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ON MONOIDS OF INJECTIVE PARTIAL COFINITE SELFMAPS

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ABSTRACT. We study the semigroup $\mathscr{I}_{\lambda}^{\mathrm{cf}}$ of injective partial cofinite selfmaps of an infinite cardinal λ. We show that $\mathscr{I}_{\lambda}^{\mathrm{cf}}$ is a bisimple inverse semigroup and each chain of idempotents in $\mathscr{I}_{\lambda}^{\mathrm{cf}}$ is contained in a bicyclic subsemigroup of $\mathscr{I}_{\lambda}^{\mathrm{cf}}$, we describe the Green relations on $\mathscr{I}_{\lambda}^{\mathrm{cf}}$ and we prove that every non-trivial congruence on $\mathscr{I}_{\lambda}^{\mathrm{cf}}$ is a group congruence. Also, we describe the structure of the quotient semigroup $\mathscr{I}_{\lambda}^{\mathrm{cf}}/\sigma$, where σ is the least group congruence on $\mathscr{I}_{\lambda}^{\mathrm{cf}}$.

1. Introduction and preliminaries

In this paper we shall denote the first infinite ordinal by ω and the cardinality of the set A by $|A|$. We shall identify all cardinals with their corresponding initial ordinals. We shall denote the set of integers by $\mathbb{Z}$ and the additive group of integers by $\mathbb{Z}(+)$.

A semigroup S is called *inverse* if for every element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\mathrm{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

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A congruence $\mathfrak{C}$ on a semigroup S is called *non-trivial* if $\mathfrak{C}$ is distinct from the universal and the identity congruences on S, and a *group congruence* if the quotient semigroup $S/\mathfrak{C}$ is a group.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. A *maximal chain* of a semilattice E is a chain which is not properly contained in any other chain of E.

The Axiom of Choice implies the existence of maximal chains in every partially ordered set. According to [13: Definition II.5.12], a chain L is called an *ω-chain* if L is isomorphic to $\{0, -1, -2, -3, \dots\}$ with the usual order $\leqslant$ or equivalently, if L is isomorphic to $(\omega, \max)$. Let E be a semilattice and $e \in E$. We put ${\downarrow}e = \{f \in E \mid f \leqslant e\}$ and ${\uparrow}e = \{f \in E \mid e \leqslant f\}$. By $(\mathcal{P}_{<\omega}(\lambda), \cup)$ we shall denote the *free semilattice with identity* over a set of cardinality $\lambda \geqslant \omega$, i.e., $(\mathcal{P}_{<\omega}(\lambda), \cup)$ is the set of all finite subsets (with the empty set) of λ with the semilattice operation "union".

If S is a semigroup, then we shall denote the Green relations on S by $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{H}$ (see [5]). A semigroup S is called *simple* if S does not contain proper two-sided ideals and *bisimple* if S has only one $\mathcal{D}$-class.

The bicyclic semigroup $\mathcal{C}(p, q)$ is the semigroup with the identity 1 generated by elements p and q subject only to the condition $pq = 1$. The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathcal{C}(p, q)$ under h is a cyclic group (see [5: Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example a well-known Andersen's result [1] states that a (0-)simple semigroup is completely (0-)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete topology [7]. The problem of embeddability of the bicycle semigroup into compact-like semigroups was studied in [2–4,8,11].

Remark 1. We observe that the bicyclic semigroup is isomorphic to the semigroup $\mathcal{C}_{\mathbb{N}}(\alpha, \beta)$ which is generated by partial transformations α and β of the set of positive integers $\mathbb{N}$, defined as follows: $(n)\alpha = n+1$ if $n \geqslant 1$ and $(n)\beta = n-1$ if $n > 1$ (see [13: Exercise IV.1.11(ii)]).

If T is a semigroup, then we say that a subsemigroup S of T is a *bicyclic subsemigroup of* T if S is isomorphic to the bicyclic semigroup $\mathcal{C}(p,q)$.

Hereafter we shall assume that λ is an infinite cardinal. If $\alpha\colon X \rightharpoonup Y$ is a partial map, then we shall denote the domain and the range of α by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of an infinite set X of cardinality λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x\in\operatorname{dom}(\alpha\beta) = \{y\in\operatorname{dom}\alpha \mid y\alpha\in\operatorname{dom}\beta\}, \quad \text{for } \alpha,\beta\in\mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the set X (see [5: Section 1.9]). The symmetric inverse semigroup was introduced by Vagner [21] and it plays a major role in the theory of semigroups.

Furthermore, we shall identify the cardinal $\lambda = |X|$ with the set X. By $\mathscr{I}_\lambda^{\mathrm{cf}}$ we shall denote a subsemigroup of injective partial selfmaps of λ with cofinite domains and ranges in $\mathscr{I}_\lambda$, i.e.,

$$\mathscr{I}_\lambda^{\mathrm{cf}} = \{\alpha \in \mathscr{I}_\lambda \mid |\lambda\setminus\operatorname{dom}\alpha| < \infty \quad \text{and} \quad |\lambda\setminus\operatorname{ran}\alpha| < \infty\}.$$

Obviously, $\mathscr{I}_\lambda^{\mathrm{cf}}$ is an inverse submonoid of the semigroup $\mathscr{I}_\lambda$. We shall call the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ the *monoid of injective partial cofinite selfmaps* of λ.

Next, by $\mathbb{I}$ we shall denote the identity and by $H(\mathbb{I})$ the group of units of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$.

It well known that each partial injective cofinite selfmap f of λ induces a homeomorphism $f^*\colon \lambda^* \to \lambda^*$ of the remainder $\lambda^* = \beta\lambda\setminus\lambda$ of the Stone-Čech compactification of the discrete space λ. Moreover, under some set theoretic axioms (like **PFA** or **OCA**), each homeomorphism of ω^* is induced by some partial injective cofinite selfmap of ω (see [15]–[20]). So the inverse semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ admits a natural homomorphism $\mathfrak{h}\colon \mathscr{I}_\lambda^{\mathrm{cf}} \to \mathcal{H}(\lambda^*)$ to the homeomorphism group $\mathcal{H}(\lambda^*)$ of λ^* and this homomorphism is surjective under certain set theoretic assumptions.

The semigroups $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ of injective isotone partial selfmaps with cofinite domains and images of positive integers and integers, respectively, were studied in [9] and [10]. There it was proved that the semigroups $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ have properties similar to the bicyclic semigroup: they are bisimple and every non-trivial homomorphic image of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is a group, and moreover, the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ has $\mathbb{Z}(+)$ as a maximal group image and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ has $\mathbb{Z}(+)\times\mathbb{Z}(+)$, respectively.

In this paper we shall study algebraic properties of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. We shall show that $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a bisimple inverse semigroup and every chain of idempotents in $\mathscr{I}_\lambda^{\mathrm{cf}}$ is contained in a bicyclic subsemigroup of $\mathscr{I}_\lambda^{\mathrm{cf}}$, we shall describe the Green relations on $\mathscr{I}_\lambda^{\mathrm{cf}}$ and we shall prove that every non-trivial congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a group congruence. Also, we shall describe the structure of the quotient semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$, where σ is the least group congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$.

2. Algebraic properties of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$

Proposition 2.1.

(i) *$\mathscr{I}_\lambda^{\mathrm{cf}}$ is a simple semigroup.*

(ii) *An element α of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ is an idempotent if and only if $(x)\alpha = x$ for every $x \in \operatorname{dom}\alpha$.*

(iii) *If $\varepsilon, \iota \in E(\mathscr{I}_\lambda^{\mathrm{cf}})$, then $\varepsilon \leqslant \iota$ if and only if $\operatorname{dom}\varepsilon \subseteq \operatorname{dom}\iota$.*

(iv) *The semilattice $E(\mathscr{I}_\lambda^{\mathrm{cf}})$ is isomorphic to $(\mathscr{P}_{<\omega}(\lambda), \cup)$ under the mapping $(\varepsilon)h = \lambda \setminus \operatorname{dom}\varepsilon$.*

(v) *Every maximal chain in $E(\mathscr{I}_\lambda^{\mathrm{cf}})$ is an ω-chain.*

(vi) *$\alpha\mathscr{R}\beta$ in $\mathscr{I}_\lambda^{\mathrm{cf}}$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$.*

(vii) *$\alpha\mathscr{L}\beta$ in $\mathscr{I}_\lambda^{\mathrm{cf}}$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$.*

(viii) *$\alpha\mathscr{H}\beta$ in $\mathscr{I}_\lambda^{\mathrm{cf}}$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and $\operatorname{ran}\alpha = \operatorname{ran}\beta$.*

(ix) *$\alpha\mathscr{D}\beta$ for all $\alpha, \beta \in \mathscr{I}_\lambda^{\mathrm{cf}}$ and hence the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ is bisimple.*

Proof.

(i) We shall show that $\mathscr{I}_\lambda^{\mathrm{cf}} \cdot \alpha \cdot \mathscr{I}_\lambda^{\mathrm{cf}} = \mathscr{I}_\lambda^{\mathrm{cf}}$ for every element $\alpha \in \mathscr{I}_\lambda^{\mathrm{cf}}$. Let α and β are arbitrary elements of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. We shall choose elements $\gamma, \delta \in \mathscr{I}_\lambda^{\mathrm{cf}}$ such that $\gamma \cdot \alpha \cdot \delta = \beta$. We put $\operatorname{dom}\gamma = \operatorname{dom}\beta$, $\operatorname{ran}\gamma = \operatorname{dom}\alpha$, $\operatorname{dom}\delta = \operatorname{ran}\alpha$ and $\operatorname{ran}\delta = \operatorname{ran}\beta$. Since the sets $\lambda \setminus \operatorname{dom}\alpha$ and $\lambda \setminus \operatorname{dom}\beta$ are finite we conclude that there exists a bijective map $f\colon \operatorname{dom}\alpha \to \operatorname{dom}\beta$. We put $\gamma = f$ and $(((x)\gamma)\,\alpha)\,\delta = (x)\beta$ for all $x \in \operatorname{dom}\beta$. Then we have that $\gamma \cdot \alpha \cdot \delta = \beta$.

Statements (ii)–(v) are trivial and they follow from the definition of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. The proofs of (vi)–(viii) follow trivially from the fact that $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a regular semigroup, and [12: Proposition 2.4.2, Exercise 5.11.2].

(ix) Let α and β be arbitrary elements of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. Since the sets $\lambda \setminus \operatorname{dom}\alpha$ and $\lambda \setminus \operatorname{ran}\beta$ are finite we conclude that there exists a bijective map $\gamma\colon \operatorname{dom}\alpha \to \operatorname{ran}\beta$. Then $\gamma \in \mathscr{I}_\lambda^{\mathrm{cf}}$ and by statements (vi) and (vii) we have that $\alpha\mathscr{R}\gamma$ and $\beta\mathscr{L}\gamma$ in $\mathscr{I}_\lambda^{\mathrm{cf}}$ and hence $\alpha\mathscr{D}\beta$ in $\mathscr{I}_\lambda^{\mathrm{cf}}$. □

We denote the group of all bijective transformations of a set of cardinality λ by $\mathscr{S}_\lambda$. Then we get the following:

Corollary 2.2. *The group of units $H(\mathbb{I})$ of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ is isomorphic to $\mathcal{S}_\lambda$.*

For any idempotents ε and ι of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ we denote:

$$H(\varepsilon,\iota)=\left\{\chi\in\mathscr{I}_\lambda^{\mathrm{cf}} \mid \chi\cdot\chi^{-1}=\varepsilon \text{ and } \chi^{-1}\cdot\chi=\iota\right\} \qquad \text{and} \qquad H(\varepsilon)=H(\varepsilon,\varepsilon).$$

Proposition 2.1(viii) implies that the set $H(\varepsilon,\iota)$ is a $\mathscr{H}$-class and the set $H(\varepsilon)$ is a a maximal subgroup in $\mathscr{I}_\lambda^{\mathrm{cf}}$ for all idempotents $\varepsilon,\iota\in\mathscr{I}_\lambda^{\mathrm{cf}}$.

Corollary 2.2 and [5: Proposition 2.20] imply the following:

Corollary 2.3. *Every maximal subgroup of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ is isomorphic to $\mathcal{S}_\lambda$.*

Proposition 2.4. $\left|\mathscr{I}_\lambda^{\mathrm{cf}}\right|=2^{|\lambda|}$.

Proof. Since $|\lambda\times\lambda|=|\lambda|$ we have that $|\mathcal{S}_\lambda|\leqslant 2^{|\lambda\times\lambda|}=2^{|\lambda|}$. Since $|\lambda\sqcup\lambda|=|\lambda|$ there exists an injective map $f\colon\mathcal{P}(\lambda)\to\mathcal{S}_{\lambda\sqcup\lambda}$ from the set $\mathcal{P}(\lambda)$ of all subset of the cardinal λ into the group $\mathcal{S}_{\lambda\sqcup\lambda}$ defined in the following way: $f(A)$ is a bijection on $\lambda\sqcup\lambda$ with support $A\sqcup A$. Then we have that $|\mathcal{S}_\lambda|\geqslant 2^{|\lambda\sqcup\lambda|}=2^{|\lambda|}$ and hence $|\mathcal{S}_\lambda|=2^{|\lambda|}$.

Since $|\mathcal{P}_{<\omega}(\lambda)|=|\mathcal{P}_{<\omega}(\lambda)\times\mathcal{P}_{<\omega}(\lambda)|=\lambda$ we conclude that [5: Theorem 2.20] and Proposition 2.1(viii) imply that

$$\left|\mathscr{I}_\lambda^{\mathrm{cf}}\right|=|\mathcal{P}_{<\omega}(\lambda)\times\mathcal{P}_{<\omega}(\lambda)\times\mathcal{S}_\lambda|=|\mathcal{P}_{<\omega}(\lambda)\times\mathcal{P}_{<\omega}(\lambda)|\cdot|\mathcal{S}_\lambda|=|\lambda|\cdot 2^{|\lambda|}=2^{|\lambda|}.$$

□

Proposition 2.5. *For every $\alpha,\beta\in\mathscr{I}_\lambda^{\mathrm{cf}}$, both sets*

$$\left\{\chi\in\mathscr{I}_\lambda^{\mathrm{cf}}\mid\alpha\cdot\chi=\beta\right\} \qquad \textit{and} \qquad \left\{\chi\in\mathscr{I}_\lambda^{\mathrm{cf}}\mid\chi\cdot\alpha=\beta\right\}$$

are finite. Consequently, every right translation and every left translation by an element of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a finite-to-one map.

Proof. We denote

$$A=\left\{\chi\in\mathscr{I}_\lambda^{\mathrm{cf}}\mid\alpha\cdot\chi=\beta\right\} \quad \text{and} \quad B=\left\{\chi\in\mathscr{I}_\lambda^{\mathrm{cf}}\mid\alpha^{-1}\cdot\alpha\cdot\chi=\alpha^{-1}\cdot\beta\right\}.$$

Then $A\subseteq B$ and the restriction of any partial map $\chi\in B$ onto $\operatorname{dom}(\alpha^{-1}\cdot\alpha)$ coincides with the partial map $\alpha^{-1}\cdot\beta$. Since every partial map from $\mathscr{I}_\lambda^{\mathrm{cf}}$ has cofinite range and cofinite domain we conclude that the set B is finite and hence so is A. □

Proposition 2.6. *Each maximal chain L of idempotents in $\mathscr{I}_\lambda^{\mathrm{cf}}$ coincides with the idempotent band $E(S)$ of a bicyclic subsemigroup S of $\mathscr{I}_\lambda^{\mathrm{cf}}$.*

P r o o f. By Proposition 2.1(iii), the chain L can be written as $L = \{\varepsilon_n\}_{n=1}^{\infty}$ where $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots$. Since every infinite subchain of an ω-chain is also an ω-chain we have that Proposition 2.1(v) implies that L is an ω-chain. Then by Proposition 2.1(iii) we get that $\operatorname{dom}\varepsilon_i \setminus \operatorname{dom}\varepsilon_{i+1} \neq \varnothing$ for all positive integers i. Also, the maximality of L implies that the set $\operatorname{dom}\varepsilon_i \setminus \operatorname{dom}\varepsilon_{i+1}$ is a singleton for all positive integers i. For every positive integer i we put $\{x_i\} = \operatorname{dom}\varepsilon_i \setminus \operatorname{dom}\varepsilon_{i+1}$. Then we put $D = \operatorname{dom}\varepsilon_1 \setminus \bigcup\limits_{i\in\mathbb{N}}\{x_i\}$ and define the partial maps $\alpha\colon \lambda \rightharpoonup \lambda$ and $\beta\colon \lambda \rightharpoonup \lambda$ as follows:

$$(x)\alpha = \begin{cases} x_{n+1}, & \text{if } x = x_n \in \operatorname{dom}\varepsilon_1 \setminus D \text{ and } n \geqslant 1; \\ x, & \text{if } x \in D; \end{cases}$$

and

$$(x)\beta = \begin{cases} x_{n-1}, & \text{if } x = x_n \in \operatorname{dom}\varepsilon_1 \setminus D \text{ and } n > 1; \\ x, & \text{if } x \in D. \end{cases}$$

Since the set $\lambda \setminus \operatorname{dom}\varepsilon_1$ is finite we have that $\alpha, \beta \in \mathscr{I}_\lambda^{\mathrm{cf}}$ and Remark 1 implies the statement of our proposition. $\square$

Proposition 2.6 and the Axiom of Choice imply the following proposition.

Proposition 2.7. *Each chain of idempotents in $\mathscr{I}_\lambda^{\mathrm{cf}}$ is contained in a bicyclic subsemigroup of $\mathscr{I}_\lambda^{\mathrm{cf}}$.*

Proposition 2.8. *Let $\mathfrak{C}$ be a congruence on the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. If there exist two non-$\mathscr{H}$-equivalent elements $\alpha, \beta \in \mathscr{I}_\lambda^{\mathrm{cf}}$ such that $\alpha\mathfrak{C}\beta$, then $\mathfrak{C}$ is a group congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$.*

P r o o f. First we suppose that α and β are distinct idempotents of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. Without loss of generality we can assume that α and β are compatible and $\alpha \leqslant \beta$. Otherwise, replace α by $\alpha \cdot \beta$. Then by Proposition 2.7 there exists a maximal chain L in $E(\mathscr{I}_\lambda^{\mathrm{cf}})$ such that L contains the elements α and β, and hence L contained in a bicyclic subsemigroup S of $\mathscr{I}_\lambda^{\mathrm{cf}}$. Then [5: Corollary 1.32] implies that $\varepsilon\mathfrak{C}\iota$ for all elements ε and ι of the chain L.

Let ν be an arbitrary idempotent of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. Obviously, if $\varepsilon, \iota \in L$ such that $\varepsilon \leqslant \iota$ then $\varepsilon \cdot \nu \leqslant \iota \cdot \nu$. Since ${\uparrow}e$ is a finite subset of the free semilattice with unity $(\mathscr{P}_{<\omega}(\lambda), \subseteq)$ for any $e \in (\mathscr{P}_{<\omega}(\lambda), \subseteq)$, we have that Proposition 2.1(iv) implies that νL is an infinite chain in $E(\mathscr{I}_\lambda^{\mathrm{cf}})$. Then we have that $\varepsilon\mathfrak{C}\iota$ for all $\varepsilon, \iota \in \nu L$. We put $L_\nu = \nu L \cup \{\nu\} \cup \{\mathbb{I}\}$. Then L_ν is a chain in $E(\mathscr{I}_\lambda^{\mathrm{cf}})$. Therefore by Proposition 2.7 we get that there exists a maximal chain $L_{\max}$ in $E(\mathscr{I}_\lambda^{\mathrm{cf}})$ which contains the chain L_ν and $L_{\max}$ is a band of a bicyclic subsemigroup S in $\mathscr{I}_\lambda^{\mathrm{cf}}$. Now [5: Corollary 1.32] implies that $\varepsilon\mathfrak{C}\iota$ for all elements ε and ι of the chain L_ν.

Hence $\nu\mathfrak{C}\mathbb{I}$ and $\alpha\mathfrak{C}\mathbb{I}$ imply that $\nu\mathfrak{C}\alpha$. Therefore all idempotents of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ are $\mathfrak{C}$-equivalent. Since the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ is inverse we conclude that quotient semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}/\mathfrak{C}$ contains only one idempotent and by [13: Lemma II.1.10] the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}/\mathfrak{C}$ is a group.

Suppose that α and β are distinct non–$\mathscr{H}$-equivalent elements of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ such that $\alpha\mathfrak{C}\beta$. Then Proposition 2.1 implies that at least one of the following conditions holds:

$$\alpha\alpha^{-1} \neq \beta\beta^{-1} \qquad \text{or} \qquad \alpha^{-1}\alpha \neq \beta^{-1}\beta.$$

By [13: Lemma III.1.1] we have that $\alpha^{-1}\mathfrak{C}\beta^{-1}$. Then $\alpha\alpha^{-1}\mathfrak{C}\alpha\beta^{-1}$ and $\beta\beta^{-1}\mathfrak{C}\alpha\beta^{-1}$ and hence $\alpha\alpha^{-1}\mathfrak{C}\beta\beta^{-1}$. Similarly we get that $\alpha^{-1}\alpha\mathfrak{C}\beta^{-1}\beta$. Then the first part of the proof implies that $\mathfrak{C}$ is a group congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$. □

Theorem 2.9. *Every non-trivial congruence on the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a group congruence.*

Proof. Let $\mathfrak{C}$ be a non-trivial congruence on the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. Let α and β be distinct $\mathfrak{C}$-equivalent elements of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. If the elements α and β are not $\mathscr{H}$-equivalent then Proposition 2.8 implies the statement of the theorem.

Suppose that $\alpha\mathscr{H}\beta$. Then [5: Theorem 2.20] implies that without loss of generality we can assume that α and β are elements of the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ and hence $\mathbb{I}\mathfrak{C}(\beta\alpha^{-1})$. We denote $\gamma = \beta\alpha^{-1}$. Since $\mathbb{I} \neq \gamma$ we conclude that there exists $x_0 \in \lambda$ such that $(x_0)\gamma \neq x_0$. We define ε to be an identity selfmap of the set $\lambda \setminus \{x_0\}$. Then $\varepsilon \in \mathscr{I}_\lambda^{\mathrm{cf}}$ and $(\varepsilon \cdot \mathbb{I})\mathfrak{C}(\varepsilon \cdot \gamma)$. Since $(x_0)\gamma \neq x_0$ we have that Proposition 2.1(viii) implies that the elements ε and $\varepsilon \cdot \gamma$ are not $\mathscr{H}$-equivalent. Then by Proposition 2.8 we get that $\mathfrak{C}$ is a group congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$. □

3. On the least group congruence on the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$

Every inverse semigroup S admits the least group congruence σ (see [13: Section III]):

$$s\sigma t \quad \text{if and only if} \quad \text{there exists an idempotent} \quad e \in S \quad \text{such that} \quad se = te.$$

Theorem 2.9 implies that every non-injective homomorphism $h\colon \mathscr{I}_\lambda^{\mathrm{cf}} \to S$ from the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ into an arbitrary semigroup S generates a group congruence $\mathfrak{h}$ on $\mathscr{I}_\lambda^{\mathrm{cf}}$. In this section we describe the structure of the quotient semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$.

Proposition 3.1. *If $\alpha\sigma\beta$ in $\mathscr{I}_\lambda^{\mathrm{cf}}$ then*

$$|\lambda \setminus \operatorname{dom}\alpha| - |\lambda \setminus \operatorname{ran}\alpha| = |\lambda \setminus \operatorname{dom}\beta| - |\lambda \setminus \operatorname{ran}\beta|.$$

P r o o f. Let ε be an idempotent of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ such that $\alpha\varepsilon = \beta\varepsilon$. We shall show that the statement of the proposition holds for the elements α and $\alpha\varepsilon$.

Without loss of generality we can assume that $\varepsilon \leqslant \alpha^{-1}\alpha$, i.e., $\operatorname{dom}\varepsilon \subseteq \operatorname{dom}(\alpha^{-1}\alpha)$. Since α is an injective partial map with $|\lambda \setminus \operatorname{dom}\alpha| < \infty$ and $|\lambda \setminus \operatorname{ran}\alpha| < \infty$, and ε is an identity map of the cofinite subset $\operatorname{dom}\varepsilon$ in λ we conclude that

$$|\lambda \setminus \operatorname{dom}\alpha| - |\lambda \setminus \operatorname{ran}\alpha| = |\lambda \setminus \operatorname{dom}(\alpha\varepsilon)| - |\lambda \setminus \operatorname{ran}(\alpha\varepsilon)|.$$

This implies the statement of the proposition. $\square$

For an arbitrary element α of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ we denote

$$\overline{d}(\alpha) = |\lambda \setminus \operatorname{dom}\alpha| \qquad \text{and} \qquad \overline{r}(\alpha) = |\lambda \setminus \operatorname{ran}\alpha|.$$

Proposition 3.2. *If α and β are arbitrary elements of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ then*

$$\overline{d}(\alpha\beta) - \overline{r}(\alpha\beta) = \overline{d}(\alpha) - \overline{r}(\alpha) + \overline{d}(\beta) - \overline{r}(\beta).$$

P r o o f. We consider four cases.

(1) First we consider the case when $\operatorname{ran}\alpha \subseteq \operatorname{dom}\beta$. We put $k = \overline{r}(\alpha) - \overline{d}(\beta)$. Then the definition of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ implies that $k \geqslant 0$, $\overline{d}(\alpha\beta) = \overline{d}(\alpha)$, $\overline{r}(\alpha\beta) = \overline{r}(\beta) - k$, and hence in this case we get that

$$\overline{d}(\alpha\beta) - \overline{r}(\alpha\beta) = \overline{d}(\alpha) - \overline{r}(\alpha) + \overline{d}(\beta) - \overline{r}(\beta).$$

(2) Suppose that the case when $\operatorname{dom}\beta \subseteq \operatorname{ran}\alpha$ holds. We put $k = \overline{d}(\beta) - \overline{r}(\alpha)$. Then the definition of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ implies that $k \geqslant 0$, $\overline{d}(\alpha\beta) = \overline{d}(\alpha) + k$, $\overline{r}(\alpha\beta) = \overline{r}(\beta)$, and hence in this case we have that

$$\overline{d}(\alpha\beta) - \overline{r}(\alpha\beta) = \overline{d}(\alpha) - \overline{r}(\alpha) + \overline{d}(\beta) - \overline{r}(\beta).$$

(3) Now we consider the case $(\lambda \setminus \operatorname{ran}\alpha) \cap (\lambda \setminus \operatorname{dom}\beta) \neq \varnothing$, $\operatorname{ran}\alpha \nsubseteq \operatorname{dom}\beta$ and $\operatorname{dom}\beta \nsubseteq \operatorname{ran}\alpha$. Then the definition of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ implies that there exist positive integers i, j and k such that $i = |(\lambda \setminus \operatorname{ran}\alpha) \setminus (\lambda \setminus \operatorname{dom}\beta)|$, $j = |(\lambda \setminus \operatorname{ran}\alpha) \cap (\lambda \setminus \operatorname{dom}\beta)|$ and $k = |(\lambda \setminus \operatorname{dom}\beta) \setminus (\lambda \setminus \operatorname{ran}\alpha)|$. Then we have that $\overline{r}(\alpha) = i + j$, $\overline{d}(\beta) = j + k$, $\overline{d}(\alpha\beta) = \overline{d}(\alpha) + k$ and $\overline{r}(\alpha\beta) = \overline{r}(\beta) + i$. Therefore, in this case we get that

$$\overline{d}(\alpha\beta) - \overline{r}(\alpha\beta) = \overline{d}(\alpha) - \overline{r}(\alpha) + \overline{d}(\beta) - \overline{r}(\beta).$$

(4) In the case when $(\lambda\setminus\operatorname{ran}\alpha)\cap(\lambda\setminus\operatorname{dom}\beta)=\varnothing$ we have that the definition of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ implies that $\overline{d}(\alpha\beta)=\overline{d}(\alpha)+\overline{d}(\beta)$, $\overline{r}(\alpha\beta)=\overline{r}(\alpha)+\overline{r}(\beta)$, and hence we get that

$$\overline{d}(\alpha\beta)-\overline{r}(\alpha\beta)=\overline{d}(\alpha)-\overline{r}(\alpha)+\overline{d}(\beta)-\overline{r}(\beta).$$

This completes the proof of the proposition. □

On the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ we define a relation $\sim_{\mathfrak{d}}$ in the following way:

$$\alpha\sim_{\mathfrak{d}}\beta \qquad \text{if and only if} \qquad \overline{d}(\alpha)-\overline{r}(\alpha)=\overline{d}(\beta)-\overline{r}(\beta),$$

for $\alpha,\beta\in\mathscr{I}_\lambda^{\mathrm{cf}}$.

Proposition 3.3. *Let λ be an infinite cardinal. Then $\sim_{\mathfrak{d}}$ is a congruence on the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ and moreover the quotient semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}/\sim_{\mathfrak{d}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$.*

Proof. Simple verifications and Proposition 3.2 imply that $\sim_{\mathfrak{d}}$ is a congruence on the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. We define a homomorphism $h\colon\mathscr{I}_\lambda^{\mathrm{cf}}\to\mathbb{Z}(+)$ by the formula $(\alpha)h=\overline{d}(\alpha)-\overline{r}(\alpha)$. Then the definitions of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ and the congruence $\sim_{\mathfrak{d}}$ on $\mathscr{I}_\lambda^{\mathrm{cf}}$, and Proposition 3.2 imply that thus defined map h is a surjective homomorphism and moreover $(\alpha)h=(\beta)h$ if and only if $\alpha\sim_{\mathfrak{d}}\beta$ in $\mathscr{I}_\lambda^{\mathrm{cf}}$. This completes the proof of the proposition. □

Proposition 3.4. *Let λ be an infinite cardinal. Then for every element β of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ such that $\overline{d}(\beta)=\overline{r}(\beta)$ there exists an element α of the group of units of $\mathscr{I}_\lambda^{\mathrm{cf}}$ such that $\alpha\sigma\beta$.*

Proof. Fix an arbitrary element β of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$. Without loss of generality we can assume that $\overline{d}(\beta)=\overline{r}(\beta)=k>0$. Let $\{x_1,\dots,x_k\}=\lambda\setminus\operatorname{dom}\beta$ and $\{y_1,\dots,y_k\}=\lambda\setminus\operatorname{ran}\beta$. We define a map $\alpha\colon\lambda\to\lambda$ in the following way:

$$(x)\alpha=\begin{cases}(x)\beta, & \text{if } x\in\operatorname{dom}\beta;\\ y_i, & \text{if } x=x_i,\ i=1,\dots,k.\end{cases}$$

Then α is an element of the group of units of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ and it is obviously that $\alpha\varepsilon=\beta\varepsilon$, where ε is the identity map of the set $\operatorname{ran}\beta$. □

For every $\alpha\in\mathscr{S}_\lambda$ we denote $\operatorname{supp}(\alpha)=\{x\in\lambda\mid (x)\alpha\neq x\}$. We define

$$\mathscr{S}_\lambda^\infty=\{\alpha\in\mathscr{S}_\lambda\mid \operatorname{supp}(\alpha)\text{ is finite}\}.$$

We observe that the Schreier–Ulam theorem (see [14: Theorem 11.3.4]) implies that $\mathscr{S}_\lambda^\infty$ is a normal subgroup of $\mathscr{S}_\lambda$ and hence $\mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty$ is a group.

Later on, when $\mathfrak{C}$ is a congruence on a semigroup S we shall denote the *natural homomorphism* generated by the congruence $\mathfrak{C}$ on S by $\pi_{\mathfrak{C}}\colon S\to S/\mathfrak{C}$.

The definition of the least group congruence σ on the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ implies the following proposition.

Proposition 3.5. *Let λ be an infinite cardinal. Then the homomorphic image $(H(\mathbb{I}))\pi_\sigma$ of the group of units $H(\mathbb{I})$ of $\mathscr{I}_\lambda^{\mathrm{cf}}$ under the natural homomorphism $\pi_\sigma\colon \mathscr{I}_\lambda^{\mathrm{cf}} \to \mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$ is isomorphic to the quotient group $\mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty$.*

Theorem 3.6. *Let λ be an infinite cardinal. Then the following conditions hold:*

(i) $(H(\mathbb{I}))\pi_\sigma = \mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty$ *is a normal subgroup of the group $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$;*

(ii) *The group $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$ contains the infinite cyclic subgroup G (i.e., the additive group of integers $\mathbb{Z}(+)$) such that $G \cap \mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty = \{e\}$, where e is the unit of the group $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$;*

(iii) $\mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty \cdot G = \mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$.

and hence the group $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$ is isomorphic to the semidirect product $\mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty \ltimes \mathbb{Z}(+)$.

Proof. (i) Since $\sigma \subseteq \sim_{\mathfrak{d}}$ we conclude that [5: Theorem 1.6] implies that there exists a unique homomorphism $g\colon \mathscr{I}_\lambda^{\mathrm{cf}}/\sigma \to G$ such that the following diagram

$$\begin{tikzcd} \mathscr{I}_\lambda^{\mathrm{cf}} \arrow[r, "\pi_\sigma"] \arrow[dr, "\pi_{\sim_{\mathfrak{d}}}"'] & \mathscr{I}_\lambda^{\mathrm{cf}}/\sigma \arrow[d, "g"] \\ & G \end{tikzcd}$$

commutes. Then by Proposition 3.5 we have that the homomorphic image $(H(\mathbb{I}))\pi_\sigma$ of the group of units $H(\mathbb{I})$ of $\mathscr{I}_\lambda^{\mathrm{cf}}$ under the natural homomorphism $\pi_\sigma\colon \mathscr{I}_\lambda^{\mathrm{cf}} \to \mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$ is isomorphic the the quotient group $\mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty$. Now Propositions 3.4 and 3.5 imply that the subgroup $(H(\mathbb{I}))\pi_\sigma = \mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty$ of the group $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$ is the kernel of the homomorphism $g\colon \mathscr{I}_\lambda^{\mathrm{cf}}/\sigma \to G$, and hence $(H(\mathbb{I}))\pi_\sigma = \mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty$ is a normal subgroup of $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$.

(ii) Fix an arbitrary $\alpha \in \mathscr{I}_\lambda^{\mathrm{cf}}$ such that $|\lambda \setminus \operatorname{dom}\alpha| = 1$ and $\operatorname{ran}\alpha = \lambda$. Then the definition of the congruence $\sim_{\mathfrak{d}}$ on $\mathscr{I}_\lambda^{\mathrm{cf}}$ implies that the element α^n is not $\sim_{\mathfrak{d}}$-equivalent to any element of the group of units $H(\mathbb{I})$ for every non-zero integer n, and hence by Proposition 3.5 we get that $((\alpha)\pi_\sigma)^n \notin \mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty$. This implies that $\{((\alpha)\pi_{\sim_{\mathfrak{d}}})^n \mid n \in \mathbb{Z}\} \cap \mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty = \{e\}$, where e is the unit of the group $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$. Also, it is obvious that $(\alpha^n)\pi_{\sim_{\mathfrak{d}}} = n \in G$ and $\{(\alpha^n)\pi_{\sim_{\mathfrak{d}}} \mid n \in \mathbb{Z}\}$ is a cyclic subgroup of $\mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty$.

(iii) Fix an arbitrary element x in $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$. Let ξ be an arbitrary element of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ be such that $(\xi)\pi_\sigma = x$. If $\overline{d}(\xi) = \overline{r}(\xi)$ then by Proposition 3.4 we have that $\xi\sigma\beta$ for some element β from the group of units of $\mathscr{I}_\lambda^{\mathrm{cf}}$, and hence we get that $x = (\beta)\pi_\sigma \cdot e \in \mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty \cdot G$, where e is the unit of the group $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$.

Suppose that $\overline{d}(\xi) - \overline{r}(\xi) = n \neq 0$. Then by Proposition 3.2 we have that $\overline{d}\left(\xi \cdot (\alpha^{-1})^n\right) - \overline{r}\left(\xi \cdot (\alpha^{-1})^n\right) = 0$. Now, Proposition 3.4 implies that the element $\xi \cdot (\alpha^{-1})^n$ is σ-equivalent to some element β of the group of units $H(\mathbb{I})$ of $\mathscr{I}_\lambda^{\mathrm{cf}}$. Then we have that $(\xi \cdot (\alpha^{-1})^n)\pi_\sigma = (\beta)\pi_\sigma$ and since $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$ is a group we get that $x = (\xi)\pi_\sigma = (\beta)\pi_\sigma \cdot (\alpha^n)\pi_\sigma \in \mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty \cdot G$. This implies that $\mathscr{S}_\lambda/\mathscr{S}_\lambda^\infty \cdot G = \mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$.

The last statement of the theorem follows from statements (i)–(iii) and [6: Exercise 2.5.3]. □

Remark 2. Proposition 3.3 implies that for every infinite cardinal λ the group $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$ has infinitely many normal subgroups and hence the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ has infinitely many group congruences.

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