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A Survey on the Generalized R. L. Moore Problem

Abstract. *We give an updated extended survey of results related to the celebrated unsolved generalized R. L. Moore problem. In particular, we address the problem of characterizing codimension one manifold factors, i.e. spaces X having the property that $X \times \mathbb{R}$ is a topological manifold. A main part of the paper is devoted to many efficient general position techniques, which have been used to solve special cases of this problem.*

Key Words: *Upper semicontinuous decomposition, Generalized manifold, Cell-like resolution, General position property, Codimension one manifold factor, Generalized Moore Problem.*

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1. Introduction

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. The Generalized R. L. Moore Problem can be formulated as follows:

Suppose that G is a cellular (or cell-like) upper semicontinuous decomposition of $\mathbb{R}^n$, where $n \geq 3$. Is then $(\mathbb{R}^n/G) \times \mathbb{R}$ homeomorphic to $\mathbb{R}^{n+1}$?

This is a classical problem that has remained unsolved for over sixty years. One major reason for the importance of this problem lies in its potential applications to manifold recognition problems such as the famous Buse-

mann Conjecture and Bing-Borsuk Conjecture – see our recent survey [37].

In 1980, Daverman published a most excellent survey on the Generalized R. L. Moore Problem [21]. In the present paper, we extend this discussion to significant developments since that point in time. Although we provide a few of the most relevant details from the earlier period, we refer the reader to Daverman's survey for a more thorough discussion. Our survey will focus on developments after 1980, especially with respect to general position strategies.

2. Background

The Generalized R. L. Moore Problem first emerged in the investigation of manifold recognition type problems in the mid 1900's. A decomposition G of a Hausdorff space S into compact subsets is *upper semicontinuous* if and only if the decomposition map $\pi : S \to S/G$ is closed. An early result of R. L. Moore states [46]:

THEOREM 2.1. (R. L. Moore Theorem) *If G is an upper semicontinuous decomposition of $\mathbb{R}^2$ into continua which do not separate $\mathbb{R}^2$, then the decomposition space $\mathbb{R}^2/G$ is homeomorphic to $\mathbb{R}^2$.*

Could this result be generalized to higher dimensions? Moore's theorem was proved by appealing to a topological characterization of the plane. Since no analogous characterizations for higher dimensional manifolds existed at the time, generalizations of his theorem to higher dimensions using the same approach were not possible.

The first breakthrough in attacking higher dimensional problems came during the 1950's when Bing developed a shrinkability criterion, which is used to define shrinkability in the following definition:

DEFINITION 2.2. An upper semicontinuous decomposition G of a space X is said to be *shrinkable* if and only if G satisfies the following shrinkability criterion: for each open cover $\mathcal{U}$ of X/G and for each open cover $\mathcal{V}$ of X there exists a homeomorphism h of X onto itself satisfying:
  1. To each $x \in X$ there corresponds $U \in \mathcal{U}$ such that $\{x, h(x)\} \subset \pi^{-1}(U)$; and
  2. To each $g \in G$ there corresponds $V \in \mathcal{V}$ such that $h(g) \subset V$.

With various minor hypotheses placed on the source space, it can be determined whether or not the decomposition map $\pi : X \to X/G$ is a near-homeomorphism, *i.e.* π can be approximated by homeomorphisms. One of the most general cases is addressed in the following theorem (see [30, 31, 41]):

THEOREM 2.3. *Suppose G is a usc decomposition of a complete metric space X. Then the decomposition map $\pi : X \to X/G$ is a near-homeomorphism if and only if G is shrinkable.*

With this and similar theorems, Bing's shrinkability criterion opened the door wide for exploring many examples of decomposition spaces.

Early on, the cellularity property was investigated for its potential as a condition on decomposition elements which might imply a similar result as Moore's Theorem for decompositions of $\mathbb{R}^3$. A subset X of an n-manifold M is said to be *cellular* in M if M contains a family of n-cells $\{C_i \mid i = 1, 2, \dots\}$ such that $C_{i+1} \subset$ int C_i and $X = \bigcap C_i$. In particular, could it be true that if G is an upper semicontinuous decomposition of $\mathbb{R}^3$ into cellular sets, then is the decomposition space $\mathbb{R}^3/G$ is homeomorphic to $\mathbb{R}^3$? In 1957, Bing [7] constructed the Dogbone space which was realized by an upper semicontinuous cellular decomposition of $\mathbb{R}^3$ but failed to be homeomorphic to $\mathbb{R}^3$, thereby demonstrating that the answer to the previous question is no. Many examples of decompositions of manifolds whose elements are cellular, or even more generally cell-like, that do not generate manifolds, have now been discovered (cf. [23] for a catalog of examples).

Shortly after the construction of his Dogbone space, Bing [8] discovered a very surprising result: the product of the Dogbone space with the real line is homeomorphic to $\mathbb{R}^4$. The inherent "tangling" of the decomposition elements preventing shrinking becomes sufficiently "unraveled" upon taking the product with $\mathbb{R}$ so that the decomposition elements can be simultaneously shrunk and the shrinkability criterion satisfied. Is this always the case? Or could elements be so tangled, that even within a product of $\mathbb{R}$ there is insufficient room to obtain the desired shrinking of elements. Thus emerged the *Generalized R. L. Moore Problem*:

PROBLEM 2.4. (Generalized R. L. Moore Problem) *If G is a cellular upper semicontinuous decomposition of $\mathbb{R}^n$, is $\mathbb{R}^n/G \times \mathbb{R}$ homeomorphic to $\mathbb{R}^{n+1}$?*

Later, it was determined that the condition "cellular" may be more appropriately replaced with "cell-like". An *absolute neighborhood retract (ANR)* is a locally contractible Peano continuum. A compact subset X of an ANR Y is *cell-like* if X is contractible in every neighborhood of itself. Cellular sets are necessarily cell-like, but the converse is not true. Moreover, a set being cell-like is inherent to the set itself, whereas cellularity depends how the set is embedded in the ambient space. It is the cell-like condition on an upper semicontinuous decomposition of a manifold that is sufficient to imply that the decomposition space is a homology manifold [48-50].

A few examples of some early results concerning the Generalized R. L. Moore Problem include the following:

The product $(E^n/G) \times E^1$ is homeomorphic to E^{n+1} in the following cases:
- G consists of a single nondegenerate element that is an arc (Andrews-Curtis [1]).
- G consists of a single nondegenerate element that is a cell (Bryant [10, 11]).
- G is an usc decomposition into points and a countable collection of arcs (Gillman-Martin [33]).
- G is an usc decomposition into points and a null sequence of cells (Meyer [42]).

These examples provide a sense of the types of problems that were most accessible through shrinking theorems.

3. Characterizations of manifolds

Essential to making progress on the Generalized R. L. Moore Problem are effective characterizations of manifolds. In dimensions 1 and 2, characterizations of manifolds are relatively simple. For example, a space is homeomorphic to S^1 if and only if it is a nondegenerate locally connected continuum (compact connected Hausdorff space) that is separated by no single point, but is separated by any pair of points [45, 54]. The *Kline Sphere Characterization Theorem* states that a space is homeomorphic to S^2 if and only if it is a nondegenerate locally connected metric continuum that is separated by any simple closed curve but not separated by any pair of points [6].

When the Generalized Moore Problem was first posed, characterizations of manifolds useful in addressing this problem in higher dimensions were lacking. However, more breakthroughs were initiated in the 1970's when Cannon proved the double suspension problem and Edwards discovered the proof of the famous Cell-like Approximation Theorem [23, 25, 30] which characterizes manifolds of dimension $n \geq 5$. Before stating the Cell-like Approximation Theorem, we shall review some of the essential properties of manifolds.

3.1. Manifolds are generalized manifolds

Throughout this paper we shall work with singular homology with integer coefficients. A *homology n-manifold* is a locally compact Hausdorff space X such that for every $x \in X$, $H_i(X, X - \{x\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ for all integers $i \geq 0$. A *homology n-manifold with boundary* is a locally compact Hausdorff space X such that either $H_i(X, X - \{x\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ or $H_i(X, X - \{x\}) \cong 0$ for all integers $i \geq 0$. The points x such that $H_i(X, X - \{x\}) \cong 0$ for all integers $i \geq 0$ are called the boundary points (cf. [9, 43]).

A *Euclidean neighborhood retract* (ENR) is a space X that embeds in $\mathbb{R}^n$ so that X is a retract of a neighborhood of itself in $\mathbb{R}^n$. ENR's are the finite-dimensional ANR's. A *generalized n-manifold* is an ENR homology n-manifold. Clearly manifolds are generalized manifolds. However, amongst many other examples, Bing's Dogbone space demonstrates that not all resolvable generalized manifolds are manifolds.

3.2. Manifolds are resolvable

A map $f : Y \to X$ is *cell-like* if for every $x \in X$, $f^{-1}(x)$ is cell-like (for more on this important class of maps see the survey [44]). A space X is said to be *resolvable* if there is a manifold M and a surjective map $f : M \to X$ which is cell-like. Such a map f is said to be a *resolution* of X. In this case, X is said to be *resolvable*. Clearly, manifolds are resolvable.

A large class of generalized 3-manifolds is known to be resolvable [51, 52]. For admissible generalized manifolds of dimension $n \geq 4$, the question of whether or not the space is resolvable can be determined by a locally defined number called the *Quinn index* (cf. [47]).

DEFINITION 3.1. Let Y be a homology n-manifold.

- If $\partial Y = \emptyset$, then Y is said to be *admissible to resolution theory* provided Y is a locally compact, connected, finite-dimensional, separable, metrizable ANR.
- If $\partial Y \neq \emptyset$, the Y is said to be *admissible to resolution theory* provided both $Y - \partial Y$ and ∂Y are admissible homology manifolds with empty boundary.

THEOREM 3.2. (Local Index Theorem) *Let $n \geq 4$ and let Y be an admissible homology n-manifold. There is a local index $i(Y) \in (1 + 8\mathbb{Z})$ defined, which has the following properties:*

1. *For every non-empty, open $U \subset Y$, $i(U) = i(Y)$;*
2. *If $\partial Y \neq \emptyset$, then $i(\partial Y) = i(Y)$;*
3. *If X is admissible, then $i(X \times Y) = i(X) \times i(Y)$; and*
4. *If $\dim Y \geq 5$, or if $\dim Y = 4$ and ∂Y is either empty or a 3-manifold, then Y is resolvable if and only if $i(Y) = 1$.*

It is unknown if all generalized n-manifolds are resolvable for $3 \leq n \leq 6$ (see [49]). Bryant, Ferry, Mio, and Weinberger [12] demonstrated the existence of nonresolvable generalized manifolds in dimensions $n \geq 7$.

3.3. Manifolds have general position properties

General position properties deal with the ability to separate mapped in objects by small adjustments, based on their dimension. We shall see

that these types of properties are very useful in detecting both manifolds and codimension one manifold factors.

The disjoint disks properties are the most basic of all types of general position properties. Let D^j denote a j-cell. A space X is said to satisfy the (m, k)-*disjoint disks property* $((m, k)$-DDP) if any two maps $f : D^m \to X$ and $g : D^k \to X$ can be approximated by maps with disjoint images. Certain (m, k)-disjoint disks properties have special names:

- The $(0, 2)$-DDP is called the *disjoint point-disk property* (DPDP).
- The $(1, 1)$-DDP is called the *disjoint arcs property* (DAP).
- The $(1, 2)$-DDP is called the *disjoint arc-disk property* (DADP).
- The $(2, 2)$-DDP is called the *disjoint disks property* (DDP).

Inherent to manifolds are the general position properties with respect to the appropriate dimensions:

PROPOSITION 3.3. *An n-manifold has the (m, k)-DDP if $m+k+1 \leq n$.*

However, general position for an arbitrary resolvable generalized manifolds may be far more restrictive. The following important result for can be found in [23, Proposition 26.3]:

PROPOSITION 3.4. *A generalized n-manifold, for $n \geq 3$, has the $(1, 1)$-DDP.*

Beyond this, little is guaranteed. Daverman and Walsh [29] poignantly demonstrated this fact in their construction of ghastly spaces, resolvable generalized manifolds of dimension $n \geq 3$ that fail to have the $(0, 2)$-DDP.

3.4. The cell-like approximation theorem

In the 1970's, major breakthroughs occurred in the manifold recognition problem. In proving the famous double suspension problem, Cannon first recognized the significance of the disjoint disks property in characterizing high dimension manifolds [16, 17]. His insight was affirmed by the monumental proof of the Edwards Cell-like Approximation Theorem (cf. [23] for the case $n > 5$ and [25] for the case $n = 5$):

THEOREM 3.5. (Cell-like Approximation Theorem) *For all $n \geq 5$, topological n-manifolds are precisely the resolvable generalized n-manifolds that have the disjoint disks property.*

It is well known that not all resolvable generalized manifolds of dimension $n \geq 5$ have the DDP (cf. [23]). Thus, as indicated previously, not all resolvable generalized manifolds are manifolds. Daverman and Repovš introduced an appropriate analogue of DDP in dimension 3 which yields analogous results [48-50] (cf. also [28]), whereas in dimension 4 we still do not have a good replacement for DDP (there exist so far only some taming theorems – cf. [5]).

4. Manifold factors

Convenient for our discussion is the terminology of manifold factors. A *codimension k manifold factor* is a space X such that $X \times \mathbb{R}^k$ is a manifold. Thus a *codimension one manifold factor* is a space X such that $X \times \mathbb{R}$ is a manifold.

Daverman has demonstrated the following intriguing result in [22]:

THEOREM 4.1. *If X is a resolvable generalized manifold of finite dimension $n \geq 3$, then $X \times \mathbb{R}^2$ is a manifold.*

It follows that all finite-dimensional resolvable generalized manifolds are codimension k manifold factors for $k \geq 2$. The question of whether or not all finite-dimensional resolvable generalized manifolds are codimension one manifold factors remains unsolved. In the terminology of manifold factors, a more general statement of the Generalized R. L. Moore Problem is the *Product with a Line Problem*:

PROBLEM 4.2. (Product with a Line Problem) *Characterize n-dimensional spaces X such that $X \times \mathbb{R}$ is an $(n+1)$-dimensional manifold.*

A necessary condition for a space to be a manifold factor is that the space is a resolvable generalized manifold [47]. In the case of spaces of dimension $n = 3$, the verification that a space is a codimension one manifold factor is still generally done by shrinking arguments. However, more flexible methods of verification in the case of $n \geq 4$ are possible due to the characterization of n-manifolds, $n \geq 5$, provided by Edward's cell-like approximation theorem.

5. Detecting codimension one manifold factors with general position properties

There are several general position properties that are useful in detecting codimension one manifold factors. Each one is designed so that if X has the said property, then $X \times \mathbb{R}$ has the disjoint disks property. The three main general position properties that we shall discuss, in order of strength in detecting codimension one manifold factors are: the disjoint arc-disk property, the disjoint homotopies property, and the disjoint concordances property or equivalently the disjoint topographical maps property. In each case, conditions known to implicate the said property are also included.

5.1. Disjoint arc-disk property

Perhaps the simplest analog to the disjoint disks property in one dimension lower is the disjoint arc-disk property DADP, mentioned earlier as the (1,2)-DDP. Daverman has shown the following [22]:

THEOREM 5.1. *If X is a resolvable generalized manifold with the DADP, then X is a codimension one manifold factor.*

Resolvable generalized manifolds that possess the disjoint arc-disk property include spaces that arise from $(n-3)$-dimensional or closed $(n-2)$-dimensional decompositions [18]. Examples include decompositions of n-manifolds arising from a classical defining sequence [23, Proposition 9.1] and decompositions of $\mathbb{R}^{n \geq 4}$ into convex sets [39].

However, not all manifolds known to be codimension one manifold factors have the disjoint arc-disk property. Examples include the totally wild flow [19], the Daverman-Walsh ghastly spaces [29] and the k-ghastly spaces [34]. Thus a more effective general position property is desired.

5.2. Disjoint homotopies property

The disjoint homotopies property has proven to be an effective detector of codimension one manifold factors in almost every known case.

Let $D = I = [0,1]$. A space X has the *disjoint homotopies property (DHP)* if every pair of path homotopies $f, g : D \times I \to X$ can be approximated by disjoint homotopies, *i.e.*, approximating maps $f', g' : D \times I \to X$ so that $f_t(D) \cap g_t(D) = \emptyset$ for all $t \in I$. The following theorems are proven in [34]:

THEOREM 5.2. *If X is a locally compact ANR with DHP, then $X \times \mathbb{R}$ has DDP.*

COROLLARY 5.3. *If X is a resolvable generalized n-manifold, where $n \geq 4$, having DHP, then X is a codimension one manifold factor.*

Strategies for obtaining approximating disjoint homotopies include:

1. Reimaging, or adjusting the image set.
2. Realigning the levels within the domain.
3. Reparameterizing the levels.

Moreover, it is sufficient to prove that two path homotopies can be approximated by disjoint homotopies in the case that one of the path homotopies is constant.

THEOREM 5.4. *If X is an ANR with the property that any constant path homotopy together with any arbitrary path homotopy can be approximated by disjoint homotopies, then X has DHP.*

These are the strategies employed to demonstrate that DHP is implied for spaces possessing one of the the next three related properties.

5.2.1. Plentiful 2-manifolds property

A space X has the *plentiful 2-manifolds property* (P2MP) if each path $\alpha : I \to X$ can be approximated by a path $\alpha' : I \to N \subset X$ where N is a 2-manifold embedded in X.

THEOREM 5.5. *Suppose X is a generalized n-manifold, $n \geq 4$, $g : D \times I \to X$ and $f : D \times I \to N \subset X$ where N is a 2-manifold embedded in X. Then f and g can be approximated by disjoint homotopies.*

COROLLARY 5.6. *If X is a resolvable generalized n-manifold, $n \geq 4$, with P2PM, then X is a codimension one manifold factor.*

Examples of spaces that have the plentiful 2-manifolds property are decomposition spaces resulting from a nested defining sequences of thickened $(n-2)$-manifolds, spaces that arise from closed $(n-2)$-dimensional decompositions, and certain k-ghastly spaces for $2 < k < n$ (cf. [34]).

5.2.2. 0-Stitched disks property

The maps of $f, g : D^2 \to X$ are said to be 0-*stitched* provided that there are 0-dimensional F_σ sets A and B contained in the interior of D^2 such that $f(D^2 - A) \cap g(D^2 - B) = \emptyset$. We say that f and g are 0-*stitched along A and B*. If Y and Z are sets in D^2 missing A and B respectively, then we say that f and g are 0-*stitched away from Y and Z*.

A space X has the 0-*stitched disks property* if any two maps $f, g : D^2 \to X$ can be approximated by maps $f', g' : D^2 \to X$ such that f' and g' are 0-stitched along sets 0-dimensional F_σ-sets A and B and away from infinite 1-skeleta $(K_j^\infty)^{(1)}$, $j = 1, 2$, of D^2 such that $f'|_{(K_1^\infty)^{(1)}} \cup g'|_{(K_2^\infty)^{(1)}}$ is $1 - 1$.

THEOREM 5.7. *If X has the 0-stitched disks property, then X has DHP.*

There are many examples of resolvable generalized manifolds $X = M/G$ in which the 0-stitched disks property can be easily verified. Suppose that M is a manifold with a sequence of triangulations $\{K_i\}$ such that $\mathrm{mesh}(K_i) \to 0$ and G is a usc decomposition arising so that the nondegeneracy set that misses the infinite 1-skeleta $(K_j^\infty)^{(1)}$ and meets the infinite 2-skeleta $(K_j^\infty)^{(2)}$ in a 0-dimensional F_σ-set. Then the resulting decomposition space will have the 0-stitched disks property. These conditions are generally easily imposed in many constructions of resolvable generalized manifolds that arise from defining sequences.

5.2.3. The method of delta-fractured maps

A map $f : D \times I \to X$ is said to be δ-*fractured* over a map $g : D \times I \to X$ if there are pairwise disjoint balls $B_1, B_2, \ldots, B_m$ in $D \times I$ such that:

1. $diam(B_i) < \delta$;
2. $f^{-1}(im(g)) \subset \bigcup_{i=1}^{m} int(B_i)$; and
3. $diam(g^{-1}(f(B_i))) < \delta$.

THEOREM 5.8. *If X is an ANR that has the property for any pair of path homotopies $f, g : D \times I \to X$, where g is a constant path homotopy, and $\delta > 0$ their are approximations $f', g' : D \times I \to X$ of f and g, respectively, such that f' is δ-fractured over g', then X has DHP.*

The strength of the method of δ-fractured maps is manifest in its application to prove that certain 2-ghastly spaces have DHP (cf. [35]).

5.3. Disjoint concordance property and the disjoint topographies property

Although the disjoint homotopies properties has proven extremely useful in detecting codimension one manifold factors, it is still unknown whether it is a necessary condition on codimension one manifold factors of dimension $n \geq 4$. However, the disjoint concordance property was shown in [25] to be both a necessary and sufficient condition on resolvable generalized manifolds of dimension $n \geq 4$ that are codimension one manifold factors.

A *path concordance* in a space X is a map $F : D \times I \to X \times I$ (where $D = I = [0, 1]$) such that $F(D \times e) \subset X \times e, e \in \{0, 1\}$. A metric space (X, ρ) satisfies the *disjoint path concordances property (DCP)* if, for any two path homotopies $f_i : D \times I \to X$ $(i = 1, 2)$ and any $\varepsilon > 0$, there exist path concordances $F_i' : D \times I \to X \times I$ such that

$$F_1'(D \times I) \cap F_2'(D \times I) = \emptyset$$

and $\rho(f_i, \text{proj}_X F_i') < \varepsilon$.

THEOREM 5.9. (Daverman and Halverson [24]) *Suppose X is a locally compact, metric ANR with DAP. Then X has DCP if and only if $X \times \mathbb{R}$ has DDP.*

COROLLARY 5.10. *A resolvable generalized manifold X of dimension $n \geq 4$ is a codimension one manifold factor if and only if X has DCP.*

The problem with the disjoint concordances property is that in its raw form, it has not had a great deal of utility. However, the disjoint topographies property, a condition equivalent to DCP but having more of the flavor of the disjoint homotopies property, has much more potential. The advantage of the disjoint topographies property over DHP is that not only does it allow for the same strategies of reimaging, realigning, and reparameterizing, but it also allows for the change in the shape of the levels.

A *topography* Υ *on* Z is a partition of Z induced by a map $\tau : Z \to I$. The *t-level of* Υ is given by

$$\Upsilon_t = \tau^{-1}(t).$$

A *topographical map pair* is an ordered pair of maps (f, τ) such that $f : Z \to X$ and $\tau : Z \to I$. The map f will be referred to as the *spatial map* and the map τ will be referred to as the *level map*. The topography associated with (f, τ) is Υ, where $\Upsilon_t = \tau^{-1}(t)$.

Note that a homotopy $f : Z \times I \to X$ has a naturally associated topography, where $\tau : Z \times I \to I$ is defined by $\tau(x, t) = t$. In particular, we may view $f : Z \times I \to X$ as being equivalent to (f, τ) and we shall refer to (f, τ) as the *natural topographical map pair associated with* f.

Suppose that for $i = 1, 2$, Υ^i is a topography on Z_i induced by τ_i and $f_i : Z_i \to X$. Then (f_1, τ_1) and (f_2, τ_2) are *disjoint topographical map pairs* provided that for all $t \in I$,

$$f_1(\Upsilon^1_t) \cap f_2(\Upsilon^2_t) = \emptyset.$$

A space X has the *disjoint topographies property (DTP)* if any two topographical map pairs (f_i, τ_i) $(i = 1, 2)$, where $f_i : D^2 \to X$, can be approximated by disjoint topographical map pairs.

THEOREM 5.11. (Halverson and Repovš [38]) *An ANR X has the disjoint topographies property if and only if $X \times \mathbb{R}$ has DTP.*

COROLLARY 5.12. *A resolvable generalized manifold X of dimension $n \geq 4$ is a codimension one manifold factor if and only if X has DTP.*

The following ribbons properties have analogs to the special properties defined for DHP. The crinkled ribbons properties are a generalization of the plentiful 2-manifolds property. The fuzzy ribbons property is a generalization of the method of δ-fractured maps (cf. [38]).

5.3.1. The crinkled ribbons properties

A generalized n-manifold X has the *crinkled ribbons property (CRP)* provided that any constant homotopy $f : K \times I \to X$, where K is a 1-complex can be approximated by a map $f' : K \times I \to X$ so that:

1. $f'(K \times \{0\}) \cap f'(K \times \{1\}) = \emptyset$; and
2. $\dim(f'(K \times I)) \leq n - 2$.

THEOREM 5.13. *If X is a resolvable generalized n-manifold, $n \geq 4$, with the crinkled ribbons property, then X is a codimension one manifold factor.*

A generalized n-manifold X has the *twisted crinkled ribbons property (CRP-T)* provided that any constant homotopy $f : D \times I$ can be approximated by a map $f' : D \times I$ so that:

1. $f'(D \times \{0\}) \cap f'(D \times \{1\})$ is a finite set of points; and
2. $\dim(f'(D \times I)) \leq n - 2$.

THEOREM 5.14. *If X is a generalized n-manifold of dimension $n \geq 4$ having the twisted crinkled ribbons property and the property that points are 1-LCC embedded in X, then X is a codimension one manifold factor.*

One should note that not all generalized manifolds of dimension $n \geq 4$ have the property that points are 1-LCC embedded. For example, the Daverman-Walsh 2-ghastly spaces are resolvable generalized manifolds that do not have the $(0, 2)$-DDP, and hence cannot satisfy the condition that points are 1-LCC embedded [29].

One application of the CRP is the result that if X is a resolvable generalized locally spherical n-manifold, $n \geq 4$, then X is a codimension one manifold factor [38]. Although this result was initially shown using shrinking arguments [20, 23], it had not been proven previously using general position techniques.

5.3.2. The fuzzy ribbons property

Because of the freedom in restructuring the levels of the topographies to obtain DTP conditions, the δ-control in the method of δ-fractured maps is not required. The analogous definition of δ-fractured maps in the setting of topographical map pairs is as follows:

Let K be a 1-complex. A topographical map pair (f, τ) is in the $\mathcal{K}$ *category* if $f : K \times I \to X$ and $\tau : K \times I \to I$ so that $K \times \{e\} \subset \tau^{-1}(e)$ for $e = 0, 1$. We denote $(f, \tau) \in \mathcal{K}$. A topographical map pair (f, τ) is in the $\mathcal{K}_c$ *category* if

1. $(f, \tau) \in \mathcal{K}$;

2. $f : K \times I \to X$ is a constant homotopy; and
3. (f, τ) is the natural topographical map pair associated with f.

Let $(f_i, \tau_i) \in \mathcal{K}$ be such that $f_i : K_i \times I \to X$ and $\tau_i : K_i \times I \to I$. Then (f_2, τ_2) is said to be *fractured* over a topographical map pair (f_1, τ_1) if there are disjoint balls $B_1, B_2, \dots, B_m$ in $K_2 \times I$ such that:

1. $f_2^{-1}(im(f_1)) \subset \bigcup_{j=1}^{m} int(B_j)$; and
2. $\tau_1 \circ f_1^{-1} \circ f_2(B_i) \neq I$.

A space X has the *fuzzy ribbons property (FRP)* provided that for any topographical map pairs, $(f_1, \tau_1) \in \mathcal{K}_c$ and $(f_2, \tau_2) \in \mathcal{K}$, and $\varepsilon > 0$ there are maps τ_i' and ε-approximations f_i' of f_i so that (f_2', τ_2') is fractured over (f_1', τ_1').

THEOREM 5.15. *If X is a generalized n-manifold of dimension $n \geq 4$ having the fuzzy ribbons property, then X is a codimension one manifold factor.*

Certain 2-ghastly spaces satisfy the FRP, such as those discussed in [35]. The same type of arguments apply, however less attention to control is needed to satisfy the FRP.

6. Epilogue

We list a few interesting unsolved problems:

QUESTION 6.1. *If G is an $(n-2)$-dimensional cell-like decomposition of an n-manifold M, where $n \geq 4$, is M/G a codimension one manifold factor?*

QUESTION 6.2. *Is every finite-dimensional resolvable generalized manifold of dimension $n \geq 4$ a codimension one manifold factor?*

Beginning in 1942, Busemann [13, 14] developed the notion of a G-space as a way of putting a Riemannian like geometry on a metric space (and also in an attempt to obtain a "synthetic description" of Finsler's spaces [32]). A *Busemann G-space* is a metric space that satisfies four basic axioms on a metric space. These axioms infer that Busemann G-spaces are homogeneous geodesic spaces with the property that small metric balls have a cone structure.

Busemann [15] conjectured that every n-dimensional Busemann G-space $(n \in \mathbb{N})$ is a topological n-manifold. This conjecture has been proven true for dimensions $n \leq 4$ [14, 40, 53]. The Busemann Conjecture is also known to be true in all dimensions under the additional hypothesis that

the Aleksandrov curvature is bounded either from below or from above [2, 3].

In the general setting, the solution to the Busemann Conjecture is determined by the answer to the following question:

QUESTION 6.3. *Are small metric spheres in n-dimensional Busemann G-spaces (n ∈ ℕ) codimension one manifold factors?*

G-homogeneous Busemann G-spaces are spaces in which the cone structure of the small metric balls are stable near their cone point. As a possible clue to the answer to Question 6.3, it has been shown that in the case of G-homogeneous Busemann G-spaces, small metric spheres are homogeneous. Moreover, these spaces need not have Aleksandrov curvature bounded either from below or from above [4]. It is unknown whether all Busemann G-spaces are G-homogeneous.

As for the prognosis, we believe that the first problem to tackle should possibly be Question 6.1 since it appears the most tractable. On the other hand, as our paper shows, the Generalized R. L. Moore Problem remains a formidable question – in spite of the great amount of work done in the last half of the century – and it will probably occupy generations to come.

REFERENCES

[1] J. J. ANDREWS – M. L. CURTIS, *n-Space modulo an arc*, Ann. of Math., (2) **75** (1962), pp. 1–7.

[2] V. N. BERESTOVSKIǏ, *Manifolds with an intrinsic metric with one-sided bounded curvature in a sense of A.D. Aleksandrov*, Mat. Fiz. Anal. Geom., **1** (1994), pp. 41-59.

[3] V. N. BERESTOVSKIǏ, *Busemann spaces with upper-bounded Aleksandrov curvature*, Algebra i Analiz, **14** (2002), pp. 3–18 (Russian); English transl.: St. Petersburg Math. J., **14** (2003), pp. 713-723.

[4] V. N. BERESTOVSKIǏ – D. M. HALVERSON – D. REPOVŠ, *Locally G-homogeneous Busemann G-spaces*, Diff. Geom. Appl., **29** (2011), pp. 299–318.

[5] M. BESTVINA – R. J. DAVERMAN – G. A. VENEMA –J. J. WALSH, *A 4-dimensional 1-LCC shrinking theorem*, Topology Appl., **110** (2001), pp. 3–20.

[6] R. H. BING, *The Kline sphere characterization problem*, Bull. Amer. Math. Soc., **52** (1946), pp. 644–653.

[7] R. H. BING, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Ann. of Math., **65** (2) (1957), pp. 484–500.

[8] R. H. BING, *The cartesian product of a certain non-manifold and a line is E^4*, Ann. of Math., **70** (2) (1959), pp. 399–412.

[9] G. BREDON, *Generalized manifolds revisited*, Topology of Manifolds, J. D. Cantrell and C. H. Edwards (eds.), Proceeding of the University of Georgia Topology of Manifolds Institute, Markham, Chicago, 1970, pp. 461–469.

[10] J. L. BRYANT, *Euclidean space modulo a cell*, Fund. Math., **63** (1968), pp. 43–51.

[11] J. L. BRYANT, *Euclidean n-space modulo an $(n-1)$-cell*, Trans. Amer. Math. Soc., **179** (1973), pp. 181–192.

[12] J. BRYANT – S. FERRY – W. MIO – S. WEINBERGER, *Topology of homology manifolds*, Ann. of Math., **143** (2) (1996), pp. 435–467.

[13] H. BUSEMANN, *Metric methods in Finsler spaces and in the foundations of geometry*, Ann. Math. Study no. **8**, Princeton University Press, Princeton, 1942.

[14] H. BUSEMANN, *On spaces in which two points determine a geodesic*, Trans. Amer. Math. Soc., **54** (1943), pp. 171–184.

[15] H. BUSEMANN, *The Geometry of Geodesics*, Academic Press, New York, 1955.

[16] J. W. CANNON, *The recognition problem: What is a topological manifold?*, Bull. Amer. Math. Soc., **84** (1978), pp. 832–866.

[17] J. W. CANNON, *The characterization of topological manifolds of dimension* ≥ 5, Proc. Int. Congr. Math. Helsinki 1978, Academia Sci. Fennica, Helsinki, 1980, pp. 449–454.

[18] J. W. CANNON – J. L. BRYANT – R. C. LACHER, *The structure of generalized manifolds having nonmanifold set of trivial dimension*, Geometric Topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), Academic Press, New York-London, 1979, pp. 261–300.

[19] J. W. CANNON – R. J. DAVERMAN, *A totally wild flow*, Indiana Univ. Math. J., **30**, 3 (1981), pp. 371-387.

[20] R. J. DAVERMAN, *Singular regular neighborhoods and local flatness in codimension one*, Proc. Amer. Math. Soc., **57** (1976), pp. 357–362.

[21] R. J. DAVERMAN, *Products of cell-like decomposition*, Topology Appl., **11** (1980), pp. 121–139.

[22] R. J. DAVERMAN, *Detecting the disjoint disks property*, Pacific J. Math., **93**, 2 (1981), pp. 277–298.

[23] R. J. DAVERMAN, *Decompositions of Manifolds*, Pure and Applied Mathematics, Vol. 124, Academic Press, Inc., Orlando, FL, 1986.

[24] R. J. DAVERMAN – D. M. HALVERSON, *Path concordances as detectors of codimension-one manifold factors*, Exotic Homology Manifolds, Oberwolfach 2003, Geom. Topol. Monogr., Vol. 9, Geom. Topol. Publ., Coventry, 2006, pp. 7–15.

[25] R. J. DAVERMAN – D. M. HALVERSON, *The cell-like approximation theorem in dimension n = 5*, Fund. Math., **197** (2007), pp. 81–121.

[26] R. J. DAVERMAN – D. REPOVŠ, *A new 3-dimensional shrinking theorem*, Trans. Amer. Math. Soc., **315** (1989), pp. 219–230.

[27] R. J. DAVERMAN – D. REPOVŠ, *General position properties that characterize 3-manifolds*, Canad. J. Math. **44**, 2 (1992), pp. 234–251.

[28] R. J. DAVERMAN – T. L. THICKSTUN, *The 3-manifold recognition problem*, Trans. Amer. Math. Soc., **358**, 12 (2006), pp. 5257-5270.

[29] R. J. DAVERMAN – J. J. WALSH, *A ghastly generalized n-manifold*, Illinois J. Math., **25**, 4 (1981), pp. 555–576.

[30] R. D. EDWARDS, *The topology of manifolds and cell-like maps*, Proc. Int. Congr. of Math., Helsinki, 1978, Acad. Sci. Fennica, Helsinki, 1980, pp. 111–127.

[31] R. D. EDWARDS – L. C. GLASER, *A method for shrinking decompositions of certain manifolds*, Trans. Amer. Math. Soc., **165** (1972), 4pp. 5–56.

[32] P. FINSLER, *Über Kurven und Flächen in allgemeinen Räumen*, Ph. D. Thesis, Göttingen, 1918.

[33] D. S. GILLMAN – J. M. MARTIN, *Countable decompositions of E^3 into points and point-like arcs*, Notices Amer. Math. Soc., **10** (1963), pp. 74–75.

[34] D. M. HALVERSON, *Detecting codimension one manifold factors with the disjoint homotopies property*, Topology Appl., **117**, 3 (2002), pp. 231–258.

[35] D. M. HALVERSON, *2-ghastly spaces with the disjoint homotopies property: the method of fractured maps*, Topology Appl., **138**, 1-3 (2004), pp. 277–286.

[36] D. M. HALVERSON, *Detecting codimension one manifold factors with 0-stitched disks*, Topology Appl., **154**, 9 (2007), pp. 1993–1998.

[37] D. M. HALVERSON – D. REPOVŠ, *The Bing-Borsuk and the Busemann Conjectures*, Math. Comm., **13**, 2 (2008), pp. 163–184.

[38] D. M. HALVERSON – D. REPOVŠ, *Detecting codimension one manifold factors with topographical techniques*, Topology Appl., **156**, 17 (2009), pp. 2870–2880.

[39] D. M. HALVERSON – D. REPOVŠ, *Decompositions of $\mathbb{R}^n (n \geq 4)$ into convex sets generate codimension one manifold factors*, Mediterr. J. Math., to appear.

[40] B. KRAKUS, *Any 3-dimensional G-space is a manifold*, Bull. Acad. Pol. Sci., **16** (1968), pp. 737–740.

[41] A. MARIN – Y. M. VISETTI, *A general proof of Bing's shrikability criterion*, Proc. Amer. Math. Soc., **53** (1975), pp. 501–507.

[42] D. V. MEYER, *More decompositions of E^3 which are factors of E^{n+1}*, Fund. Math., **67** (1970), pp. 49–65.

[43] W. J. R. MITCHELL, *Defining the boundary of a homology manifold*, Proc. Amer. Math. Soc., **110** (1990), pp. 509–513.

[44] W. J. R. MITCHELL – D. REPOVŠ, *Topology of cell-like mappings*, Proc. Conf. Diff. Geom. and Topol., Cala Gonone 1988, Suppl. Rend. Fac. Sci. Nat. Univ. Cagliari, **58** (1988), pp. 265–300.

[45] R. L. MOORE, *Concerning upper semicontinuous collections of continua which do not separate a given set*, Proc. Nat. Acad. Sci., 10 (1924), pp. 356–360.

[46] R. L. MOORE, *Concerning upper semicontinuous collections of continua*, Trans. Amer. Math. Soc., **27** (1925), pp. 416–428.

[47] F. QUINN, *An obstruction to the resolution of homology manifolds*, Mich. Math. J., **34** (1987), pp. 285–291.

[48] D. REPOVŠ, *The recognition problem for topological manifolds*, Geometric and Algebraic Topology, J. Krasinkiewicz, S. Spież, and H. Toruǹczyk (eds.), PWN, Warsaw 1986, pp. 77–108.

[49] D. REPOVŠ, *Detection of higher dimensional topological manifolds among topological spaces*, Giornate di Topologia e Geometria Delle Varietà, Bologna 1990, M. Ferri, Ed., Univ. degli Studi di Bologna 1992, pp. 113–143.

[50] D. REPOVŠ, *The recognition problem for topological manifolds: A survey*, Kodai Math. J., **17**, 3 (1994), pp. 538–548.

[51] T. L. THICKSTUN, *An extension of the loop theorem and resolution of generalized 3-manifolds with 0-dimensional singular set*, Invent. Math., **78** (1984), pp. 161–222.

[52] T. L. THICKSTUN, *Resolutions of generalized 3-manifolds whose singular sets have general position dimension one*, Topology Appl., **138**, 1-3 (2004), pp. 61-95.
[53] P. THURSTON, *4-dimensional Busemann G-spaces are 4-manifolds*, Diff. Geom. Appl., **6**, 3 (1996), pp. 245–270.
[54] R. L. WILDER, *Topology of Manifolds*, Amer. Math. Soc. Colloq. Publ., **32**, Providence R.I., 1963.

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