Studia Scientiarum Mathematicarum Hungarica 48 (3), 342–353 (2011)
DOI: 10.1556/SScMath.48.2011.3.1176

TOPOLOGICAL MONOIDS OF MONOTONE INJECTIVE PARTIAL SELFMAPS OF $\mathbb{N}$ WITH COFINITE DOMAIN AND IMAGE

OLEG GUTIK[1] and DUŠAN REPOVŠ[2]

[1] Department of Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv, 79000, Ukraine
e-mail: o_gutik@franko.lviv.ua, ovgutik@yahoo.com

[2] Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, P.O.B. 2964, Ljubljana, 1001, Slovenia
e-mail: dusan.repovs@guest.arnes.si

Communicated by L. Márki

(Received October 9, 2009; accepted January 22, 2010)

Abstract

In this paper we study the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ of partial cofinal monotone bijective transformations of the set of positive integers $\mathbb{N}$. We show that the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. We also prove that every locally compact topology τ on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that $\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau\big)$ is a topological inverse semigroup, is discrete. Finally, we describe the closure of $\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau\big)$ in a topological semigroup.

1. Introduction and preliminaries

Our purpose is to study the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ of partial cofinal monotone bijective transformations of the set of positive integers $\mathbb{N}$. We shall show that the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its nontrivial group homomorphisms are either isomorphisms or group homomorphisms. We shall also prove that every locally compact topology τ on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that $\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau\big)$ is a

---

2000 *Mathematics Subject Classification.* Primary 22A15, 20M20; Secondary 20M18, 54H15.

Key words and phrases. Topological semigroup, topological inverse semigroup, bicyclic semigroup, semigroup of bijective partial transformations, closure.

topological inverse semigroup is discrete and we shall describe the closure of $\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau\big)$ in a topological semigroup.

In this paper all spaces will be assumed to be Hausdorff. Furthermore we shall follow the terminology of [5, 6, 8]. We shall denote the first infinite cardinal by ω and the cardinality of the set A by $|A|$. If Y is a subspace of a topological space X and $A \subseteq Y$, then we shall denote the topological closure and the interior of A in Y by $\mathrm{cl}_Y(A)$ and $\mathrm{Int}_Y(A)$, respectively.

An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\mathrm{inv} : S \to S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion.*

If S is a semigroup, then we shall denote the *band* (i.e. the subset of idempotents) of S by $E(S)$. If the band $E(S)$ is a nonempty subset of S, then the semigroup operation on S determines the partial order $\leqq$ on $E(S)$: $e \leqq f$ if and only if $ef = fe = e$. This order is called *natural.*

A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or *chain* if the semilattice operation admits a linear natural order on E. A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [11, Definition II.5.12] a chain L is called an ω-chain if L is isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leqq$. Let E be a semilattice and $e \in E$. We denote ${\downarrow}e = \{f \in E \mid f \leqq e\}$ and ${\uparrow}e = \{f \in E \mid e \leqq f\}$.

A *topological* (*inverse*) *semigroup* is a topological space together with a continuous multiplication (and an inversion, respectively). Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of a set X of cardinality λ together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \mathrm{dom}(\alpha\beta) = \{y \in \mathrm{dom}\,\alpha \mid y\alpha \in \mathrm{dom}\,\beta\}$, for $\alpha, \beta \in \mathscr{I}_\lambda$. The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the set X (see [6]). The symmetric inverse semigroup was introduced by Wagner [12] and it plays a major role in the theory of semigroups.

Let $\mathbb{N}$ be the set of all positive integers. We shall denote the semigroup of monotone, non-decreasing, injective partial transformations of $\mathbb{N}$ such that the sets $\mathbb{N} \setminus \mathrm{dom}\,\varphi$ and $\mathbb{N} \setminus \mathrm{rank}\,\varphi$ are finite for all $\varphi \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ by $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. Obviously, $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an inverse subsemigroup of the semigroup $\mathscr{I}_\omega$. The semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is called *the semigroup of cofinite monotone partial bijections* of $\mathbb{N}$.

We shall denote every element α of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ by $\begin{pmatrix} n_1 & n_2 & n_3 & n_4 & \ldots \\ m_1 & m_2 & m_3 & m_4 & \ldots \end{pmatrix}$ and this means that α maps the positive integer n_i into m_i for all $i = 1, 2, 3, \ldots$. In this case the following conditions hold:

(*i*) the sets $\mathbb{N} \setminus \{n_1, n_2, n_3, n_4, \ldots\}$ and $\mathbb{N} \setminus \{m_1, m_2, m_3, m_4, \ldots\}$ are finite; and

(*ii*) $n_1 < n_2 < n_3 < n_4 < \ldots$ and $m_1 < m_2 < m_3 < m_4 < \ldots$.

We observe that an element α of the semigroup $\mathscr{I}_\omega$ is an element of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ if and only if it satisfies the conditions (*i*) and (*ii*).

The bicyclic semigroup $\mathscr{C}(p,q)$ is the semigroup with the identity 1, generated by elements p and q, subject only to the condition $pq = 1$. The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathscr{C}(p,q)$ under h is a cyclic group (see [6, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups.

For example, the well-known result of Andersen [1] states that a (0–)simple semigroup is completely (0–)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete topology and a topological semigroup S can contain $\mathscr{C}(p,q)$ only as an open subset [7]. Neither stable nor Γ-compact topological semigroups can contain a copy of the bicyclic semigroup [2, 10]. Also, the bicyclic semigroup does not embed into a countably compact topological inverse semigroup [9].

Moreover, the conditions were given in [3] and [4] when a countable compact or pseudocompact topological semigroup does not contain the bicyclic semigroup. However, Banakh, Dimitrova and Gutik constructed, using set-theoretic assumptions (Continuum Hypothesis or Martin Axiom), an example of a Tychonoff countably compact topological semigroup which contains the bicyclic semigroup [4].

We remark that the bicyclic semigroup is isomorphic to the semigroup $\mathscr{C}_{\mathbb{N}}(\alpha,\beta)$ which is generated by partial transformations α and β of the set of positive integers $\mathbb{N}$, defined as follows:

$$(n)\alpha = n+1 \quad \text{if } n \geqq 1, \qquad \text{and} \qquad (n)\beta = n-1 \quad \text{if } n > 1.$$

Therefore the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ contains an isomorphic copy of the bicyclic semigroup $\mathscr{C}(p,q)$.

2. Algebraic properties of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$

PROPOSITION 2.1. *The following properties hold:*

(*i*) $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ *is a simple semigroup.*

(*ii*) $\alpha\mathscr{R}\beta$ $(\alpha\mathscr{L}\beta)$ *in* $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$ $(\operatorname{rank}\alpha = \operatorname{rank}\beta)$.

(*iii*) $\alpha\mathscr{H}\beta$ *in* $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ *if and only if* $\alpha=\beta$.

(*iv*) *For every* $\varepsilon,\iota\in E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ *there exists* $\alpha\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ *such that* $\alpha\alpha^{-1}=\varepsilon$ *and* $\alpha^{-1}\alpha=\iota$.

(*v*) $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ *is a bisimple semigroup.*

(*vi*) *If* $\varepsilon,\iota\in E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$, *then* $\varepsilon\leqq\iota$ *if and only if* $\operatorname{dom}\varepsilon\subseteqq\operatorname{dom}\iota$.

(*vii*) *The semilattice* $E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ *is isomorphic to* $(\mathscr{P}_{<\omega}(\mathbb{N}),\subseteqq)$ *under the mapping* $(\varepsilon)h=\mathbb{N}\setminus\operatorname{dom}\varepsilon$.

(*viii*) *Every maximal chain in* $E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ *is an* ω*-chain.*

Proof. (*i*) Let

$$\alpha=\begin{pmatrix} n_1 & n_2 & n_3 & n_4 & \dots\\ m_1 & m_2 & m_3 & m_4 & \dots\end{pmatrix} \quad\text{and}\quad \beta=\begin{pmatrix} k_1 & k_2 & k_3 & k_4 & \dots\\ l_1 & l_2 & l_3 & l_4 & \dots\end{pmatrix}$$

be any elements of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$, where $n_i,m_i,k_i,l_i\in\mathbb{N}$ for $i=1,2,3,\dots$. We put

$$\gamma=\begin{pmatrix} k_1 & k_2 & k_3 & k_4 & \dots\\ n_1 & n_2 & n_3 & n_4 & \dots\end{pmatrix} \quad\text{and}\quad \delta=\begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \dots\\ l_1 & l_2 & l_3 & l_4 & \dots\end{pmatrix}.$$

Then we have that $\gamma\alpha\delta=\beta$. Therefore $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\cdot\alpha\cdot\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})=\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ for any $\alpha\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ and hence $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is a simple semigroup.

Statement (*ii*) follows from definitions of relations $\mathscr{R}$ and $\mathscr{L}$ and Theorem 1.17 of [6]. Also, (*ii*) implies (*iii*). For the idempotents

$$\varepsilon=\begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \dots\\ m_1 & m_2 & m_3 & m_4 & \dots\end{pmatrix} \quad\text{and}\quad \iota=\begin{pmatrix} l_1 & l_2 & l_3 & l_4 & \dots\\ l_1 & l_2 & l_3 & l_4 & \dots\end{pmatrix}$$

we put $\alpha=\begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \dots\\ l_1 & l_2 & l_3 & l_4 & \dots\end{pmatrix}$. Then $\alpha\alpha^{-1}=\varepsilon$ and $\alpha^{-1}\alpha=\iota$, and hence (*iv*) holds. Also, (*v*) follows from (*ii*). All other assertions are trivial. □

Proposition 2.2. *For every* $\alpha,\beta\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$, *both sets* $\big\{\chi\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\mid\alpha\cdot\chi=\beta\big\}$ *and* $\big\{\chi\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\mid\chi\cdot\alpha=\beta\big\}$ *are finite. Consequently, every right translation and every left translation by an element of the semigroup* $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ *is a finite-to-one map.*

PROOF. We denote

$$A=\left\{\chi\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\mid\alpha\cdot\chi=\beta\right\}\text{ and }B=\left\{\chi\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\mid\alpha^{-1}\cdot\alpha\cdot\chi=\alpha^{-1}\cdot\beta\right\}.$$

Then $A\subseteqq B$ and the restriction of any partial map $\chi\in B$ to $\operatorname{dom}(\alpha^{-1}\cdot\alpha)$ coincides with the partial map $\alpha^{-1}\cdot\beta$. Since every partial map from $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is monotone and non-decreasing, the set B is finite and hence so is A. $\square$

For every $\gamma\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ $M_{\mathrm{dom}}(\gamma)=\min\{n\in\mathbb{N}\mid m\in\operatorname{dom}\gamma\text{ for all }m\geqq n\}$ and $M_{\mathrm{ran}}(\gamma)=\min\{n\in\mathbb{N}\mid m\in\operatorname{ran}\gamma\text{ for all }m\geqq n\}$ and put $M(\gamma)=\max\left\{M_{\mathrm{dom}}(\gamma),M_{\mathrm{ran}}(\gamma)\right\}$.

LEMMA 2.3. *For every idempotent ε of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ there exists an idempotent $\varepsilon_0\in E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)\setminus E\big(\mathscr{C}_{\mathbb{N}}(\alpha,\beta)\big)$ such that the following conditions hold:*

(1) $\varepsilon_0\leqq\varepsilon$;

(2) *ε_0 is the unity of a subsemigroup $\mathscr{C}$ of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$, which is isomorphic to the bicyclic semigroup; and*

(3) $\mathscr{C}\cap\mathscr{C}_{\mathbb{N}}(\alpha,\beta)=\varnothing$.

PROOF. Let ε be an arbitrary idempotent of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. We put $n_0=M(\varepsilon)+1$ and

$$\varepsilon_0=\begin{pmatrix} n_0{-}1 & n_0{+}1 & n_0{+}2 & n_0{+}3 & \cdots\\ n_0{-}1 & n_0{+}1 & n_0{+}2 & n_0{+}3 & \cdots\end{pmatrix}.$$

We define the partial monotone bijections $\widetilde{\alpha}\colon\mathbb{N}\rightharpoonup\mathbb{N}$ and $\widetilde{\beta}\colon\mathbb{N}\rightharpoonup\mathbb{N}$ as follows:

$$\widetilde{\alpha}=\begin{pmatrix} n_0{-}1 & n_0{+}1 & n_0{+}2 & n_0{+}3 & \cdots\\ n_0{-}1 & n_0{+}2 & n_0{+}3 & n_0{+}4 & \cdots\end{pmatrix}\quad\text{and}$$

$$\widetilde{\beta}=\begin{pmatrix} n_0{-}1 & n_0{+}2 & n_0{+}3 & n_0{+}4 & \cdots\\ n_0{-}1 & n_0{+}1 & n_0{+}2 & n_0{+}3 & \cdots\end{pmatrix}.$$

Let $\mathscr{C}$ a semigroup generated by the elements $\widetilde{\alpha}$ and $\widetilde{\beta}$. Then $\mathscr{C}$ satisfies the conditions (2)—(3) of the lemma and $\varepsilon_0=\widetilde{\alpha}\cdot\widetilde{\beta}$ is the identity of the semigroup $\mathscr{C}$ such that $\varepsilon_0\leqq\varepsilon$. $\square$

LEMMA 2.4. *For every $\lambda\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ there exist $\mu\in\mathscr{C}_{\mathbb{N}}(\alpha,\beta)$ and $\varepsilon\in E\big(\mathscr{C}_{\mathbb{N}}(\alpha,\beta)\big)$ such that $\lambda\cdot\varepsilon=\mu\cdot\varepsilon$ and $\varepsilon\cdot\lambda=\varepsilon\cdot\mu$.*

PROOF. The definition of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ implies that for every $\gamma \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ the notions $M_{\mathrm{dom}}(\gamma)$, $M_{\mathrm{ran}}(\gamma)$ and $M(\gamma)$ exist and they are unique, and hence they are well-defined.

We define partial maps $\mu : \mathbb{N} \rightharpoonup \mathbb{N}$ and $\varepsilon : \mathbb{N} \rightharpoonup \mathbb{N}$ as follows: $\operatorname{dom}\mu = \big\{ n \in \mathbb{N} \mid n \geqq M(\lambda) \big\}$ and $(i)\mu = (i)\lambda$ for all $i \in \operatorname{dom}\mu$ and $\operatorname{dom}\varepsilon = \big\{ n \in \mathbb{N} \mid n \geqq M(\lambda) \big\}$ and $(i)\mu = i$ for all $i \in \operatorname{dom}\mu$. Then we have $\lambda \cdot \varepsilon = \mu \cdot \varepsilon$ and $\varepsilon \cdot \lambda = \varepsilon \cdot \mu$. □

The proof of the following lemma is similar to the proof of Lemma 2.4.

LEMMA 2.5. *For every idempotent $\varphi \in E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ there exists $\varepsilon \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$ such that $\varphi \cdot \varepsilon \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$. More than, $\psi \cdot \varepsilon \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$ for every $\psi \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$ such that $\psi \leqq \varepsilon$.*

LEMMA 2.6. *For every idempotent $\varepsilon \in E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ there exists $\varphi \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$ such that $\varphi \leqq \varepsilon$.*

PROOF. The definition of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ implies that there exists a maximal positive integer n_{ε} such that $n_{\varepsilon} - 1 \notin \operatorname{dom}\varepsilon$. We put

$$\varphi = \begin{pmatrix} n_{\varepsilon} & n_{\varepsilon}{+}1 & n_{\varepsilon}{+}2 & n_{\varepsilon}{+}3 & \cdots \\ n_{\varepsilon} & n_{\varepsilon}{+}1 & n_{\varepsilon}{+}2 & n_{\varepsilon}{+}3 & \cdots \end{pmatrix}.$$

Then we get $\varphi \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$ and $\varphi \leqq \varepsilon$. □

LEMMA 2.7. *For every element $\lambda \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ there exists an idempotent ε of the subsemigroup $\mathscr{C}_{\mathbb{N}}(\alpha, \beta)$ such that $\lambda \cdot \varepsilon \cdot \lambda^{-1}, \lambda^{-1} \cdot \varepsilon \cdot \lambda \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$.*

PROOF. By Lemma 2.4 there exists $\mu \in \mathscr{C}_{\mathbb{N}}(\alpha, \beta)$ and $\varepsilon \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$ such that $\lambda \cdot \varepsilon = \mu \cdot \varepsilon$ and $\varepsilon \cdot \lambda = \varepsilon \cdot \mu$. Therefore, we have that

$$\begin{aligned} \lambda \cdot \varepsilon \cdot \lambda^{-1} &= \lambda \cdot \varepsilon \cdot \varepsilon \cdot \lambda^{-1} = (\lambda \cdot \varepsilon) \cdot (\lambda \cdot \varepsilon)^{-1} = (\mu \cdot \varepsilon) \cdot (\mu \cdot \varepsilon)^{-1} \\ &= \mu \cdot \varepsilon \cdot \varepsilon \cdot \mu^{-1} = \mu \cdot \varepsilon \cdot \mu^{-1} \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big) \end{aligned}$$

and similarly $\lambda^{-1} \cdot \varepsilon \cdot \lambda = \mu^{-1} \cdot \varepsilon \cdot \mu \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$. □

LEMMA 2.8. *Let S be a semigroup and $h : \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to S$ a homomorphism such that $(\varepsilon)h = (\varphi)h$ for some distinct idempotents $\varepsilon, \varphi \in E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$. Then $(\varepsilon)h = (\psi)h$, for every $\psi \in E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$.*

PROOF. We consider the following cases:

(1) $\varepsilon, \varphi \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\big)$;

(2) ε and φ are distinct comparable idempotents of $E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$;

(3) ε and φ are distinct incomparable idempotents of $E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$.

Suppose case (1) holds. Then by Corollary 1.32 of [6] we have that $(\chi)h = (\varepsilon)h$ for every $\chi \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha,\beta)\big)$. Let ψ be any idempotent of $E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ such that $\psi \notin E\big(\mathscr{C}_{\mathbb{N}}(\alpha,\beta)\big)$. Then by Proposition 2.1(v) there exists $\gamma \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that $\gamma\cdot\gamma^{-1} = \varepsilon$ and $\gamma^{-1}\cdot\gamma = \psi$. By Lemma 2.7 there exist $\varepsilon_0, \varepsilon_1 \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha,\beta)\big)$ such that $\gamma^{-1}\cdot\varepsilon_0\cdot\gamma = \varepsilon_1 \in E\big(\mathscr{C}_{\mathbb{N}}(\alpha,\beta)\big)$. Therefore, we have that

$$\begin{aligned}(\psi)h &= (\psi\cdot\psi)h = \left(\gamma^{-1}\cdot\gamma\cdot\gamma^{-1}\cdot\gamma\right)h = \left(\gamma^{-1}\cdot\varepsilon\cdot\gamma\right)h \\ &= \left(\gamma^{-1}\right)h\cdot(\varepsilon)h\cdot(\gamma)h = \left(\gamma^{-1}\right)h\cdot(\varepsilon_0)h\cdot(\gamma)h = \left(\gamma^{-1}\cdot\varepsilon_0\cdot\gamma\right)h \\ &= (\varepsilon_1)h = (\varepsilon)h.\end{aligned}$$

Suppose case (2) holds. Without loss of generality we can assume that $\varepsilon \leqq \varphi$. Then $(\varepsilon)h = (\varepsilon_1)h = (\varphi)h$ for every idempotent $\varepsilon_1 \in E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ such that $\varepsilon \leqq \varepsilon_1 \leqq \varphi$. Therefore, without loss of generality we can assume that $\operatorname{dom}\varphi \setminus \operatorname{dom}\varepsilon$ is singleton. Let $\{n_{\varepsilon}^{\varphi}\} = \operatorname{dom}\varphi \setminus \operatorname{dom}\varepsilon$. Let j be the minimal integer of $\operatorname{dom}\varepsilon$ such that $(i)\varepsilon = i$ for all $i \geqq j$. We put $\varepsilon_0 = \begin{pmatrix} n_{\varepsilon}^{\varphi} & j & j{+}1 & j{+}2 & \cdots \\ n_{\varepsilon}^{\varphi} & j & j{+}1 & j{+}2 & \cdots \end{pmatrix}$ and $\lambda = \begin{pmatrix} n_{\varepsilon}^{\varphi} & j & j{+}1 & j{+}2 & \cdots \\ j{-}1 & j & j{+}1 & j{+}2 & \cdots \end{pmatrix}$. Then $\lambda^{-1}\cdot\varepsilon_0\cdot\varphi\cdot\varepsilon_0\cdot\lambda$ and $\lambda^{-1}\cdot\varepsilon_0\cdot\varepsilon\cdot\varepsilon_0\cdot\lambda$ are distinct idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}(\alpha,\beta)$. Therefore, we have that

$$\begin{aligned}\left(\lambda^{-1}\cdot\varepsilon_0\cdot\varphi\cdot\varepsilon_0\cdot\lambda\right)h &= \left(\lambda^{-1}\cdot\varepsilon_0\right)h\cdot(\varphi)h\cdot(\varepsilon_0\cdot\lambda)h \\ &= \left(\lambda^{-1}\cdot\varepsilon_0\right)h\cdot(\varepsilon)h\cdot(\varepsilon_0\cdot\lambda)h = \left(\lambda^{-1}\cdot\varepsilon_0\cdot\varepsilon\cdot\varepsilon_0\cdot\lambda\right)h,\end{aligned}$$

and hence case (1) holds.

Suppose case (3) holds. Then we have that $(\varepsilon)h = (\varepsilon\cdot\varepsilon)h = (\varepsilon)h\cdot(\varepsilon)h = (\varepsilon)h\cdot(\varphi)h = (\varepsilon\cdot\varphi)h$. Since the idempotents ε and φ are distinct and incomparable we conclude that $\varepsilon\cdot\varphi < \varepsilon$ and $\varepsilon\cdot\varphi < \varphi$, and hence case (2) holds. □

THEOREM 2.9. *Let S be a semigroup and $h\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to S$ a non-annihilating homomorphism. Then either h is a monomorphism or $\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)h$ is a cyclic subgroup of S.*

PROOF. Suppose that $h\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to S$ is not an isomorphism “into”. Then $(\alpha)h = (\beta)h$, for some distinct $\alpha, \beta \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. Since $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an inverse semigroup we conclude that

$$\left(\alpha^{-1}\right)h = \left((\alpha)h\right)^{-1} = \left((\beta)h\right)^{-1} = \left(\beta^{-1}\right)h$$

and hence $\left(\alpha\alpha^{-1}\right)h = \left(\beta\beta^{-1}\right)h$. Therefore the assertion of Lemma 2.8 holds. Since every homomorphic image of an inverse semigroup is an inverse semigroup we conclude that $\left(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\right)h$ is a subgroup of S.

Since the map $h\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to S$ is a group homomorphism we have that h generates a group congruence $\mathfrak{h}$ on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. If $\mathfrak{c}$ is any congruence on the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ then the mapping $\mathfrak{c} \mapsto \mathfrak{c} \vee \mathfrak{g}$ maps the congruence $\mathfrak{c}$ onto a group congruence $\mathfrak{c} \vee \mathfrak{g}$, where $\mathfrak{g}$ is the least group congruence on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ (cf. [11, Section III]).

Such a mapping is a map from the lattice of all congruences of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ onto the lattice of all group congruences of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ [11]. By Lemma III.5.2 of [11], the elements γ and δ of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ are $\mathfrak{g}$-equivalent if and only if there exists an idempotent $\varepsilon \in E\left(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\right)$ such that $\gamma \cdot \varepsilon = \delta \cdot \varepsilon$. Lemma 2.4 implies that for every $\gamma \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ there exists $\delta \in \mathscr{C}_{\mathbb{N}}(\alpha, \beta)$ such that $\gamma\mathfrak{g}\delta$. Therefore the least group congruence $\mathfrak{g}$ on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ induces the least group congruence on its subsemigroup $\mathscr{C}_{\mathbb{N}}(\alpha, \beta)$.

We observe that $\gamma\mathfrak{g}\delta$ in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ (or in $\mathscr{C}_{\mathbb{N}}(\alpha, \beta)$) if and only if there exists a positive integer i such that the restrictions of the partial mapping γ and δ onto the set $\{i, i+1, i+2, \ldots\}$ coincide. Then we define the map $f\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to \mathbb{Z}_{+}$ onto the additive group of integers as follow:

$$(1) \qquad (\gamma)f = n \quad \text{if} \;\; (i)\gamma = i + n \;\; \text{for infinitely many positive integers} \;\; i.$$

The definition of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ implies that such a map f is well-defined. The map $f\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to \mathbb{Z}_{+}$ generates the least group congruence $\mathfrak{g}$ on the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ and hence f is a group homomorphism. This completes the proof of the theorem. □

REMARK 2.10. We observe that the following conditions are equivalent:

(*i*) $\gamma\mathfrak{g}\delta$ in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$;

(*ii*) there exists $\varepsilon \in E\left(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\right)$ such that $\varepsilon \cdot \gamma = \varepsilon \cdot \delta$;

(*iii*) there exists $\varepsilon \in E\left(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\right)$ such that $\varepsilon \cdot \gamma = \varepsilon \cdot \delta$; and

(*iv*) there exists $\varepsilon \in E\left(\mathscr{C}_{\mathbb{N}}(\alpha, \beta)\right)$ such that $\gamma \cdot \varepsilon = \delta \cdot \varepsilon$.

3. On topological semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$

LEMMA 3.1. *If E is an infinite semilattice satisfying that $\uparrow e$ is finite for all $e \in E$, then the only locally compact, Hausdorff topology relative to which E is a topological semilattice is the discrete topology.*

PROOF. Assume that E has a locally compact, Hausdorff topology under which it is a topological semilattice, and that E has a non-isolated point e in this topology. Since E is Hausdorff, every neighbourhood of e has infinitely many elements. Let K be a compact neighbourhood of e. Then there is a net $\langle e_i \rangle_{i\in\mathscr{I}} \subseteqq \operatorname{Int}_E(K) \setminus \{e\}$ with $\lim_i e_i = e$. Since E is a topological semilattice, $e = e^2 = e \cdot \lim_i e_i = \lim_i (e \cdot e_i)$, and since $\uparrow e$ is finite, we can assume that $e \cdot e_i < e$ are distinct for all $i \in \mathscr{I}$. Since $e \in \operatorname{Int}_E(K)$, we can also assume $e \cdot e_i \in \operatorname{Int}_E(K)$ for all i. Thus, we can assume $e_i < e$ satisfy $e_i \in \operatorname{Int}_E(K)$ for all i.

Choose one such e_j, and then note that $e_j = e_j \cdot e = e_j \cdot \lim_i e_i = \lim_i\, (e_j \cdot e_i)$. The same argument we just gave for e then implies that $e_i < e_j$ for all i, that $\lim_i e_i = e_j$, and that $e_i \in \operatorname{Int}_E(K)$ for all i. We let $e_1 = e_j$, and now repeat the argument. Since K is compact, this sequence has a limit point, s in K, and the continuity of the semilattice operation implies s is another idempotent and that $s < e_n$ for all n. But then $\uparrow s$ is infinite, contrary to our hypothesis. Hence E cannot have a nonisolated point, so it is discrete. □

Proposition 2.2 and Lemma 3.1 imply the following:

LEMMA 3.2. *If $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is a locally compact Hausdorff topological semigroup, then $E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ with the induces from $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ topology is a discrete topological semilattice.*

THEOREM 3.3. *Every locally compact Hausdorff topology on the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is a topological inverse semigroup, is discrete.*

PROOF. By Lemma 3.2, the band $E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ is a discrete topological space. Since $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is a topological inverse semigroup, the maps $h\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ and $f\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)$ defines by the formulae $(\alpha)h = \alpha \cdot \alpha^{-1}$ and $(\alpha)f = \alpha^{-1} \cdot \alpha$ are continuous and for every two idempotents ε and φ of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ there exists a unique element χ in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that $\chi \cdot \chi^{-1} = \varepsilon$ and $\chi^{-1} \cdot \chi = \varphi$, we have that every element of the topological semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an isolated point of the topological space $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. □

The following theorem describes the closure of the discrete semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ in a topological semigroup.

THEOREM 3.4. *If a topological semigroup S contains $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ as a proper, dense discrete subsemigroup, then $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an open subsemigroup of S and $S\setminus\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an ideal of S.*

PROOF. The first assertion of the theorem follows from Theorem 3.3.9 of [8].

Suppose that $\chi\in S\setminus\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ and $\alpha\in S$. If $\chi\cdot\alpha\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ then there exist open neighbourhoods $U(\chi)$ and $U(\alpha)$ of χ and α in S, respectively, such that $U(\chi)\cdot U(\alpha)=\{\chi\cdot\alpha\}$. We observe that the set $U(\chi)\cap\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is infinite and fix any point $\mu\in U(\alpha)\cap\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. Hence we have

$$\big(U(\chi)\cap\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)\cdot\mu\subseteqq\big(U(\chi)\cap\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)\cdot\big(U(\alpha)\cap\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big)=\{\chi\cdot\alpha\}.$$

This contradicts Proposition 2.2. The obtained contradiction implies that $\chi\cdot\alpha\in S\setminus\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$.

The proof of the assertion $\alpha\cdot\chi\in S\setminus\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is similar. □

Theorems 3.3 and 3.4 imply the following:

COROLLARY 3.5. *If a topological semigroup S contains $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ as a proper, dense locally compact subsemigroup, then $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an open subsemigroup of S and $S\setminus\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an ideal of S.*

The following example shows that a remainder of a closure of the discrete (and hence of a locally compact topological inverse) semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ in a topological inverse semigroup contains only a zero element.

EXAMPLE 3.6. Let $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ be a discrete topological semigroup and let S be the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ with adjoined zero $\mathscr{O}$, i.e. $\mathscr{O}\cdot\alpha=\alpha\cdot\mathscr{O}=\mathscr{O}\cdot\mathscr{O}=\mathscr{O}$ for all $\alpha\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. We define a topology τ on S as follows:

(a) all non-zero elements of the semigroup S are isolated points in (S,τ); and

(b) the family $\mathscr{B}(\mathscr{O})=\big\{U_i(\mathscr{O})\mid i=1,2,3,\dots\big\}$, where $U_i(\mathscr{O})=\{\mathscr{O}\}\cup\big\{\alpha\in\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\mid|\mathbb{N}\setminus\operatorname{dom}\alpha|\geqq i \text{ and } |\mathbb{N}\setminus\operatorname{ran}\alpha|\geqq i\big\}$, determines a base of the topology τ at the point $\mathscr{O}\in S$.

We observe that (S,τ) is a topological inverse semigroup which contains $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ as a dense subsemigroup and τ is not a locally compact topology on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$.

We recall that a topological space X is said to be:

- *countably compact* if each closed discrete subspace of X is finite;
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded;
- *sequentially compact* if each sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ has a convergent subsequence.

A topological semigroup S is called Γ-compact if for every $x \in S$ the closure of the set $\{x, x^2, x^3, \dots\}$ is a compactum in S (see [10]). Since the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ contains the bicyclic semigroup as a subsemigroup the results obtained in [2], [3], [4], [9], [10] imply that *if a topological semigroup S satisfies one of the following conditions: (i) S is compact; (ii) S is Γ-compact; (iii) the square $S \times S$ is countably compact; (iv) S is a countably compact topological inverse semigroup; or (v) the square $S \times S$ is a Tychonoff pseudocompact space, then S does not contain the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$.*

The following example shows that the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ embeds into a locally compact topological semigroup as a discrete subsemigroup.

Example 3.7. Let $\mathbb{Z}_+$ be the additive group of integers. Let $h \colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to \mathbb{Z}_+$ be a group homomorphism defined by the formula (1). On the set $S = \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \sqcup \mathbb{Z}_+$ we define the semigroup operation '$*$' as follows:

$$x * y = \begin{cases} x \cdot y, & \text{if } x, y \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}); \\ x + (y)h, & \text{if } x \in \mathbb{Z}_+ \text{ and } y \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}); \\ (x)h + y, & \text{if } y \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \text{ and } x \in \mathbb{Z}_+; \\ (x)h + (y)h, & \text{if } x, y \in \mathbb{Z}_+, \end{cases}$$

where '$\cdot$' and '$+$' are the semigroup operation in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ and the group operation in $\mathbb{Z}_+$, respectively. The semigroup S is called the *adjunction semigroup of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ and $\mathbb{Z}_+$ relative to homomorphism h* (see [5, Vol. 1, pp. 77–80]).

Let $\leqq_c$ be the canonical partial order on the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ (see [11, Section II.1]), i.e.

$$\alpha \leqq_c \beta \quad \text{if and only if there exists} \quad \varepsilon \in E\big(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})\big) \quad \text{such that} \quad \alpha = \beta \cdot \varepsilon.$$

We observe that if $\alpha \leqq_c \beta$ in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$, then $(\alpha)h = (\beta)h$. For every $x \in \mathbb{Z}_+$ and $\alpha \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that $(\alpha)h = x$ we put

$$U_\alpha(x) = \{x\} \cup \big\{\beta \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \mid (\beta)h = x \text{ and } \alpha \not\leqq_c \beta\big\}.$$

We define a topology τ on S as follows:

(i) all elements of the subsemigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ are isolated points in (S,τ); and

(ii) the family $\mathscr{B}(x) = \{U_\alpha(x) \mid (\alpha)h = x\}$ determines a base of the topology at the point $x \in \mathbb{Z}_+$.

Simple verifications show that (S,τ) is a 0-dimensional locally compact topological inverse semigroup.

Acknowledgements. This research was supported by the Slovenian Research Agency grants P1-0292-0101, J1-2057-0101 and BI-UA/09-10/005. We thank the referee for many comments and suggestions. In particular, we are very grateful to the referee for the new statement and proof of Lemma 3.1.

REFERENCES

[1] Andersen, O., *Ein Bericht über die Struktur abstrakter Halbgruppen,* PhD Thesis, Hamburg (1952).

[2] Anderson, L. W., Hunter, R. P. and Koch, R. J., Some results on stability in semigroups, *Trans. Amer. Math. Soc.,* **117** (1965), 521–529. *MR* **30**#2095

[3] Banakh, T., Dimitrova S. and Gutik, O., The Rees–Suschkiewitsch Theorem for simple topological semigroups, *Mat. Stud.,* **31(2)** (2009), 211–218. *MR* **2011c**:22003

[4] Banakh, T., Dimitrova S. and Gutik, O., Embedding the bicyclic semigroup into countably compact topological semigroups, *Topology Appl.,* **157** (2010), no. 18, 2803–2814. (arXiv:0811.4276). *MR* 2729339

[5] Carruth, J. H., Hildebrant, J. A. and Koch, R. J., *The Theory of Topological Semigroups,* Vol. I, Marcel Dekker, Inc., New York and Basel (1983); Vol. II, Marcel Dekker, Inc., New York and Basel (1986). *MR* **87j**:22007 and *MR* 0691307 (**84g**:22002)

[6] Clifford, A. H. and Preston, G. B., *The Algebraic Theory of Semigroups,* Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I. (1961); Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I. (1967). *MR* 0132791 (**24**#A2627) and *MR* 0218472 (**36**#1558)

[7] Eberhart, C. and Selden, J., On the closure of the bicyclic semigroup, *Trans. Amer. Math. Soc.,* **144** (1969), 115–126. *MR* **40**#5767

[8] Engelking, R., *General Topology,* 2nd ed., Heldermann, Berlin (1989). *MR* **91c**:54001

[9] Gutik, O. and Repovš, D., On countably compact 0-simple topological inverse semigroups, *Semigroup Forum,* **75(2)** (2007), 464–469. *MR* **2008g**:22001

[10] Hildebrant, J. A. and Koch, R. J., Swelling actions of Γ-compact semigroups, *Semigroup Forum,* **33** (1988), no. 1, 65–85. *MR* **87e**:22007

[11] Petrich, M., *Inverse Semigroups,* John Wiley & Sons, New York (1984). *MR* **85k**:20001

[12] Wagner, V. V., Generalized groups, *Dokl. Akad. Nauk SSSR,* **84** (1952), 1119–1122 (in Russian).