Hamdani et al. *Boundary Value Problems* (2020) 2020:150
https://doi.org/10.1186/s13661-020-01447-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Multiplicity of solutions for a class of fractional $p(x,\cdot)$-Kirchhoff-type problems without the Ambrosetti–Rabinowitz condition

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Abstract

We are interested in the existence of solutions for the following fractional $p(x,\cdot)$-Kirchhoff-type problem:

$$\begin{cases} M\left(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s}}\,dx\,dy\right)(-\Delta)^s_{p(x,\cdot)}u=f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 2$ is a bounded smooth domain, $s\in(0,1)$, $p:\overline{\Omega}\times\overline{\Omega}\to(1,\infty)$, $(-\Delta)^s_{p(x,\cdot)}$ denotes the $p(x,\cdot)$-fractional Laplace operator, $M:[0,\infty)\to[0,\infty)$, and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ are continuous functions. Using variational methods, especially the symmetric mountain pass theorem due to Bartolo–Benci–Fortunato (Nonlinear Anal. 7(9):981–1012, 1983), we establish the existence of infinitely many solutions for this problem without assuming the Ambrosetti–Rabinowitz condition. Our main result in several directions extends previous ones which have recently appeared in the literature.

MSC: Primary 35R11; secondary 35J20; 35J60

Keywords: Fractional $p(x,\cdot)$-Kirchhoff-type problems; $p(x,\cdot)$-fractional Laplace operator; Ambrosetti–Rabinowitz type conditions; Symmetric mountain pass theorem; Cerami compactness condition; Fractional Sobolev spaces with variable exponent; Multiplicity of solutions

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N\geq 2$. Let us consider the following fractional $p(x,\cdot)$-Kirchhoff-type problem:

$$\begin{cases} M\left(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s}}\,dx\,dy\right)(-\Delta)^s_{p(x,\cdot)}u=f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{1.1}$$

where $0 < s < 1$, $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$ is a continuous function with $sp(x,y) < N$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, and M, f are continuous functions satisfying certain growth conditions to be stated later on.

The fractional $p(x,\cdot)$-Laplacian operator $(-\Delta)^s_{p(x,\cdot)}$ is, up to normalization factors by the Riesz potential, defined as follows: for each $x \in \Omega$,

$$(-\Delta)^s_{p(x,\cdot)}\varphi(x) = \text{p.v.} \int_\Omega \frac{|\varphi(x) - \varphi(y)|^{p(x,y)-2}(\varphi(x) - \varphi(y))}{|x-y|^{N+sp(x,y)}}\,dy, \tag{1.2}$$

along any $\varphi \in C_0^\infty(\Omega)$, where *p.v.* is the commonly used abbreviation for the *principal value*.

Throughout this paper, we shall assume that $M : \mathbb{R}_0^+ := [0, +\infty) \to \mathbb{R}_0^+$ is a continuous function satisfying the following conditions:

(M_1): there exist $\tau_0 > 0$ and $\gamma \in (1, (p_s^*)^-/p^+)$ such that

$$tM(t) \le \gamma \widehat{M}(t), \quad \text{for all } t \ge \tau_0,$$

where

$$\widehat{M}(t) = \int_0^t M(\tau)\,d\tau$$

and p^+ and p^- will be defined in Sect. 2;

(M_2): for every $\tau > 0$ there exists $\kappa = \kappa(\tau) > 0$ such that

$$M(t) \ge \kappa, \quad \text{for all } t \ge \tau.$$

Obviously, the conditions (M_1) and (M_2) are fulfilled for the model case:

$$M(t) = a + b\theta t^{\theta-1}, \quad \text{where } a, b \ge 0 \text{ and } a + b > 0. \tag{1.3}$$

It is worth pointing out that condition (M_2) was originally used to establish multiplicity of solutions for a class of higher order $p(x)$-Kirchhoff equations [11].

In recent years, a lot of attention has been given to problems involving fractional and nonlocal operators. This type of operators arises in a natural way in many different applications, e.g., image processing, quantum mechanics, elastic mechanics, electrorheological fluids (see [8, 15, 16, 34] and the references therein).

In their pioneering paper, Bahrouni and Rădulescu [6] studied qualitative properties of the fractional Sobolev space $W^{s,q(x),p(x,y)}(\Omega)$, where Ω is a smooth bounded domain. Their results have been applied in the variational analysis of a class of nonlocal fractional problems with several variable exponents.

Recently, by means of approximation and energy methods, Zhang and Zhang [38] have established the existence and uniqueness of nonnegative renormalized solutions for such problems. When $s = 1$, the operator degrades to integer order. It has been extensively studied in the literature; see for example [9, 10, 18, 19, 21] and the references therein. In particular, when $p(x,\cdot)$ is a constant, this operator is reduced to the classical fractional p-Laplacian operator.

For studies concerning this operator, we refer to [31, 32, 39–41]. We emphasize that, unless the functions $p(x,\cdot)$ and $q(x)$ are constants, the space $W^{s,q(x),p(x,y)}(\Omega)$ does not coincide with the Sobolev space $W^{s,p(x)}(\Omega)$ when s is a natural number; see [16, 23]. However, because of various applications in physics and in mathematical finance, the study of nonlocal problems in such spaces is still very interesting.

On the other hand, a lot of interest has in recent years been devoted to the study of Kirchhoff-type problems. More precisely, in 1883 Kirchhoff [24] established a model given by the following equation:

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{p_0}{\lambda}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.4}$$

a generalization of the well-known D'Alembert wave equation for free vibrations of elastic strings, where ρ, p_0, λ, E, L are constants which represent some physical meanings, respectively.

In the study of problem (1.1), the following Ambrosetti–Rabinowitz condition given in [3] has been widely used:

(AR): There exists a constant $\mu > p^+$ such that

$$tf(x,t)\geq \mu F(x,t)>0, \quad \text{where } F(x,t)=\int_0^t f(x,s)\,ds.$$

Clearly, if the (AR) condition holds, then

$$F(x,t)\geq c_1|t|^{\mu}-c_2, \quad \text{for all } (x,t)\in\Omega\times\mathbb{R}, \tag{1.5}$$

where c_1, c_2 are two positive constants.

It is well known that (AR) condition is very important for ensuring the boundedness of the Palais–Smale sequence. When the nonlinear term f satisfies the (AR) condition, many results have been obtained by using the critical point theory and variational methods; see for example [1, 2, 4, 5, 12–14, 17, 20, 28, 29, 36, 37]. In particular, Ali *et al.* [1] and Azroul *et al.* [5] have established the existence of nontrivial weak solutions for a class of fractional $p(x,\cdot)$-Kirchhoff-type problems by using the mountain pass theorem of Ambrosetti and Rabinowitz, direct variational approach, and Ekeland's variational principle.

Since the (AR) condition implies condition (1.5), one cannot deal with problem (1.1) by using the mountain pass theorem directly if $f(x,t)$ is p^+-asymptotically linear at ∞, i.e.

$$\lim_{|t|\to\infty}\frac{f(x,t)}{|t|^{p^+-1}}=l, \quad \text{uniformly in } x\in\Omega, \tag{1.6}$$

where l is a constant. For this reason, in recent years some authors have studied problem (1.1) by trying to omit the condition (AR); see for example [18, 22, 27].

Not having the (AR) condition brings great difficulties, so it is natural to consider if this kind of fractional problems have corresponding results even if the nonlinearity does not satisfy the (AR) condition. In fact, in the absence of Kirchhoff's interference, Lee et al. [26] have obtained infinitely many solutions to a fractional $p(x)$-Laplacian equation without assuming the (AR) condition, by using the fountain theorem and the dual fountain theorem.

Inspired by the above work, we consider in this paper the fractional $p(x,\cdot)$-Kirchhoff-type problem without the (AR) condition. Our situation is different from [1, 5] since our Kirchhoff function M belongs to a larger class of functions, whereas the nonlinear term f is p^+-asymptotically linear at ∞.

More precisely, let us assume that f satisfies the following global conditions:

(F_1): $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous with $F(x,t)\geq 0$, for all $(x,t)\in\Omega\times\mathbb{R}$, where

$$F(x,t)=\int_0^t f(x,s)\,ds;$$

(F_2): there exist a function $\alpha\in C(\overline{\Omega})$, $p^+<\alpha^-\leq\alpha(x)<p_s^*(x)$, for all $x\in\Omega$, and a number $\Lambda_0>0$ such that, for each $\lambda\in(0,\Lambda_0)$, $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$f(x,t)\leq(\lambda+\epsilon)|t|^{\overline{p}(x)-1}+C_\epsilon|t|^{\alpha(x)-1},\quad \text{for all } (x,t)\in\Omega\times\mathbb{R};$$

(F_3): the following is uniformly satisfied on $\overline{\Omega}$:

$$\lim_{|t|\to\infty}\frac{F(x,t)}{|t|^{p^+\gamma}}=\infty;$$

(F_4): there exist constants $\mu>p^+\gamma$ and $\varpi_0>0$ such that

$$F(x,t)\leq\frac{1}{\mu}f(x,t)t+\varpi_0|t|^{p^-},\quad \text{for all } (x,t)\in\Omega\times\mathbb{R};$$

where γ is given by (M_1);

(F_5): the following holds:

$$f(x,-t)=-f(x,t),\quad \text{for all } (x,t)\in\Omega\times\mathbb{R}.$$

A simple computation proves that the function

$$f(x,t)=|t|^{p^+\gamma-2}t\ln^{\alpha(x)}\bigl(1+|t|\bigr),\quad \text{where } \alpha(x)>1, \tag{1.7}$$

does not satisfy the (AR) condition. However, it is easy to see that $f(x,t)$ in (1.7) satisfies conditions (F_1)–(F_5).

We can now state the definition of (weak) solutions for problem (1.1) (see Sect. 2 for details):

Definition 1.1 A function $u\in E_0=W_0^{s,q(x),p(x,y)}(\Omega)$ is called a (weak) solution of problem (1.1), if for every $w\in E_0$ it satisfies the following:

$$M\bigl(\sigma_{p(x,y)}(u)\bigr)\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+p(x,y)s}}\,dx\,dy$$
$$-\int_\Omega f(x,u)w\,dx=0,$$

where

$$\sigma_{p(x,y)}(u)=\int_{\Omega\times\Omega}\frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy.$$

The main result of our paper is the following theorem.

Theorem 1.1 *Let $q(x)$, $p(x,y)$ be continuous variable functions such that $sp(x,y) < N$, $p(x,y) = p(y,x)$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and $q(x) \geq p(x,x)$ for all $x \in \overline{\Omega}$. Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (F_1)–(F_5) and that $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function satisfying conditions (M_1) and (M_2). Then there exists $\Lambda > 0$ such that, for each $\lambda \in (0, \Lambda)$, problem (1.1) has a sequence $\{u_n\}_n$ of nontrivial solutions.*

The paper is organized as follows. In Sect. 2, we shall introduce the necessary properties of variable exponent Lebesgue spaces and fractional Sobolev spaces with variable exponent. In Sect. 3, we shall verify the Cerami compactness condition. Finally, in Sect. 4, we shall prove Theorem 1.1 by means of a version of the mountain pass theorem.

2 Fractional Sobolev spaces with variable exponent

For a smooth bounded domain Ω in $\mathbb{R}^N$, we consider a continuous function $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$. We assume that p is symmetric, that is,

$$p(x,y) = p(y,x), \quad \text{for all } (x,y) \in \overline{\Omega} \times \overline{\Omega}$$

and

$$1 < p^- := \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \leq p(x,y) \leq p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < \infty.$$

We also introduce a continuous function $q : \overline{\Omega} \to \mathbb{R}$ such that

$$1 < q^- := \min_{x\in\overline{\Omega}} q(x) \leq q(x) \leq q^+ := \max_{x\in\overline{\Omega}} q(x) < \infty.$$

We first give some basic properties of variable exponent Lebesgue spaces. Set

$$C_+(\overline{\Omega}) = \{ r \in C(\overline{\Omega}) : 1 < r(x) \text{ for all } x \in \overline{\Omega} \}.$$

Given $r \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{r(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable: } \int_\Omega |u(x)|^{r(x)}\,dx < \infty \right\},$$

and this space is endowed with the Luxemburg norm,

$$|u|_{r(x)} = \inf\left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{r(x)} dx \leq 1 \right\}.$$

Then $(L^{r(x)}(\Omega), |\cdot|_{r(x)})$ is a separable reflexive Banach space; see [25, Theorem 2.5 and Corollaries 2.7 and 2.12].

Let $\widetilde{r} \in C_+(\overline{\Omega})$ be the conjugate exponent of q, that is,

$$\frac{1}{r(x)} + \frac{1}{\widetilde{r}(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

We shall need the following Hölder inequality, whose proof can be found in [25, Theorem 2.1]. Assume that $v \in L^{r(x)}(\Omega)$ and $u \in L^{\widetilde{r}(x)}(\Omega)$. Then

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{r^-}+\frac{1}{\widetilde{r}^-}\right)|u|_{r(x)}|v|_{\widetilde{r}(\cdot)} \leq 2|u|_{r(x)}|v|_{\widetilde{r}(x)}.$$

A modular of the $L^{r(x)}(\Omega)$ space is defined by

$$\varrho_{r(x)}: L^{r(x)}(\Omega) \to \mathbb{R}, \qquad u \mapsto \varrho_{r(x)}(u) = \int_\Omega \left|u(x)\right|^{r(x)}dx.$$

Assume that $u \in L^{r(x)}(\Omega)$ and $\{u_n\} \subset L^{r(x)}(\Omega)$. Then the following assertions hold (see [16]):

$$\begin{aligned}
&(1)\quad |u|_{r(x)} < 1 \quad (\text{resp.}, = 1, > 1) \quad \Leftrightarrow \quad \varrho_{r(x)}(u) < 1 \quad (\text{resp.}, = 1, > 1),\\
&(2)\quad |u|_{r(x)} < 1 \quad \Rightarrow \quad |u|_{r(x)}^{r^+} \leq \varrho_{r(x)}(u) \leq |u|_{r(x)}^{r^-},\\
&(3)\quad |u|_{r(x)} > 1 \quad \Rightarrow \quad |u|_{r(x)}^{r^-} \leq \varrho_{r(x)}(u) \leq |u|_{r(x)}^{r^+},\\
&(4)\quad \lim_{n\to\infty}|u_n|_{r(x)} = 0 \quad (\text{resp.}, = \infty) \quad \Leftrightarrow \quad \lim_{n\to\infty}\varrho_{r(x)}(u_n) = 0 \quad (\text{resp.}, = \infty),\\
&(5)\quad \lim_{n\to\infty}|u_n - u|_{r(x)} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty}\varrho_{r(x)}(u_n - u) = 0.
\end{aligned}$$

Given $s \in (0,1)$ and the functions $p(x,y)$, $q(x)$ as we mentioned above, the fractional Sobolev space with variable exponents via the Gagliardo approach $E = W^{s,q(x),p(x,y)}(\Omega)$ is defined as follows:

$$E = \left\{u \in L^{q(x)}(\Omega): \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < \infty, \text{ for some } \mu > 0\right\}.$$

Let

$$[u]_{s,p(x,y)} = \inf\left\{\mu > 0: \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < 1\right\},$$

be the variable exponent Gagliardo seminorm and define

$$\|u\|_E = [u]_{s,p(x,y)} + |u|_{q(x)}.$$

Then E equipped with the norm $\|\cdot\|_E$ becomes a Banach space.

Proposition 2.1 *The following properties hold*:
(1) *If* $1 \leq [u]_{s,p(x,y)} < \infty$, *then*

$$\left([u]_{s,p(x,y)}\right)^{p_-} \leq \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \leq \left([u]_{s,p(x,y)}\right)^{p^+}.$$

(2) *If* $[u]_{s,p(x,y)} \leq 1$, *then*

$$\left([u]_{s,p(x,y)}\right)^{p^+} \leq \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \leq \left([u]_{s,p(x,y)}\right)^{p_-}.$$

Given $u \in W^{s,q(x),p(x,y)}(\Omega)$, we set

$$\rho(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_{\Omega} |u|^{q(x)}\,dx$$

and

$$\|u\|_{\rho} = \inf\left\{\mu > 0 : \rho\left(\frac{u}{\mu}\right) \leq 1\right\}.$$

It is well known that $\|\cdot\|_{\rho}$ is a norm which is equivalent to the norm $\|\cdot\|_{W^{s,q(x),p(x,y)}(\Omega)}$. By Lemma 2.2 in [38], $(W^{s,q(x),p(x,y)}(\Omega), \|\cdot\|_{\rho})$ is uniformly convex and $W^{s,q(x),p(x,y)}(\Omega)$ is a reflexive Banach space.

We denote our workspace $E_0 = W_0^{s,q(x),p(x,y)}(\Omega)$, the closure of $C_0^{\infty}(\Omega)$ in E. Then E_0 is a reflexive Banach space with the norm

$$\|\cdot\|_{E_0} = [u]_{s,p(x,y)}.$$

A thorough variational analysis of the problems with variable exponents has been developed in the monograph by Rădulescu and Repovš [33]. The following result provides a compact embedding into variable exponent Lebesgue spaces.

Theorem 2.1 (see [38]) *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $s \in (0,1)$. Let $q(x)$, $p(x,y)$ be continuous variable exponents such that*

$$sp(x,y) < N, \quad \textit{for } (x,y) \in \overline{\Omega}\times\overline{\Omega} \quad \textit{and} \quad q(x) \geq p(x,x), \quad \textit{for all } x \in \overline{\Omega}.$$

Assume that $\tau : \overline{\Omega} \longrightarrow (1,\infty)$ is a continuous function such that

$$p^*(x) = \frac{Np(x,x)}{N - sp(x,x)} > \tau(x) \geq \tau^- > 1, \quad \textit{for all } x \in \overline{\Omega}.$$

Then there exists a constant $C = C(N,s,p,q,r,\Omega)$ such that, for every $u \in W^{s,q(x),p(x,y)}$,

$$|u|_{\tau(x)} \leq C\|u\|_E. \tag{2.1}$$

That is, the space $W^{s,q(x),p(x,y)}(\Omega)$ is continuously embeddable in $L^{\tau(x)}(\Omega)$. Moreover, this embedding is compact. In addition, if $u \in W_0^{s,q(x),p(x,y)}$, the following inequality holds:

$$|u|_{\tau(x)} \leq C\|u\|_{E_0}.$$

Theorem 2.2 (see [6]) *For all $u, v \in E_0$, we consider the operator $I : E_0 \to E_0^*$ such that*

$$\langle I(u), v\rangle = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy.$$

Then the following properties hold:

(1) *I is a bounded and strictly monotone operator.*
(2) *I is a mapping of type (S_+), that is,*

$$\text{if } u_n \rightharpoonup u \in E_0 \quad \text{and} \quad \limsup_{n\to\infty} I(u_n)(u_n - u) \le 0, \quad \text{then } u_n \to u \in E_0.$$

(3) *$I : E_0 \to E_0^*$ is a homeomorphism.*

3 The cerami compactness condition

Let us consider the Euler–Lagrange functional associated to problem (1.1), defined by $J_\lambda : E_0 \to \mathbb{R}$

$$J_\lambda(u) = \widehat{M}\big(\sigma_{p(x,y)}(u)\big) - \int_\Omega F(x,u)\,dx. \tag{3.1}$$

Note that J_λ is a $C^1(E_0,\mathbb{R})$ functional and

$$\begin{aligned}\langle J_\lambda'(u), w\rangle = M\big(\sigma_{p(x,y)}(u)\big) \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+p(x,y)s}}\,dx\,dy \\ - \int_\Omega f(x,u)w\,dx \end{aligned} \tag{3.2}$$

for all $w \in E_0$. Therefore critical points of J_λ are weak solutions of problem (1.1).

In order to prove our main result (Theorem 1.1), we recall the definition of the Cerami compactness condition [30].

Definition 3.1 We say that J_λ satisfies the Cerami compactness condition at the level $c \in \mathbb{R}$ ($(\mathrm{Ce})_c$ condition for short), if every sequence $\{u_n\}_n \subset E_0$, i.e., $J_\lambda(u_n) \to c$ and

$$\big\|J_\lambda'(u_n)\big\|_{E_0{}^*}\big(1 + \|u_n\|_{E_0}\big) \to 0, \quad \text{as } n \to \infty,$$

admits a strongly convergent subsequence in E_0. If J_λ satisfies the $(\mathrm{Ce})_c$ condition for any $c \in \mathbb{R}$ then we say that J_λ satisfies the Cerami compactness condition.

Claim 3.1 *Under assumptions of Theorem 1.1, every $(Ce)_c$ sequence $\{u_n\}_n \subset E_0$ of J_λ is bounded in E_0.*

Proof Let $\{u_n\}_n$ be a $(Ce)_c$-sequence of J_λ. Then

$$J_\lambda(u_n) \to c \quad \text{and} \quad \big\|J_\lambda'(u_n)\big\|_{E_0{}^*}\big(1 + \|u_n\|_{E_0}\big) \to 0. \tag{3.3}$$

First, we prove that the sequence $\{u_n\}_n$ is bounded in E_0. To this end, we argue by contradiction. So suppose that $\|u_n\|_{E_0} \to \infty$, as $n \to \infty$. We define the sequence $\{v_n\}_n$ by

$$v_n = \frac{u_n}{\|u_n\|_{E_0}}, \quad n \in \mathbb{N}.$$

It is clear that $\{v_n\}_n \subset E_0$ and $\|v_n\|_{E_0} = 1$ for all $n \in \mathbb{N}$. Passing, if necessary, to a subsequence, we may assume that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } E_0, \\ &v_n \to v \quad \text{in } L^{\tau(x)}(\Omega), 1 \leq \tau(x) < p^*(x), \\ &v_n(x) \to v(x) \quad \text{a.e. on } \Omega. \end{aligned} \tag{3.4}$$

Let $\Omega_{\natural} := \{x \in \Omega : v(x) \neq 0\}$. If $x \in \Omega_{\natural}$, then it follows from (3.4) that

$$\lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} \frac{u_n}{\|u_n\|_{E_0}} = v(x) \neq 0.$$

This means that

$$\left|u_n(x)\right| = \left|v_n(x)\right| \|u_n\|_{E_0} \to +\infty \quad \text{a.e. on } \Omega_{\natural}, \text{ as } n \to \infty.$$

Moreover, it follows by condition (F_3) and Fatou's lemma that, for each $x \in \Omega_{\natural}$,

$$+\infty = \lim_{n\to\infty} \int_\Omega \frac{|F(x,u_n(x))|}{|u_n(x)|^{p+\gamma}} \frac{|u_n(x)|^{p+\gamma}}{\|u_n(x)\|_{E_0}^{p+\gamma}}\,dx = \lim_{n\to\infty} \int_\Omega \frac{|F(x,u_n(x))||v_n(x)|^{p+\gamma}}{|u_n(x)|^{p+\gamma}}\,dx. \tag{3.5}$$

Condition (M_1) gives

$$\widehat{M}(t) \leq \widehat{M}(1)t^\gamma, \quad \text{for all } t \geq 1. \tag{3.6}$$

Now, since $\|u_n\|_{E_0} > 1$, it follows by (3.1), (3.3) and (3.6) that

$$\begin{aligned} \int_\Omega F(x,u_n)\,dx &\leq \widehat{M}(\sigma_{p(x,y)u_n}) + C \\ &\leq \frac{\widehat{M}(1)}{(p^-)^\gamma} \left(\sigma_{p(x,y)}(u_n)\right)^\gamma + C \\ &\leq \frac{\widehat{M}(1)}{(p^-)^\gamma} \|u_n\|_{E_0}^{\gamma p^+} + C, \end{aligned}$$

for all $n \in \mathbb{N}$. We can now conclude that

$$\lim_{n\to\infty} \int_\Omega \frac{F(x,u_n)}{\|u_n\|_{E_0}^{p+\gamma}}\,dx \leq \lim_{n\to\infty} \left(\frac{\widehat{M}(1)}{(p^-)^\gamma} + \frac{C}{\|u_n\|_{E_0}^{\gamma p^+}} \right). \tag{3.7}$$

From (3.5) and (3.7) we obtain

$$+\infty \leq \frac{\widehat{M}(1)}{(p^-)^\gamma},$$

which is a contradiction. Therefore

$$|\Omega_{\natural}| = 0 \quad \text{and} \quad v(x) = 0 \quad \text{a.e. on } \Omega.$$

It follows from (M_1), (M_2), (F_4) and $v_n \to v = 0$ in $L^{p^-}(\Omega)$ that

$$\begin{aligned}
&\frac{1}{\|u_n\|_{E_0}^{p^-}}\left(J_\lambda(u_n) - \frac{1}{\mu}J'_\lambda(u_n)u_n\right) \\
&\quad\geq \frac{1}{\|u_n\|_{E_0}^{p^-}}\left[\widehat{M}\left(\sigma_{p(x,y)}(u_n)\right) - \int_\Omega F(x,u_n)\,dx\right. \\
&\qquad\left. - \frac{1}{\mu}M\left(\sigma_{p(x,y)}(u_n)\right)\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s}}\,dx\,dy + \frac{1}{\mu}\int_\Omega f(x,u_n)u_n\,dx\right] \\
&\quad\geq \frac{1}{\|u_n\|_{E_0}^{p^-}}\left[\frac{1}{\gamma}M\left(\sigma_{p(x,y)}(u_n)\right)\sigma_{p(x,y)}(u_n)\right. \\
&\qquad\left. - \frac{1}{\mu}M\left(\sigma_{p(x,y)}(u_n)\right)\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s}}\,dx\,dy - \varpi_0\int_\Omega |u_n|^{p^-}\,dx\right] \\
&\quad\geq \frac{1}{\|u_n\|_{E_0}^{p^-}}\left[\left(\frac{1}{\gamma p^+} - \frac{1}{\mu}\right)M\left(\sigma_{p(x,y)}(u_n)\right)\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s}}\,dx\,dy\right. \\
&\qquad\left. - \varpi_0\int_\Omega |u_n|^{p^-}\,dx\right] \\
&\quad\geq \left(\frac{1}{\gamma p^+} - \frac{1}{\mu}\right)\kappa - \lambda\varpi_0\int_\Omega |v_n|^{p^-}\,dx,
\end{aligned}$$

which means that

$$0 \geq \left(\frac{1}{\gamma p^+} - \frac{1}{\mu}\right)\kappa, \quad \text{as } n \to \infty.$$

This is a contradiction. As a consequence, we can conclude that Cerami sequence $\{u_n\}_n$ is indeed bounded. This completes the proof of Claim 3.1. □

We now complete the verification of the Cerami compactness condition $(Ce)_c$ for J_λ.

Claim 3.2 *The functional J_λ satisfies condition $(Ce)_c$ in E_0.*

Proof Let $\{u_n\}_n$ be a $(Ce)_c$ sequence for J_λ in E_0. Claim 3.1 asserts that $\{u_n\}_n$ is bounded in E_0. By Theorem 2.1, the embedding $E_0 \hookrightarrow L^{\tau(x)}(\Omega)$ is compact, where $1 \leq \tau(x) < p^*(x)$. Since E_0 is a reflexive Banach space, passing, if necessary, to a subsequence, still denoted by $\{u_n\}_n$, there exists $u \in E_0$ such that

$$u_n \rightharpoonup u \quad \text{in } E_0, \qquad u_n \to u \quad \text{in } L^{\tau(x)}(\Omega), \qquad u_n(x) \to u(x), \quad \text{a.e. on } \Omega. \tag{3.8}$$

By virtue of (3.3), we get

$$\begin{aligned}
&\left\langle J'_\lambda(u_n), u_n - u\right\rangle \\
&\quad= M\left(\sigma_{p(x,y)}(u_n)\right) \\
&\qquad\times \int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))((u_n(x)-u(x))-(u_n(y)-u(y)))}{|x-y|^{N+p(x,y)s}}\,dx\,dy \\
&\qquad- \int_\Omega f(x,u_n)(u_n-u)\,dx \to 0.
\end{aligned} \tag{3.9}$$

Now, by condition (F_2),

$$\left|f(x,u_n)\right| \le (\lambda+\epsilon)|u_n|^{\overline{p}(x)-1} + C_\epsilon |u_n|^{\alpha(x)-1}. \tag{3.10}$$

It follows from (3.8), (3.10) and Proposition 2.1 that

$$\begin{aligned}
&\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right| \\
&\quad\le \int_\Omega (\lambda+\epsilon)|u_n|^{\overline{p}(x)-1}|u_n-u|\,dx + \int_\Omega C_\epsilon |u_n|^{\alpha(x)-1}|u_n-u|\,dx \\
&\quad\le (\lambda+\epsilon)\left||u_n|^{\overline{p}(x)-1}\right|_{\frac{\overline{p}(x)}{\overline{p}(x)-1}}|u_n-u|_{\overline{p}(x)} + C_\epsilon\left||u_n|^{\alpha(x)-1}\right|_{\frac{\alpha(x)}{\alpha(x)-1}}|u_n-u|_{\alpha(x)} \\
&\quad\le (\lambda+\epsilon)\max\left\{\|u_n\|_{E_0}^{p_+-1},\|u_n\|_{E_0}^{p_--1}\right\}|u_n-u|_{p(x)} \\
&\qquad + C_\epsilon \max\left\{\|u_n\|_{E_0}^{\alpha_+-1},\|u_n\|_{E_0}^{\alpha_--1}\right\}_{\frac{\alpha(x)}{\alpha(x)-1}}|u_n-u|_{\alpha(x)} \\
&\quad\to 0, \quad \text{as } n\to\infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)\,dx = 0. \tag{3.11}$$

Therefore we can infer from (3.9) and (3.11) that

$$\begin{aligned}
&M\big(\sigma_{p(x,y)}(u_n)\big) \\
&\quad\times \int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))((u_n(x)-u(x))-(u_n(y)-u(y)))}{|x-y|^{N+p(x,y)s}}\,dx\,dy \\
&\quad\to 0.
\end{aligned}$$

Since $\{u_n\}_n$ is bounded in E_0, using (M_2), we can conclude that the sequence of positive real numbers $\{M(\sigma_{p(x,y)}(u_n))\}$ is bounded from below by some positive number for n large enough. Invoking Theorem 2.2, we can deduce that $u_n \to u$ strongly in E_0. This completes the proof of Claim 3.2. □

4 Proof of Theorem 1.1

To prove Theorem 1.1, we shall use the following symmetric mountain pass theorem.

Theorem 4.1 (see [7, 35]) *Let $X = Y \oplus Z$ be an infinite-dimensional Banach space, where Y is finite-dimensional, and let $I \in C^1(X,\mathbb{R})$. Suppose that:*

(1) *I satisfies $(Ce)_c$-condition, for all $c > 0$;*
(2) *$I(0) = 0$, $I(-u) = I(u)$, for all $u \in X$;*
(3) *there exist constants $\rho, a > 0$ such that $I|_{\partial B_\rho \cap Z} \ge a$;*
(4) *for every finite-dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $I(u) \le 0$ on $\widetilde{X}\backslash B_R$.*

Then I possesses an unbounded sequence of critical values.

Let us first verify that functional J_λ satisfies the mountain pass geometry.

Claim 4.1 *Under the hypotheses of Theorem 1.1, there exists* $\Lambda > 0$ *such that, for each* $\lambda \in (0, \Lambda)$, *we can choose* $\rho > 0$ *and* $a > 0$ *such that*

$$J_\lambda(u) \geq a > 0, \quad \textit{for all } u \in E_0 \textit{ with } \|u\| = \rho.$$

Proof Let $\rho \in (0,1)$ and $u \in E_0$ be such that $\|u\|_{E_0} = \rho$. By assumption (F_2), for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\left|F(x,t)\right| \leq \frac{(\lambda+\epsilon)}{\overline{p}(x)}|t|^{\overline{p}(x)} + \frac{C_\epsilon}{\alpha(x)}|t|^{\alpha(x)}, \quad \text{for all } x \in \Omega, t \in \mathbb{R}. \tag{4.1}$$

Moreover, (M_2) gives

$$\widehat{M}(t) \geq \widehat{M}(1)t^\gamma, \quad \text{for all } t \in [0,1], \tag{4.2}$$

whereas (M_1) implies that $\widehat{M}(1) > 0$. Thus, using (4.1), (4.2) and (2.1), we obtain, for all $u \in E_0$, with $\|u\|_{E_0} = \rho$,

$$\begin{aligned} J_\lambda(u) &= \widehat{M}\big(\sigma_{p(x,y)}(u)\big) - \int_\Omega F(x,u)\,dx \\ &\geq \frac{\widehat{M}(1)}{(p^+)^\gamma}\big(\sigma_{p(x,y)}(u)\big)^\gamma - \int_\Omega \frac{\lambda+\epsilon}{\overline{p}(x)}|u|^{\overline{p}(x)}\,dx - \int_\Omega \frac{C_\epsilon}{\alpha(x)}|u|^{\alpha(x)}\,dx \\ &\geq \frac{\widehat{M}(1)}{(p^+)^\gamma}\min\big\{\|u\|_{E_0}^{\gamma p^+}, \|u\|_{E_0}^{\gamma p^-}\big\} - \frac{\epsilon+\lambda}{p^-}\max\big\{|u|_{p(x)}^{p^+}, |u|_{p(x)}^{p^-}\big\} \\ &\quad - \frac{C_\epsilon}{\alpha^-}\max\big\{|u|_{\alpha(x)}^{\alpha^+}, |u|_{\alpha(x)}^{\alpha^-}\big\} \\ &\geq \frac{\widehat{M}(1)}{(p^+)^\gamma}\|u\|_{E_0}^{\gamma p^+} - c_1(\epsilon+\lambda)\|u\|_E^{p^-} - c_2\|u\|_{E_0}^{\alpha^-} \\ &\geq \rho^{\gamma p^+}\left(\frac{\widehat{M}(1)}{(p^+)^\gamma} - c_1(\epsilon+\lambda)\rho^{p^- - \gamma p^+} - c_2\rho^{\alpha^- - \gamma p^+}\right), \end{aligned} \tag{4.3}$$

where $\rho = \|u\|_{E_0}$. Since $\epsilon > 0$ is arbitrary, let us choose

$$\epsilon = \frac{\widehat{M}(1)}{2c_1(p^+)^\gamma}\rho^{\gamma p^+ - p^-} > 0. \tag{4.4}$$

Then, by (4.3) and (4.4), we obtain

$$J_\lambda(u) \geq \rho^{\gamma p^+}\left(\frac{\widehat{M}(1)}{2(p^+)^\gamma} - \lambda c_1\rho^{p^- - \gamma p^+} - c_2\rho^{\alpha^- - \gamma p^+}\right). \tag{4.5}$$

Now, for each $\lambda > 0$, we define a continuous function, $g_\lambda : (0,\infty) \to \mathbb{R}$,

$$g_\lambda(s) = \lambda c_1 s^{p^- - \gamma p^+} + c_2 s^{\alpha^- - \gamma p^+}.$$

Since $1 < p^- < \gamma p^+ < \alpha^-$, it follows that

$$\lim_{s\to 0^+} g_\lambda(s) = \lim_{s\to +\infty} g_\lambda(s) = +\infty.$$

Thus we can find the infimum of g_λ. Note that equating

$$g'_\lambda(s) = \lambda c_1\big(p^- - \gamma p^+\big)s^{p^- - \gamma p^+ - 1} + c_2\big(\alpha^- - \gamma p^+\big)s^{\alpha^- - \gamma p^+ - 1} = 0,$$

we get

$$s_0 = s = \widetilde{C}\lambda^{\frac{1}{\alpha^- - p^-}},$$

where

$$\widetilde{C} := \left(\frac{c_1(\gamma p^+ - p^-)}{c_1(\alpha^- - \gamma p^+)}\right)^{\frac{1}{\alpha^- - p^-}} > 0.$$

Clearly, $s_0 > 0$. It can also be checked that $g''_\lambda(s_0) > 0$ and hence the infimum of $g_\lambda(s)$ is achieved at s_0.

Now, observing that

$$g_\lambda(s_0) = \big(c_1\widetilde{C}^{p^- - \gamma p^+} + c_2\widetilde{C}^{\alpha^- - \gamma p^+}\big)\lambda^{\frac{\alpha^- - \gamma p^+}{\alpha^- - p^-}} \to 0, \quad \text{as } \lambda \to 0^+,$$

we can infer from (4.5) that there exists $0 < \Lambda < \Lambda_0$ (see (F_2)) such that all for all $\lambda \in (0, \Lambda)$ we can choose ρ small enough and $\alpha > 0$ such that

$$J_\lambda(u) \geq a > 0, \quad \text{for all } u \in E_0 \text{ with } \|u\|_{E_0} = \rho.$$

This completes the proof of Claim 4.1. □

Claim 4.2 *Under the hypotheses of Theorem 1.1, for every finite-dimensional subspace $W \subset E_0$ there exists $R = R(W) > 0$ such that*

$$J_\lambda(u) \leq 0, \quad \textit{for all } u \in W, \textit{ with } \|u\|_{E_0} \geq R.$$

Proof In view of (F_3), we know that, for all $A > 0$, there exists $C_A > 0$ such that

$$F(x,t) \geq A|t|^{\gamma p^+} - C_A, \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}. \tag{4.6}$$

Again, (M_2) gives

$$\widehat{M}(t) \leq \widehat{M}(1)t^\gamma, \quad \text{for all } t \geq 1, \tag{4.7}$$

with $\widehat{M}(1) > 0$ by (M_1). By (4.6) and (4.7) we have

$$\begin{aligned} J_\lambda(u) &= \widehat{M}\big(\sigma_{p(x,y)}(u)\big) - \int_\Omega F(x,u)\,dx \\ &\leq \frac{\widehat{M}(1)}{(p^-)^\gamma}\big(\sigma_{p(x,y)}(u)\big)^\gamma - A\int_\Omega |u|^{\gamma p^+}\,dx + C_A|\Omega| \\ &\leq \frac{\widehat{M}(1)}{(p^-)^\gamma}\|u\|_{E_0}^{\gamma p^+} - A\int_\Omega |u|^{\gamma p^+}\,dx + C_A|\Omega|. \end{aligned}$$

Consequently, since $\|u\|_{E_0} > 1$, all norms on the finite-dimensional space W are equivalent, so there is $C_W > 0$ such that

$$\int_\Omega |u|^{\gamma p^+}\,dx \ge C_W \|u\|_{E_0}^{\gamma p^+}.$$

Let $R = R(W) > 0$. Then for all $u \in W$ with $\|u\|_{E_0} \ge R$ we obtain

$$J_\lambda(u) \le \|u\|_{E_0}^{p^+\gamma}\left(\frac{\widehat{M}(1)}{(p^-)^\gamma} - AC_W\right) + C_A|\Omega|. \tag{4.8}$$

So choosing in inequality 4.8,

$$A = \frac{2\widehat{M}(1)}{C_W (p^-)^\gamma},$$

we can conclude that

$$J_\lambda(u) \le 0, \quad \text{for all } u \in W \text{ with } \|u\|_{E_0} \ge R.$$

This completes the proof of Claim 4.2. □

Proof of Theorem 1.1 Obviously, $J_\lambda(0) = 0$ and by condition (F_5), J_λ is an even functional. Invoking Claims 3.1, 3.2, 4.1, and 4.2, and Theorem 4.1, we can now conclude that there indeed exists an unbounded sequence of solutions of problem (1.1). This completes the proof of Theorem 1.1. □

Acknowledgements
The authors would like to thank the referees for their comments and suggestions.

Funding
The first author was supported by the Tunisian Military Research Center for Science and Technology Laboratory. The second author was supported by the Fundamental Research Funds for Central Universities (2019B44914) and the National Key Research and Development Program of China (2018YFC1508100), the China Scholarship Council (201906710004), and the Jilin Province Undergraduate Training Program for Innovation and Entrepreneurship (201910204085). The third author was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) (Grant N.101.02.2017.04). The fourth author was supported by the Slovenian Research Agency grants P1-0292, N1-0114, N1-0083, N1-0064, and J1-8131.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that they made their contributions equally. All authors read and approved the final manuscript.

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Received: 1 May 2020 Accepted: 7 September 2020 Published online: 15 September 2020

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