = Q$, $\pi_1(X_1) = Q^{+,-}$, $\pi_1(X_2) = Q^{-,-}$, and $\pi_1(X_3) = Q$. In particular, it follows from [1], that $\Gamma^3(\sigma(f)) = 0$ for filtration in (1.8).

In [5] was introduced the notion of a type of any element $x \in L_n(\pi)$ and it was proved that the elements of the first and the second type cannot be realized by normal maps of closed manifolds. For any element x of the second type there exists an "infinite" Browder-Livesay filtration for which $\Gamma^k(x)$ is nontrivial for all $k \ge 0$. Every element $x \in L_n(\pi)$ which lays in the subgroup generated by surgery obstructions of normal maps of closed manifolds has the third type. And if an element x has the third type, then for any Browder-Livesay filtration there exists a finite k, such that $\Gamma^k(x)$ is trivial. In all known to the authors cases $\Gamma^3(x)$ is trivial for the elements of the third type.

In the present paper, we prove that $\Gamma^3(\sigma(f))$ is trivial for any Browder-Livesay filtration of a manifold X^{4k+1} with $\pi_1(X) \cong Q$, where $\sigma(f) \in L_1(Q)$. We compute splitting obstruction groups LN_* and surgery obstruction groups LP_* for various inclusions $\rho \to Q$ of index 2, describe natural maps in the braids of exact sequences in (1.5), and make more precise several results about surgery obstruction groups of the group Q.

We use the surgery and splitting obstruction groups equipped with decoration "s" and we do not mention this in designations.

2. Browder-Livesay filtrations for a manifold X^{4k+1} with $\pi_1(X) = Q$

Let Q be the quaternion group from (1.1). We define an orientation homomorphism $w : Q \to \{\pm 1\}$ on generators x and y and denote it by superscripts in the following way:

$$w(x) = w(y) = 1, \quad (Q,w) = Q^+;$$

$$w(x) = 1, w(y) = -1, \quad (Q,w) = Q^{+,-};$$

$$w(x) = -1, w(y) = 1, \quad (Q,w) = Q^{-,+};$$

and

$$w(x) = -1, w(y) = -1, \quad (Q,w) = Q^{-,-} \quad \text{(in this case } w(xy) = 1\text{)}.$$

Let $\rho = \mathbb{Z}/4$ be the cyclic group with generator t. There are only two homomorphisms of orientation on the group ρ (homomorphisms $\mathbb{Z}/4$ into the group ± 1), and we shall write $\rho^+ = \mathbb{Z}/4^+$ in the case of trivial orientation and $\rho^- = \mathbb{Z}/4^-$ in the opposite case. The group Q has only three different subgroups of index 2 generated by x, y, and xy. All these subgroups are isomorphic to $\mathbb{Z}/4$.

Consider the Browder-Livesay filtration

$$X_3 \subset X_2 \subset X_1 \subset X_0 = X, \quad (\dim X_3 \geq 5) \tag{2.1}$$

of a manifold X with $\pi_1(X) \cong Q$.

Filtration in (2.1) yields a commutative diagram

$$\tag{2.2}$$

where every triangle corresponds to the square F_i, since we have isomorphisms of oriented groups $\rho_i = \pi_1(\partial U_{i+1}) \cong \pi_1(X_i \setminus X_{i+1})$ for $0 \leq i \leq 2$.

The diagram in (2.2) is commutative as a diagram of groups, and the skew maps are inclusions of index 2 preserving the orientations. The horizontal maps in (2.2) preserve the orientation on the images of skew maps and reverse the orientation outside these images. Two diagrams as in (2.2) are isomorphic if there exists an isomorphism between them preserving orientations.

Theorem 2.1. *Let X^n, $n \geq 8$, be a closed topological manifold with $\pi_1(X) \cong Q$. Then the diagram (2.2) is isomorphic to one of the following diagrams*

$$
\begin{array}{ccccccccc}
 & \mathbb{Z}/4^+ & & & \mathbb{Z}/4^- & & & \mathbb{Z}/4^+ & \\
k\swarrow & & \searrow k & j\swarrow & & \searrow j & i\swarrow & & \searrow i \\
Q^{-,+} & \overset{\cong}{\longrightarrow} & Q^{-,-} & & \overset{\cong}{\longrightarrow} & Q^{+,-} & & \overset{\cong}{\longrightarrow} & Q^+
\end{array}
\tag{2.3}
$$

where skew homomorphisms are given on the generator t by the maps $i(t) = x$, $j(t) = y$, $k(t) = xy$;

$$
\begin{array}{ccccccccc}
 & \mathbb{Z}/4^+ & & & \mathbb{Z}/4^+ & & & \mathbb{Z}/4^+ & \\
k\swarrow & & \searrow k & \swarrow & & \searrow & \swarrow & & \searrow \\
Q^{-,+} & \overset{\cong}{\longrightarrow} & Q^+ & & \overset{\cong}{\longrightarrow} & Q^{+,-} & & \overset{\cong}{\longrightarrow} & Q^+
\end{array}
\tag{2.4}
$$

where $k\colon \mathbb{Z}/4 \to Q$ is given on the generator t by the map $t \to y$ and other skew homomorphisms are given on the generator t by the map $t \to x$;

$$
\begin{array}{ccccccccc}
 & \mathbb{Z}/4^+ & & & \mathbb{Z}/4^+ & & & \mathbb{Z}/4^+ & \\
\swarrow & & \searrow & \swarrow & & \searrow & \swarrow & & \searrow \\
Q^{+,-} & \overset{\cong}{\longrightarrow} & Q^+ & & \overset{\cong}{\longrightarrow} & Q^{+,-} & & \overset{\cong}{\longrightarrow} & Q^+
\end{array}
\tag{2.5}
$$

where all homomorphisms $\mathbb{Z}/4 \to Q$ are given on the generator t by the map $t \to x$;

$$
\begin{array}{ccccccccc}
 & \mathbb{Z}/4^- & & & \mathbb{Z}/4^- & & & \mathbb{Z}/4^+ & \\
k\swarrow & & \searrow k & j\swarrow & & \searrow j & i\swarrow & & \searrow i \\
Q^{-,+} & \overset{\cong}{\longrightarrow} & Q^{-,-} & & \overset{\cong}{\longrightarrow} & Q^{+,-} & & \overset{\cong}{\longrightarrow} & Q^+
\end{array}
\tag{2.6}
$$

where skew homomorphisms are given on the generator t by the maps $i(t) = x$, $j(t) = y$, $k(t) = y$;

$$
\begin{array}{ccccccccc}
 & \mathbb{Z}/4^- & & & \mathbb{Z}/4^- & & & \mathbb{Z}/4^+ & \\
k\swarrow & & \searrow k & j\swarrow & & \searrow j & i\swarrow & & \searrow i \\
Q^{-,+} & \overset{\cong}{\longrightarrow} & Q^{-,-} & & \overset{\cong}{\longrightarrow} & Q^{+,-} & & \overset{\cong}{\longrightarrow} & Q^+
\end{array}
\tag{2.7}
$$

where skew homomorphisms are given on the generator t by the maps $i(t) = x$, $j(t) = y$, $k(t) = x$.

    For any diagram from the list (2.3)–(2.7) there exists a Browder-Livesay filtration (2.1) of the manifold X^n which gives this diagram.

Proof. There exist isomorphisms of the oriented groups

$$Q^{+,-} \cong Q^{-,+} \cong Q^{-,-}$$

since there exist automorphisms of the group Q permuting x, y, and xy. We have only three index 2 subgroups of Q generated by x, y, and xy, which are isomorphic to $\mathbb{Z}/4$. From this follows that all skew maps in (2.3) are inclusions of $\mathbb{Z}/4$ to Q preserving orientation. Now a consideration of various cases which agree with orientation conditions on the horizontal maps gives the first statement of the proposition. Consider the characteristic map $\phi\colon X^n \to RP^N$ (N sufficiently large) of the subgroup

$i(\mathbb{Z}/4^+) = i(\rho^+) \subset Q^+$ generated by x such that $\mathrm{Ker}\,\phi = \rho^+$. The transversal preimage of RP^{N-1} gives a submanifold $X_1 \subset X$ such that $X_1 \subset X$ is a Browder-Livesay pair with the commutative triangle

$$
\begin{array}{ccc}
 & \rho^+ & \\
 \swarrow & & \searrow \\
Q^+ \cong \pi_1(X) & \longrightarrow & \pi_1(X_1) \cong Q^{+,-}
\end{array}
$$

of groups (see [3] and [5]). Iteration of this construction gives the second statement of the theorem. $\qquad\square$

Every diagram from Theorem 2.1 induces diagram (1.7). The surgery obstruction groups fitting into the diagram (1.7) are well known [10]:

$$
\begin{array}{ccccccc}
i = & 0 & 1 & 2 & 3 \\
L_i(\mathbb{Z}/4^+) \cong & \mathbb{Z}^3 & 0 & \mathbb{Z} \oplus \mathbb{Z}/2 & \mathbb{Z}/2 \\
L_i(\mathbb{Z}/4^-) \cong & 0 & 0 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 \\
L_i(Q) \cong & (\mathbb{Z})^5 & |4| & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 \\
L_i(Q^{+,-}) \cong & |4| & 0 & \mathbb{Z} \oplus \mathbb{Z}/2 & |4|,
\end{array}
$$

where $|4|$ denotes a two-group of order 4.

Let $n = 4k+1$ and $\pi_1(X^n) \cong Q$. In this case commutative diagram (2.3) induces a diagram (see [1])

$$
\begin{array}{cccc}
0 = L_1(\mathbb{Z}/4^+) & \xrightarrow{0} & L_1(Q^+) & \\
 & & \nearrow \quad \searrow^p & \\
 & LP_0(F_0) & \Gamma\downarrow & L_1(\mathbb{Z}/4^+ \to Q^+) \\
 & & \searrow \quad \nearrow_r & \\
LN_0(\mathbb{Z}/4^+ \to Q^+) & \xrightarrow{0} & L_0(Q^{+,-}) & \\
 & & \| & \\
0 = L_0(\mathbb{Z}/4^-) & \xrightarrow{0} & L_0(Q^{+,-}) & \\
 & & \nearrow \quad \searrow^p & \\
 & LP_3(F_1) & \Gamma\downarrow & L_0(\mathbb{Z}/4^- \to Q^{+,-}) \quad (2.8) \\
 & & \searrow \quad \nearrow_r & \\
LN_3(\mathbb{Z}/4^- \to Q^{+,-}) & \xrightarrow{0} & L_3(Q^{-,-}) & \\
 & & \| & \\
\mathbb{Z}/2 \cong L_3(\mathbb{Z}/4^+) & \xrightarrow{\phi(mono)} & L_3(Q^{-,-}) & \\
 & & \nearrow \quad \searrow & \\
 & LP_2(F_2) & \Gamma\downarrow & L_3(\mathbb{Z}/4^+ \to Q^{-,-}) \\
 & & \searrow \quad \nearrow & \\
LN_2(\mathbb{Z}/4^+ \to Q^{-,-}) & \to & L_2(Q^{+,+}) &
\end{array}
$$

where the groups $LN_0(\mathbb{Z}/4^+ \to Q^+) = 0$ and $LN_3(\mathbb{Z}/4^- \to Q^{+,-}) = 0$ were computed in [1].

All maps p and r in (2.8) are monomorphisms, and the surgery obstruction $\sigma(f) \in L_1(Q^+)$ has the following properties [1]:

$$
p(\sigma(f)) = r(\sigma(g)) \quad \text{for some element} \quad \sigma(g) \in L_0(Q^{+,-}),
$$

$$p(\sigma(g)) = r(\sigma(h)) \text{ for some element } \sigma(h) \in L_3(Q^{-,-}),$$

and

$$\sigma(h) = \phi(a)$$

where $a \in L_3(\mathbb{Z}/4^+) \cong \mathbb{Z}/2$ is the nontrivial element.

Note, that the results of [1] immediately imply that $\Gamma^3(\sigma(f)) = 0$ for the diagram (2.3).

3. Computing Γ^i for Browder-Livesay filtrations

In this section we prove the following result: let X^{4k+1} ($4k + 1 \geq 9$) be a manifold with $\pi_1(X) \cong Q$. Consider the Browder-Livesay filtration of X with a diagram of inclusion (2.3) – (2.7). Then $\Gamma^3(\sigma(f)) = 0$ for the Cappell-Shaneson surgery obstruction $\sigma(f) \in L_1(Q^+)$.

The statement of the theorem for diagram (2.3) follows from diagram (2.8) and results of [1]. We shall give the statement of the theorem for other diagrams (2.4) – (2.7) in Theorems 3.1 and 3.4 of this section.

We introduce the following notations (see [10]):

$$T^{-,-} = \mathbb{Z}[Q^{-,-}], \quad \hat{T}_2^{-,-} = \hat{\mathbb{Z}}_2[Q^{-,-}],$$
$$R^+ = \mathbb{Z}[\mathbb{Z}/4^+], \quad \hat{R}_2^+ = \hat{\mathbb{Z}}_2[\mathbb{Z}/4^+],$$

and similarly for other orientations. Denote by $L_n^{rel}(T^{-,-})$ the relative groups $L_n(T^{-,-} \longrightarrow T_2^{-,-})$ fitting into the relative exact sequence

$$\rightarrow L_n(T^{-,-}) \longrightarrow L_n(\hat{T}_2^{-,-}) \longrightarrow L_n(T^{-,-} \longrightarrow T_2^{-,-}) \longrightarrow L_{n-1}(T^{-,-}) \rightarrow$$

and similarly for other group rings and orientations. ¿From the isomorphisms $Q^{+,-} \cong Q^{-,+} \cong Q^{-,-}$ which were described in Section 2, we obtain isomorphisms

$$T^{-,-} \cong T^{-,+} \cong T^{+,-},$$
$$\hat{T}_2^{-,-} \cong \hat{T}_2^{+,-} \cong \hat{T}_2^{-,+}$$

and isomorphisms

$$L_n^Y(T^{-,-} \longrightarrow T_2^{-,-}) \cong L_n^Y(T^{+,-} \longrightarrow T_2^{+,-}) \cong L_n^Y(T^{-,+} \longrightarrow T_2^{-,+}).$$

We shall consider the L-groups with decorations "prime". For the case of a group ring $\mathbb{Z}[\pi]$ these L-groups coincide with surgery obstruction groups. We have isomorphisms (see [1] and [10])

$$LN_n(\mathbb{Z}/4^+ \rightarrow Q^{+,+}) \cong L_n\left(\mathbb{Z}[\mathbb{Z}/4], \mathrm{Id}, -t^2\right) = L_n(B)$$

and the relative exact sequence

$$\rightarrow L_n\left(\mathbb{Z}[\mathbb{Z}/4], \mathrm{Id}, -t^2\right) \rightarrow L_n(\hat{\mathbb{Z}}_2[\mathbb{Z}/4], \mathrm{Id}, -t^2) \rightarrow L_0^{rel}\left(\mathbb{Z}[\mathbb{Z}/4], \mathrm{Id}, -t^2\right) \rightarrow$$

Denote

$$L_n(B) = L_n\left(\mathbb{Z}[\mathbb{Z}/4], \mathrm{Id}, -t^2\right)$$

and

$$L_n(\hat{B}_2) = L_n\left(\hat{\mathbb{Z}}_2[\mathbb{Z}/4], \mathrm{Id}, -t^2\right).$$

By [1], we have

$$LN_n(\mathbb{Z}/4^- \xrightarrow{j} Q^{-,-}) \cong L_n\left(\mathbb{Z}[\mathbb{Z}/4], D, \pm t^2\right) = L_n(A)$$

where $D(t) = -t$. In particular, we have an isomorphism

$$LN_n(\mathbb{Z}/4^- \xrightarrow{j} Q^{-,-}) \cong LN_{n+2}(\mathbb{Z}/4^- \xrightarrow{j} Q^{-,-}).$$

Denote

$$L_n(A) = L_n\left(\mathbb{Z}[\mathbb{Z}/4], D, \pm t^2\right)$$

and

$$L_n(\hat{A}_2) = L_n\left(\hat{\mathbb{Z}}_2[\mathbb{Z}/4], D, \pm t^2\right).$$

These groups fit into the relative exact sequence

$$\to L_n\left(\mathbb{Z}[\mathbb{Z}/4], D, \pm t^2\right) \to L_n(\hat{\mathbb{Z}}_2[\mathbb{Z}/4], D, \pm t^2) \to L_n^{rel}\left(\mathbb{Z}[\mathbb{Z}/4], D, \pm t^2\right) \to$$

Braid of exact sequences (1.5) also exists for relative groups (see [6] and [10]). For the inclusions $\mathbb{Z}/4^\pm \to Q^{+,\pm}$, all relative groups and all groups for the the group ring over the ring $\hat{\mathbb{Z}}_2$ are known [10]. In particular, we have

n	$=$	0	1	2	3
$L_n^Y(R^+ \to R_2^+)$	$\cong$	0	$\mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2$	0	$\mathbb{Z}$
$L_n^Y(T^+ \to T_2^+)$	$\cong$	0	$\mathbb{Z}^5 \oplus (\mathbb{Z}/2)^4$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
$L_n^Y(B \to \hat{B}_2)$	$\cong$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0	$(\mathbb{Z})^2 \oplus (\mathbb{Z}/2)^2$
$L_n^Y(R^- \to R_2^-)$	$\cong$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	0	0
$L_n^Y(T^{-,-} \to T_2^{-,-})$	$\cong$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	0	$\mathbb{Z}$
$L_n^Y(A \to \hat{A}_2)$	$\cong$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

(3.1)

Consider the diagram

$$
\begin{array}{cccc}
0 = L_1(\mathbb{Z}/4^+) & \xrightarrow{0} & L_1(Q^+) & \\
& s\nearrow & \searrow \cong & \\
& \Gamma\downarrow & & L_1(\mathbb{Z}/4^+ \to Q^+) \\
& q\searrow & \cong \nearrow & \\
0 = LN_0(\mathbb{Z}/4^+ \to Q^+) & \xrightarrow{0} & L_0(Q^{+,-}) & \\
& & \| & \\
L_0(\mathbb{Z}/4^+) & \xrightarrow{Im=\mathbb{Z}/2} & L_0(Q^{+,-}) & \\
& s\nearrow & \searrow & \\
& \Gamma\downarrow & & L_0(\mathbb{Z}/4^+ \to Q^{+,-}) = \mathbb{Z}/2 \\
& q\searrow & \nearrow & \\
\mathbb{Z}/2 = LN_3(\mathbb{Z}/4^+ \to Q^{+,-}) & \xrightarrow{mono} & L_3(Q^{+,+}) = (\mathbb{Z}/2)^2 & \\
& & \| & \\
L_3(\mathbb{Z}/4^+) & \xrightarrow{mono} & L_3(Q^{+,+}) & \\
& s\nearrow & \searrow^{epi} & \\
& \Gamma\downarrow & & L_3(\mathbb{Z}/4^+ \to Q^{-,-}) = \mathbb{Z}/2 \\
& q\searrow & \nearrow & \\
LN_2(\mathbb{Z}/4^+ \to Q^{+,+}) & \longrightarrow & L_2(Q^{+,-}). &
\end{array}
$$

(3.2)

which is defined by diagram (2.4). Diagram (3.2) is similar to diagram (2.8). In the group $L_0(Q^{+,-})$ lies the element $\sigma(g)$ which is the image of the element $\sigma(f) \in L_1(Q^+)$ under the upper map Γ in (3.1) as follows from [1].

Theorem 3.1. *In diagram* (3.2), *the element* $\sigma(g) \in L_0(Q^{+,-})$ *lies in the image of the map*

$$L_0(\mathbb{Z}/4^+) \longrightarrow L_0(Q^{+,-}),$$

and hence $\Gamma^2(\sigma(f)) = 0$. *From this it immediately follows that* $\Gamma^2(\sigma(f)) = 0$ *for diagrams* (2.4) *and* (2.5).

Proof. Follows by Lemma 3.2 and Lemma 3.3.

Lemma 3.2. *Let* a *be the nontrivial element of the relative group* $L_0(T^{-,-} \to \hat{T}_2^{-,-}) = \mathbb{Z}/2$ *and*

$$\partial \colon L_0(T^{-,-} \to \hat{T}_2^{-,-}) \to L_3(T^{-,-})$$

be a map from the corresponding relative exact sequence. Then

$$\partial(a) = \sigma(h) = \phi(x), \quad where \quad x \in L_3(\mathbb{Z}/4^+), \quad x \neq 0.$$

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}/4^+ & \to & Q^{+,+} \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 & \to & \mathbb{Z}/2^+ \oplus \mathbb{Z}/2^+
\end{array}
\tag{3.3}
$$

in which vertical maps are natural projections. By [10], the vertical maps in (3.3) induce a retraction of the braid of exact sequences of relative groups for the top inclusion in (3.3) to the corresponding braid of exact sequences of relative groups for the bottom inclusion in (3.3).

As the kernel of this retraction we obtain a braid of exact sequences with known groups and easily computed maps. All relative groups for the inclusion $\mathbb{Z}/2^+ \to \mathbb{Z}/2^+ \oplus \mathbb{Z}/2^+$ are known from [10]:

$$
\begin{array}{ccccccc}
n & = & 0 & 1 & & 2 & 3 \\
L_n(\mathbb{Z}[\mathbb{Z}/2] \to \hat{\mathbb{Z}}_2[\mathbb{Z}/2]) & \cong & 0 & (\mathbb{Z} \oplus \mathbb{Z}/2)^2 & 0 & 0 \\
L_n(\mathbb{Z}[(\mathbb{Z}/2)^2] \to \hat{\mathbb{Z}}_2[(\mathbb{Z}/2)^2]) & \cong & 0 & (\mathbb{Z} \oplus \mathbb{Z}/2)^4 & 0 & 0.
\end{array}
$$

All relative groups for the inclusion $\mathbb{Z}/4^+ \to Q^{+,+}$ were given in (3.1) (see [10]). From this we conclude that the map

$$\mathbb{Z}/2 = L_0^{rel}\big(\mathbb{Z}[\mathbb{Z}/4], \mathrm{Id}, -t^2\big) \xrightarrow{\xi} L_0^{rel}(T^{+,-}) = \mathbb{Z}/2 \tag{3.4}$$

of relative groups is an isomorphism. The map ξ from (3.4) lies in the commutative diagram

$$
\begin{array}{ccccc}
0 = L_0\big(\mathbb{Z}[\mathbb{Z}/4], \mathrm{Id}, -t^2\big) & \longrightarrow & L_0(\hat{\mathbb{Z}}_2[\mathbb{Z}/4], \mathrm{Id}, -t^2) & \xrightarrow{mono} & L_0^{rel}\big(\mathbb{Z}[\mathbb{Z}/4], \mathrm{Id}, -t^2\big) = \mathbb{Z}/2 \\
\downarrow 0 & & \downarrow & & \cong \downarrow \xi \\
|4| \cong L_0(T^{+,-}) & \xrightarrow{epi} & L_0(\hat{T}_2^{+,-})(= \mathbb{Z}/2) & \xrightarrow{0} & L_0^{rel}(T^{+,-}) = \mathbb{Z}/2
\end{array}
\tag{3.5}
$$

in which top row is a part of relative exact sequence for an inclusion $B \to \hat{B}_2$, the left group in top row is trivial by [1], the bottom row is the part of the relative exact sequence for an inclusion $T^{+,-} \to \hat{T}_2^{+,-}$, and the group and the maps in the bottom row were described in [10]. Since the map ξ is an isomorphism, we obtain from commutativity of diagram (3.5) that

$$L_0(\hat{\mathbb{Z}}_2[\mathbb{Z}/4], \mathrm{Id}, -t^2) = 0.$$

Consider a part

$$
\begin{array}{ccccccc}
\mathbb{Z}/2 = L_3(\hat{R}_2^+) & \to & L_3(\hat{T}_2^{-,-})(=\mathbb{Z}/2) & \to & L_3(\hat{\mathbb{Z}}_2[\mathbb{Z}/4],\mathrm{Id},-t^2) & \to & L_1(\hat{T}_2^{+,+}) \\
\downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma \\
0 = L_0(\hat{\mathbb{Z}}_2[\mathbb{Z}/4],\mathrm{Id},-t^2) & \to & L_2(\hat{T}_2^{+,+})(=\mathbb{Z}/2) & \to & L_2(\hat{R}_2) & \overset{\cong}{\to} & L_2(\hat{T}_2^{-,-})
\end{array}
\tag{3.6}
$$

of the two-row diagram for the inclusion $\hat{R}_2^+ \to \hat{T}_2^{-,-}$ which corresponds to the inclusion $\mathbb{Z}/4^+ \to Q^{-,-}$ given by the map $t \to xy$. The rows of diagram (3.6) are chain complexes with isomorphic homology groups. The right bottom horizontal map is an isomorphism $\mathbb{Z}/2 \to \mathbb{Z}/2$ (Arf-invariant), hence the homology group in the member $L_2(\hat{T}_2^{+,+})$ is $\mathbb{Z}/2$. Hence in top row the homology group in the member $L_3(\hat{T}_2^{-,-})$ is $\mathbb{Z}/2$ and hence the induced by the inclusion $\mathbb{Z}/4^+ \to Q^{-,-}$ $(t \to xy)$ map

$$
\alpha : \mathbb{Z}/2 = L_3(\hat{R}_2^+) \to L_3(\hat{T}_2^{-,-}) = \mathbb{Z}/2
\tag{3.7}
$$

is trivial.

The inclusion $\mathbb{Z}/4^+ \to Q^{-,-}$ $(t \to xy)$ induces the map of relative exact sequences, and we obtain the commutative diagram

$$
\begin{array}{ccccc}
0 = L_0(R^+ \to \hat{R}_2^+) & \overset{0}{\to} & L_3(R^+) = \mathbb{Z}/2 & \overset{\cong}{\to} & L_3(\hat{R}_2^+) = \mathbb{Z}/2 \\
\downarrow & & \phi\downarrow mono & & 0\downarrow \\
\mathbb{Z}/2 = L_0(T^{-,-} \to \hat{T}_2^{-,-}) & \overset{mono}{\underset{\partial}{\to}} & L_3(T^{-,-}) = |4| & \overset{epi}{\to} & L_3(\hat{T}_2^{-,-}) = \mathbb{Z}/2
\end{array}
\tag{3.8}
$$

where the right hand vertical map is trivial as we have proved. Now the statement of Lemma 3.2 follows from diagram (3.8). $\square$

For the inclusion $\mathbb{Z}/4^- \to Q^{+,-}$ $(t \to y)$ we consider the following part of diagram (3.2)

$$
\begin{array}{ccc}
L_0(Q^{+,-}) & & \\
 & \searrow^{p} & \\
 & & L_0(\mathbb{Z}/4^- \to Q^{+,-}) \\
 & \nearrow_{r} & \\
L_3(Q^{-,-}) & &
\end{array}
\tag{3.9}
$$

where $\sigma(g) \in L_0(Q^{+,-})$ and $\sigma(h) \in L_3(Q^{-,-})$. We have a natural map of the corresponding diagram of relative groups to diagram (3.2) and hence a map of a diagram of relative groups to diagram (3.9). Thus we obtain the commutative diagram

$$
\begin{array}{ccc}
0 \neq b \in (\mathbb{Z}/2)^2 = L_1(T^{+,-} \to \hat{T}_2^{+,-}) & \overset{Im=\mathbb{Z}/2}{\underset{\delta}{\to}} & L_0(Q^{+,-}) \ni \sigma(g) \\
\downarrow epi & & p\downarrow mono \\
\mathbb{Z}/2 = L_1\begin{pmatrix} R^- & \to & \hat{R}_2^- \\ \downarrow & & \downarrow \\ T^{+,-} & \to & \hat{T}_2^{+,-} \end{pmatrix} & \overset{mono}{\to} & L_0(\mathbb{Z}/4^- \to Q^{+,-}) \\
\uparrow\cong & & r\uparrow mono \\
0 \neq a \in \mathbb{Z}/2 = L_0(T^{-,-} \to \hat{T}_2^{-,-}) & \overset{mono}{\underset{\partial}{\to}} & L_3(Q^{-,-}) \ni \sigma(h)
\end{array}
\tag{3.10}
$$

in which the maps p and r are monomorphisms by [1]. The middle horizontal map in (3.10) is a monomorphism since the diagonal map

$$
L_0(T^{-,-} \to \hat{T}_2^{-,-}) \to L_0(\mathbb{Z}/4^- \to Q^{+,-})
$$

from bottom square is a monomorphism as follows from Lemma 3.2. The left hand upper vertical map in (3.10) is an epimorphism by [10]. Hence from commutativity of (3.10) we obtain that there exists an element

$$b \in L_1(T^{+,-} \to \hat{T}_2^{+,-})$$

such that $\delta(b) = \sigma(g) \in L_0(Q^{+,-})$.

Consider the map of relative exact sequence of the inclusion $R^+ \to \hat{R}_2^+$ to relative exact sequence of the inclusion $T^{+,-} \to \hat{T}_2^{+,-}$ which is induced by the inclusion $\mathbb{Z}/4^+ \to Q^{+,-}$. We can write down the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}^3 \oplus (\mathbb{Z}/2)^3 = L_1(R^+ \to \hat{R}_2^+) & \xrightarrow{epi} & L_0(R^+) = \mathbb{Z}^3 \\
\downarrow Im = \mathbb{Z}/2 & & \downarrow \\
0 \neq b \in (\mathbb{Z}/2)^2 = L_1(T^{+,-} \to \hat{T}_2^{+,-}) & \xrightarrow[\delta]{Im = \mathbb{Z}/2} & L_0(T^{+,-}) = |4| \ (\ni \sigma(g) = \delta(b))
\end{array}
\tag{3.11}
$$

where the left hand vertical map follows from [10].

Lemma 3.3. *The map*

$$\mathbb{Z}^3 = L_0(R^+) \xrightarrow{i_*} L_0(T^{+,-})$$

in diagram (3.11) *has the image* $\mathbb{Z}/2$ *and hence the element* $\sigma(g)$ *lies in the image of this map.*

Proof. Commutative diagram (3.3) induces a commutative diagram

$$
\begin{array}{ccc}
LN_1(\mathbb{Z}/4^+ \to Q^{+,+}) & \longrightarrow & L_3(Q^{+,+}) = (\mathbb{Z}/2)^2 \\
\downarrow epi & & \downarrow mono \\
LN_1(\mathbb{Z}/2^+ \to \mathbb{Z}/2^+ \oplus \mathbb{Z}/2^+) & \xrightarrow{mono} & L_3(\mathbb{Z}/2^+ \oplus \mathbb{Z}/2^+) = (\mathbb{Z}/2)^3 \\
|| & & \\
L_3(\mathbb{Z}/2^+) = \mathbb{Z}/2 & &
\end{array}
\tag{3.12}
$$

in which the right vertical map is a monomorphism by [1] and the left vertical map is an epimorphism as follows from the long exact sequence of the pull-back diagram (see [1])

$$
\begin{array}{ccc}
(\mathbb{Z}[\mathbb{Z}/4], Id, -t^2) & \to & (\mathbb{Z}[i], Id, 1) \\
\downarrow & & \downarrow \\
(\mathbb{Z}[\mathbb{Z}/2], Id, -1) & \to & (\mathbb{Z}/2[\mathbb{Z}/2], Id, 1),
\end{array}
$$

where $LN_i(\mathbb{Z}/4^+ \to Q^{+,+}) = L_i(\mathbb{Z}[\mathbb{Z}/4], Id, -t^2)$. Thus the image of the upper horizontal map in (3.12) is $\mathbb{Z}/2$. Now consider the following part of the two-row diagram for the inclusion $\mathbb{Z}/4^+ \to Q^{+,-}$

$$
\begin{array}{ccccccc}
0 = L_1(Q^{+,-}) & \longrightarrow & LN_1(\mathbb{Z}/4^+ \to Q^{+,+}) & \xrightarrow{Im = \mathbb{Z}/2} & L_3(Q^{+,+}) = (\mathbb{Z}/2)^2 & \longrightarrow & L_3(\mathbb{Z}/4^+) \\
\downarrow \Gamma & & \downarrow \Gamma & & \downarrow \Gamma & & \downarrow \Gamma \\
\mathbb{Z}^5 = L_0(Q^{+,+}) & \xrightarrow{coker = \mathbb{Z}/2} & L_0(\mathbb{Z}/4^+) = \mathbb{Z}^3 & \xrightarrow{i_*} & L_0(Q^{+,-}) = |4| & \longrightarrow & LN_0 = 0
\end{array}
\tag{3.13}
$$

in which rows are chain complexes with isomorphic homology groups and vertical maps Γ are isomorphisms of homologies. The map

$$\mathbb{Z}^5 = L_0(Q^{+,+}) \to L_0(\mathbb{Z}/4^+) = \mathbb{Z}^3$$

has cokernel $\mathbb{Z}/2$ as follows from consideration of the corresponding diagram of relative groups. It follows now that image i_* in (3.13) is $\mathbb{Z}/2$ and the lemma is proved.   □

Now Theorem 3.1 is proved.   □

On the base of diagram (2.6) we can write down the following commutative diagram of surgery obstruction groups

$$
\begin{array}{lllll}
0 = L_1(\mathbb{Z}/4^+) & \overset{0}{\to} & L_1(Q^+) & & \\
 & & \nearrow \quad \searrow p & & \\
 & & \Gamma \downarrow \qquad L_1(\mathbb{Z}/4^+ \to Q^+) & & \\
 & & \searrow \quad \nearrow r & & \\
0 = LN_0(\mathbb{Z}/4^+ \to Q^+) & \overset{0}{\to} & L_0(Q^{+,-}) & & \\
 & & \| & & \\
0 = L_0(\mathbb{Z}/4^+) & \overset{j_*=0}{\longrightarrow} & L_0(Q^{+,-}) & & \\
 & & \nearrow \quad \searrow p & & \\
 & & \Gamma \downarrow \qquad L_0(\mathbb{Z}/4^- \to Q^{+,-}) & & \quad (3.14) \\
 & & \searrow \quad \nearrow r & & \\
LN_3(\mathbb{Z}/4^- \to Q^{+,-}) & \overset{0}{\to} & L_3(Q^{-,-}) & & \\
 & & \| & & \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \cong L_3(\mathbb{Z}/4^-) & \overset{j_*}{\longrightarrow} & L_3(Q^{-,-}) & & \\
 & & \nearrow \quad \searrow & & \\
 & & \Gamma \downarrow \qquad L_3(\mathbb{Z}/4^- \to Q^{-,-}) & & \\
 & & \searrow \quad \nearrow & & \\
LN_2(\mathbb{Z}/4^- \to Q^{-,-}) & \longrightarrow & L_2(Q^{+,-}) & &
\end{array}
$$

which is similar to diagram (2.8). By [1], $\sigma(g) \in L_0(Q^{+,-})$, $\sigma(h) \in L_3(Q^{-,-})$, and $p(\sigma(g)) = r(\sigma(h))$.

Theorem 3.4. *The map*

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 = L_3(\mathbb{Z}/4^-) \overset{j_*}{\longrightarrow} L_3(Q^{-,-})$$

in diagram (3.14) is an isomorphism. Hence $\Gamma^3(\sigma(f)) \in L_2(Q^{+,-})$ is trivial for diagram (3.14). From this it follows that $\Gamma^3(\sigma(f)) = 0$ for diagrams (2.6) and (2.7).

Proof. For the inclusion $j : \mathbb{Z}/4^- \to Q^{+,-}$, consider a two-row diagram of relative L-groups, which is part of the braid of exact sequences of relative L-groups

$$
\begin{array}{cccccc}
\mathbb{Z}/2 & \mathbb{Z}/2 & 0 & 0 & 0 & 0 \\
\| & \| & \| & \| & \| & \| \\
\overset{j_*^!}{\to} L_0^{rel}(R^-) \to L_0^{rel}(T^-) \to & L_2^{rel}(A) \to & L_2^{rel}(T^-) \to & L_2^{rel}(R^-) \to & L_2^{rel}(T^-) \to \\
\downarrow\Gamma \qquad \downarrow\Gamma & \downarrow\Gamma & \downarrow\Gamma & \downarrow\Gamma & \downarrow\Gamma \\
\to L_3^{rel}(A) \to L_3^{rel}(T^-) \to & L_3^{rel}(R^-) \to & L_3^{rel}(T^-) \to & L_1^{rel}(A) \to & L_1^{rel}(T^-) \to \\
\| \qquad \| & \| & \| & \| & \| \\
\mathbb{Z} \qquad \mathbb{Z} & 0 & \mathbb{Z} & \mathbb{Z} & (\mathbb{Z}/2)^2
\end{array}
\quad (3.15)
$$

$$
\begin{array}{cc}
0 & \mathbb{Z}/2 \\
\| & \| \\
\to L_0^{rel}(A) \longrightarrow L_0^{rel}(T^-) \overset{j^!}{\to} \\
\downarrow\Gamma \qquad\qquad \downarrow\Gamma \\
\to L_1^{rel}(R^-) \overset{j_*}{\longrightarrow} L_1^{rel}(T^-) \to \\
\| \qquad\qquad \| \\
(\mathbb{Z}/2)^2 \qquad (\mathbb{Z}/2)^2
\end{array}
$$

Now we shall prove Lemmas 3.5, 3.6, 3.7, and 3.8 which yields a proof of Theorem 3.4.

Lemma 3.5. *In diagram* (3.15), *the map*

$$\mathbb{Z} = L_3^{rel}(A) \longrightarrow L_3^{rel}(T^-) = \mathbb{Z}$$

is isomorphism, and the map

$$\mathbb{Z} = L_3^{rel}(T^-) \longrightarrow L_1^{rel}(A) = \mathbb{Z}$$

is a multiplication by 2 with coimage $\mathbb{Z}/2$.

Proof. Suppose that the map

$$\mathbb{Z}/2 = L_0^{rel}(T^-) \overset{j^!}{\longrightarrow} L_0^{rel}(R^-) = \mathbb{Z}/2$$

in diagram (3.15) is an isomorphism. Then, comparing homology of top and bottom rows of diagram (3.15), we obtain a contradiction. Hence the map

$$\mathbb{Z}/2 = L_0^{rel}(R^-) \overset{j_*}{\longrightarrow} L_0^{rel}(T^-) = \mathbb{Z}/2$$

in top row of (3.15) is an isomorphism. From this the result follows by diagram chasing in diagram (3.15). $\qquad\square$

For an inclusion $j : \mathbb{Z}/4^- \to Q^-$, we have a two-row diagram of L-groups

$$
\begin{array}{ccccccccccc}
0 & & |4| & & ? & & \mathbb{Z}\oplus\mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}\oplus\mathbb{Z}/2 \\
\| & & \| & & \| & & \| & & \| & & \| \\
\to L_0(\mathbb{Z}/4^-) & \to & L_0(Q^-) & \to & LN_2 & \to & L_2(Q^-) & \to & L_2(\mathbb{Z}/4^-) & \to & L_2(Q^-) \to \\
\downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma \\
\to LN_3 & \to & L_3(Q^-) & \to & L_3(\mathbb{Z}/4^-) & \to & L_3(Q^-) & \to & LN_1 & \to & L_1(Q^-)\to \\
\| & & \| & & \| & & \| & & \| & & \| \\
0 & & |4| & & (\mathbb{Z}/2)^2 & & |4| & & 0 & & 0
\end{array}
\qquad (3.16)
$$

$$
\begin{array}{cc}
? & |4| \\
\| & \| \\
\to LN_0 \longrightarrow L_0(Q^-) \overset{j^!}{\to} \\
\downarrow\Gamma \qquad\qquad \downarrow\Gamma \\
\to L_1(\mathbb{Z}/4-) \longrightarrow L_1(Q^-) \to \\
\| \qquad\qquad \| \\
0 \qquad\qquad 0
\end{array}
$$

Lemma 3.6. *In diagram* (3.16), *the map*

$$\mathbb{Z}/2 = L_2(\mathbb{Z}/4^-) \overset{j_*}{\longrightarrow} L_2(Q^-) = \mathbb{Z} \oplus \mathbb{Z}/2$$

is a monomorphism, and hence the map

$$L_2(Q^-) \overset{j^!}{\longrightarrow} L_2(\mathbb{Z}/4^-)$$

is trivial.

Proof. Consider the commutative diagram of oriented groups

$$
\begin{array}{ccc}
\mathbb{Z}/4^- & \xrightarrow{j(t)=y} & Q^{+,-} \\
\downarrow{\scriptstyle t^2=1} & & \downarrow{\scriptstyle x^2=y^2=1} \\
\mathbb{Z}/2^- & \longrightarrow & \mathbb{Z}/2^+ \oplus \mathbb{Z}/2^-.
\end{array}
\tag{3.17}
$$

Diagram (3.17) induces the following diagram of L-groups

$$
\begin{array}{ccc}
\mathbb{Z}/2 = L_2(\mathbb{Z}/4^-) & \xrightarrow{j_*} & L_2(Q^{+,-}) \\
\downarrow{\scriptstyle \cong} & & \downarrow \\
\mathbb{Z}/2 = L_2(\mathbb{Z}/2^-) & \xrightarrow{\cong} & L_2(\mathbb{Z}/2^+ \oplus \mathbb{Z}/2^-) = \mathbb{Z}/2
\end{array}
\tag{3.18}
$$

in which the left vertical map is isomorphism (Arf-invariant is preserved), and the bottom horizontal map is an isomorphism (the bottom horizontal map is an inclusion of a direct summand). Now we obtain the monomorphism of top row in (3.18). The result follows. $\qquad\square$

Lemma 3.7. *We have isomorphisms* [6]

$$
LN_2(\mathbb{Z}/4^- \to Q_3^-) \cong LN_0(\mathbb{Z}/4^- \to Q_3^-) \cong \mathbb{Z} \oplus \mathbb{Z}/2
$$

and the map

$$
\mathbb{Z} \cong L_{2n+1}^{rel}(A) \to L_{2n}(A) \cong LN_{2n}(\mathbb{Z}/4^- \to Q_3^-)
$$

is an inclusion of a direct summand.

Proof. Consider a commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z} = L_3^{rel}(A) & \xrightarrow{\cong} & \mathbb{Z} = L_3^{rel}(T^-) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial_1} \\
LN_2 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}/2 = L_2(Q^-)
\end{array}
\tag{3.19}
$$

in which vertical maps fit into the corresponding relative exact sequences. The right vertical map ∂_1 in (3.19) is an inclusion of a direct summand by [10], and upper horizontal map is an isomorphism by Lemma 3.5. Hence the left vertical map in (3.19) is an inclusion of a direct summand. Now from diagram (3.16) we obtain the diagram

$$
\begin{array}{ccccccc}
|4| & & ? & & \mathbb{Z}/2 & & \\
\| & & \| & & \| & & \\
0 \to L_0(Q^-) & \to & LN_2/\{\mathrm{Im}\ \partial\} & \to & L_2(Q^-)/\{\mathrm{Im}\ \partial_1\} & \to 0 \\
\downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} & & \\
0 \to L_3(Q^-) & \to & L_3(\mathbb{Z}/4^-) & \xrightarrow{j_*} & L_3(Q^-) & \to 0 \\
\| & & \| & & \| & & \\
|4| & & (\mathbb{Z}/2)^2 & & |4| & &
\end{array}
\tag{3.20}
$$

in which the homology groups in the corresponding places of upper and bottom rows are isomorphic by isomorphisms Γ. Hence we have the isomorphism

$$
LN_2/\{\mathrm{Im}\ \partial\} \cong \mathbb{Z}/2
$$

and

$$LN_2 \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

The isomorphism $LN_2 \cong LN_0$ follows from 2-periodicity.　　　　　　　□

Lemma 3.8. *The map*

$$\mathbb{Z} \oplus \mathbb{Z}/2 = LN_2 \to L_2(Q^{+,-}) = \mathbb{Z} \oplus \mathbb{Z}/2$$

in diagram (3.16) *is an isomorphism*

Proof. Consider the commutative diagram

$$
\begin{array}{ccccc}
\mathbb{Z}\oplus\mathbb{Z}/2\cong LN_0(\mathbb{Z}/4^-{\to}Q^{-,-}) & \stackrel{\cong}{\longrightarrow} & L_0\big(\mathbb{Z}[\mathbb{Z}/4],D,t^2\big) & \stackrel{epi}{\longrightarrow} & L_0(Q^{+,-}) \\
\downarrow{epi} & & \downarrow{t^2=1} & & \downarrow{x^2=y^2=1} \\
\mathbb{Z}/2\cong LN_0(\mathbb{Z}/2^-{\to}\mathbb{Z}/2^-\oplus\mathbb{Z}/2^-) & \stackrel{\cong}{\longrightarrow} & L_0(\mathbb{Z}/2^-) & \stackrel{\cong}{\longrightarrow} & L_0(\mathbb{Z}/2^+\oplus\mathbb{Z}/2^-)
\end{array}
\tag{3.21}
$$

in which the top right hand horizontal map is an epimorphism as follows from (3.16). Hence the left vertical map provides an isomorphism of the torsion part $\mathbb{Z}/2$ of the group LN_0 on the group $L_0(\mathbb{Z}/2^-) = \mathbb{Z}/2$ (the direct summand $\mathbb{Z}$ of the group LN_0 maps trivially since it factors acroos the corresponding map of relative groups which is trivial).

Now we can write down a commutative diagram

$$
\begin{array}{ccccc}
\mathbb{Z}\oplus\mathbb{Z}/2\cong LN_2(\mathbb{Z}/4^-{\to}Q^{-,-}) & \stackrel{\cong}{\longrightarrow} & L_2\big(\mathbb{Z}[\mathbb{Z}/4],D,t^2\big) & \longrightarrow & L_2(Q^{+,-})\cong\mathbb{Z}\oplus\mathbb{Z}/2 \\
\downarrow{epi} & & \downarrow{t^2=1} & & \downarrow{x^2=y^2=1} \\
\mathbb{Z}/2\cong LN_2(\mathbb{Z}/2^-{\to}\mathbb{Z}/2^-\oplus\mathbb{Z}/2^-) & \stackrel{\cong}{\longrightarrow} & L_2(\mathbb{Z}/2^-) & \stackrel{\cong}{\longrightarrow} & L_2(\mathbb{Z}/2^+\oplus\mathbb{Z}/2^-)
\end{array}
\tag{3.22}
$$

in which the left vertical map induces an isomorphism of the torsion part $\mathbb{Z}/2$ of the group LN_2 on the group $L_2(\mathbb{Z}/2^-) = \mathbb{Z}/2$ by 2-periodicity. The direct summand $\mathbb{Z}$ of the group LN_2 maps isomorphically to the direct summand $\mathbb{Z}$ of the group $L_2(Q^{+,-})$ as follows from Lemma 3.5 and Lemma 3.7. The diagram (3.22) provides a similar result about torsion subgroups. Lemma 3.8 is proved. Now Theorem 3.4 follows.　　　　　　　□

　　This finishes the proof of Theorem 3.4.　　　　　　　□

4.　Braids of exact sequences

In this section we give explicit results about braids of exact sequences of L_*-groups for various index 2 inclusions $\rho \to Q$. The results of this section are direct corollaries of computations in Section 3 and [1], [6], and [10].

Theorem 4.1. *For an inclusion* $j\colon \mathbb{Z}/4^- \to Q^-$, *we have the following two-row diagram of L-groups*

$$
\begin{array}{ccccccccc}
0 & & (\mathbb{Z}/2)^2 & & \mathbb{Z}\oplus\mathbb{Z}/2 & & \mathbb{Z}\oplus\mathbb{Z}/2 & & \mathbb{Z}/2 \\
\| & & \| & & \| & & \| & & \| \\
\to L_0(\mathbb{Z}/4^-) \longrightarrow & L_0(Q^-) & \stackrel{0}{\longrightarrow} & LN_2 & \stackrel{\cong}{\longrightarrow} & L_2(Q^-) & \stackrel{j^!=0}{\longrightarrow} & L_2(\mathbb{Z}/4^-)\to \\
\downarrow\Gamma & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma \\
\to LN_3 & \longrightarrow L_3(Q^-) & \stackrel{0}{\longrightarrow} & L_3(\mathbb{Z}/4^-) & \stackrel{\cong}{\longrightarrow} & L_3(Q^-) & \longrightarrow & LN_1\to \\
\| & \| & & \| & & \| & & \| \\
0 & (\mathbb{Z}/2)^2 & & (\mathbb{Z}/2)^2 & & (\mathbb{Z}/2)^2 & & 0
\end{array}
\tag{4.1}
$$

$$
\begin{array}{ccccc}
\mathbb{Z}\oplus\mathbb{Z}/2 & & \mathbb{Z}\oplus\mathbb{Z}/2 & & (\mathbb{Z}/2)^2 \\
\| & & \| & & \| \\
\xrightarrow[mono]{j_*} L_2(Q^-) & \xrightarrow{Coker=(\mathbb{Z}/2)^2} & LN_0 & \xrightarrow{epi} & L_0(Q^-)\xrightarrow{j^!=0} \\
\downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma \\
\longrightarrow L_1(Q^-) & \longrightarrow & L_1(\mathbb{Z}/4^-) \longrightarrow & & L_1(Q^-)\rightarrow \\
\| & & \| & & \| \\
0 & & 0 & & 0
\end{array}
$$

Corollary 4.2. *There exist isomorphisms*

$$L_0(Q^-) \cong L_3(Q^-) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Corollary 4.3. *Let $Y \subset X$ be be a Browder-Livesay pair of manifolds with a push-out square of fundamental groups*

$$
\Psi = \begin{array}{ccc}
\pi_1(\partial U) & \xrightarrow{\cong} & \pi_1(X \setminus Y) \\
\downarrow & & \downarrow \\
\pi_1(Y) & \longrightarrow & \pi_1(X)
\end{array}
\quad = \quad
\begin{array}{ccc}
\mathbb{Z}/4^- & \longrightarrow & \mathbb{Z}/4^- \\
\downarrow & & \downarrow \\
Q^- & \longrightarrow & Q^-
\end{array}
$$

in the corresponding splitting problem. Then we have the following isomorphisms

$$LP_n(\Psi) = (\mathbb{Z}/2)^2, \mathbb{Z}/2, \mathbb{Z} \oplus (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^2$$

for $n = 0, 1, 2, 3 \mod 4$, respectively.

Theorem 4.4. *For an inclusion $j\colon \mathbb{Z}/4^+ \to Q^{+,-}$, we have the following two-row diagram of L-groups*

$$
\begin{array}{ccccccccc}
0 & & 0 & & \mathbb{Z}/2 & & (\mathbb{Z}/2)^2 & & \mathbb{Z}/2 \\
\| & & \| & & \| & & \| & & \| \\
\rightarrow L_1(\mathbb{Z}/4) & \longrightarrow & L_1(Q^-) & \xrightarrow{0} & LN_1 & \xrightarrow{mono} & L_3(Q) & \xrightarrow{0} & L_3(\mathbb{Z}/4)\rightarrow \\
\downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma \\
\rightarrow LN_2 & \xrightarrow{mono} & L_0(Q) & \xrightarrow{Coker=\mathbb{Z}/2} & L_0(\mathbb{Z}/4) & \xrightarrow{Im=\mathbb{Z}/2} & L_0(Q^-) & \longrightarrow & LN_0\rightarrow \\
\| & & \| & & \| & & \| & & \| \\
\mathbb{Z}^2 & & \mathbb{Z}^5 & & \mathbb{Z}^3 & & (\mathbb{Z}/2)^2 & & 0
\end{array}
\tag{4.2}
$$

$$
\begin{array}{ccccc}
& & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 \\
& & \| & \| & \| \\
\xrightarrow{mono} & L_3(Q^-) & \xrightarrow{0} LN_3 & \xrightarrow{\cong} L_1(Q)\xrightarrow{0} & \\
& \downarrow\Gamma & \downarrow\Gamma & \downarrow\Gamma & \\
\longrightarrow & L_2(Q) & \xrightarrow{0} L_2(\mathbb{Z}/4) & \xrightarrow{\cong} L_2(Q^-)\xrightarrow{0} & \\
& \| & \| & \| & \\
& \mathbb{Z}/2 & \mathbb{Z}\oplus\mathbb{Z}/2 & \mathbb{Z}\oplus\mathbb{Z}/2 &
\end{array}
$$

where $LN_i = LN_i(\mathbb{Z}/4 \to Q^+) \cong LN_{i+2}(\mathbb{Z}/4 \to Q^-)$.

Corollary 4.5. *There exists an isomorphism*

$$L_1(Q^+) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Corollary 4.6. *Let $Y \subset X$ be a Browder-Livesay pair of manifolds with a push-out square of fundamental groups*

$$
\Psi_1 = \begin{array}{ccc}
\pi_1(\partial U) & \xrightarrow{\cong} & \pi_1(X \setminus Y) \\
\downarrow & & \downarrow \\
\pi_1(Y) & \longrightarrow & \pi_1(X)
\end{array}
\quad = \quad
\begin{array}{ccc}
\mathbb{Z}/4^+ & \longrightarrow & \mathbb{Z}/4^+ \\
\downarrow & & \downarrow \\
Q^{+,+} & \longrightarrow & Q^{+,-}
\end{array}
$$

in the corresponding splitting problem. Then we have the following isomorphisms

$$LP_n(\Psi_1) = \mathbb{Z}^2, \mathbb{Z} \oplus (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3$$

for $n = 0, 1, 2, 3 \mod 4$, respectively.

Theorem 4.7. *For an inclusion* $j \colon \mathbb{Z}/4^+ \to Q^+$, *we have the following two-row diagram of L-groups*

$$
\begin{array}{ccccccccc}
0 & & (\mathbb{Z}/2)^2 & & (\mathbb{Z}/2)^2 & & (\mathbb{Z}/2)^2 & & \mathbb{Z}/2 \\
\| & & \| & & \| & & \| & & \| \\
\to L_1(\mathbb{Z}/4) & \longrightarrow & L_1(Q) & \overset{0}{\longrightarrow} & LN_3 & \overset{\cong}{\longrightarrow} & L_3(Q^-) & \overset{0}{\longrightarrow} & L_3(\mathbb{Z}/4^-) \overset{mono}{\longrightarrow} \\
\downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma \\
\to LN_0 & \longrightarrow & L_0(Q^-) & \overset{0}{\longrightarrow} & L_0(\mathbb{Z}/4) & \overset{mono}{\longrightarrow} & L_0(Q) & \overset{epi}{\longrightarrow} & LN_2 \overset{0}{\longrightarrow} \\
\| & & \| & & \| & & \| & & \| \\
0 & & (\mathbb{Z}/2)^2 & & \mathbb{Z}^3 & & \mathbb{Z}^5 & & \mathbb{Z}^2
\end{array}
$$
(4.3)

$$
\begin{array}{ccccccc}
(\mathbb{Z}/2)^2 & & \mathbb{Z}/2 & & 0 \\
\| & & \| & & \| \\
\overset{mono}{\longrightarrow} L_3(Q) & \overset{0}{\longrightarrow} & LN_1 & \longrightarrow & L_1(Q^-)\to \\
\downarrow\Gamma & & \downarrow\Gamma & & \downarrow\Gamma \\
\overset{0}{\longrightarrow} L_2(Q^-) & \overset{Coker=(\mathbb{Z}/2)^2}{\underset{Ker=\mathbb{Z}/2}{\longrightarrow}} & L_2(\mathbb{Z}/4) & \overset{epi}{\longrightarrow} & L_2(Q)\to \\
\| & & \| & & \| \\
\mathbb{Z}\oplus\mathbb{Z}/2 & & \mathbb{Z}\oplus\mathbb{Z}/2 & & \mathbb{Z}/2
\end{array}
$$

where $LN_i = LN_i(\mathbb{Z}/4 \to Q^+) \cong LN_{i+2}(\mathbb{Z}/4 \to Q^-)$.

Corollary 4.8. *Let* $Y \subset X$ *be be a Browder-Livesay pair of manifolds with a push-out square of fundamental groups*

$$
\Psi_2 = \begin{array}{ccc}
\pi_1(\partial U) & \overset{\cong}{\longrightarrow} & \pi_1(X \setminus Y) \\
\downarrow & & \downarrow \\
\pi_1(Y) & \longrightarrow & \pi_1(X)
\end{array}
= \begin{array}{ccc}
\mathbb{Z}/4^+ & \longrightarrow & \mathbb{Z}/4^+ \\
\downarrow & & \downarrow \\
Q^{+,-} & \longrightarrow & Q^{+,+}
\end{array}
$$

in the corresponding splitting problem. Then we have the following isomorphisms

$$LP_n(\Psi_2) = (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^2, \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2$$

for $n = 0, 1, 2, 3 \mod 4$, respectively.

Acknowledgment

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