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Simply Connected 3-Manifolds with a Dense Set of Ends of Specified Genus

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Abstract. We show that for every sequence (n_i), where each n_i is either an integer greater than 1 or is ∞, there exists a simply connected open 3-manifold M with a countable dense set of ends $\{e_i\}$ so that, for every i, the genus of end e_i is equal to n_i. In addition, the genus of the ends not in the dense set is shown to be less than or equal to 2. These simply connected 3-manifolds are constructed as the complements of certain Cantor sets in S^3. The methods used require careful analysis of the genera of ends and new techniques for dealing with infinite genus.

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1. Introduction

The ends of an open 3-manifold M are determined by any properly nested sequence $C_1 \subset C_2 \subset \ldots$ of compact 3-manifolds C_i whose union is M. Such a sequence is called an exhaustion of M. The components of $M \setminus C_i$ form an inverse sequence with the bonding maps given by inclusions.

Each end e is a point in this inverse sequence. An end e is thus associated with a sequence $D_1 \supset D_2 \ldots$ where D_i is a component of $M \setminus C_i$. The *genus* of the end e associated with the sequence (C_i), denoted by $g(e, (C_i))$, is defined to be $\sup\{g(D_i)\}$, where $g(D_i)$ is the genus of the boundary of D_i. The *genus* of the end e, denoted by $g(e)$, is the minimum of $g(e, (C_i))$, taken over all possible exhaustions (C_i) of the manifold M.

The endpoint (or Freudenthal) compactification of M adds a point to M for each end. We say that a set of ends is dense (in the set of all ends)

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if it is dense in the set of all ends of the endpoint compactification. For background on endpoint compactifications and the theory of ends, see [8, 9,14,19]. For an alternate proof using defining sequences of the result that every homeomorphism of the open 3-manifold extends to a homeomorphism of its endpoint compactification, see [10].

It is easy to construct 3-manifolds with a finite number of ends of specified genus. For example, if H is a surface of genus k, then $H \times \mathbb{R}$ has two ends of genus k. It is more difficult to construct simply connected examples or examples with more complicated end structures. The examples we produce will be complements of certain Cantor sets in S^3. The Cantor sets will have simply connected complements and the end sets of the complements will be these Cantor sets. See Kirkor [15], DeGryse and Osborne [7], Ancel and Starbird [1], and Wright [24] for further discussion of wild (non-standard) Cantor sets with simply connected complement. For more recent results on Cantor sets with simply connected complements, see [11,12].

The main theorem we will prove in this paper is the following.

Theorem 1.1. *Let* $\mathfrak{S} = (n_1, n_2, \ldots)$ *be a sequence where each* n_i *is either an integer greater than* 1 *or is* ∞. *Then, there is a simply connected open 3-manifold* $M = M_{\mathfrak{S}}$ *with uncountably many ends so that the following holds: There is a countable dense set* D *of ends* $\{e_1, e_2, \ldots\}$ *in the endpoint compactification of* M *so that, for every* i, *the genus of* e_i *is* n_i. *The genus of each end not in* D *is less than or equal to 2.*

This theorem will follow as a corollary of the following result about Cantor sets in S^3.

Theorem 1.2. *Let* $\mathfrak{S} = (n_1, n_2, \ldots)$ *be a sequence where each* n_i *is either an integer greater than* 1 *or is* ∞. *Then, there is a Cantor set* $C = C_{\mathfrak{S}}$ *in* S^3 *with simply connected complement so that the following holds: There is a countable dense set* D *of points* $\{x_1, x_2, \ldots\}$ *in* C *so that, for every* i, *the local genus of* C *at* x_i *is* n_i. *The local genus of* C *at each point not in* D *is less than or equal to 2.*

Section 2 provides details about the local genus of points in a Cantor set in S^3 (introduced in [25]) and relates this to the genus of an end of the 3-manifold that is the complement of the Cantor set. Section 3 gives two replacement constructions we will need in the construction of our examples. Section 4 gives the main construction of our examples. Section 5 shows that the examples we construct give a proof of the theorems from Sect. 1. Section 6 lists some question arising from our results.

For the remainder of this paper, we assume all surfaces referred to are closed (i.e. compact, connected and without boundary) and that all 1-handles are orientable and unknotted.

2. Local Genus of Points in a Cantor Set

We review the definition and some facts from [25] about the local genus of a point in a Cantor set in S^3. At the end of this section, we relate the local

genus of a point x in a Cantor set $C \subset S^3$ to the genus of an end x of the complementary 3-manifold $S^3 \setminus C$. We use IntX and FrX to denote the topological interior and boundary of a subset X of a space Y.

Let C be a Cantor set in S^3. A defining sequence for C is a nested sequence (M_i) of compacta $M_i \subset C$ whose intersection is C such that each M_i consists of pairwise disjoint handlebodies and so that, for every i, $M_{i+1} \subset \operatorname{Int} M_i$. The genus of a handlebody H is denoted by $g(H)$.

Let $\mathcal{D}(C)$ be the set of all defining sequences for C. Let $(M_i) \in \mathcal{D}(C)$. For any $x \in C$, we denote by $M_i^{\{x\}}$ the component of M_i which contains x. Define

$$g_x(C; (M_i)) = \sup\{g(M_i^{\{x\}}); \ i \geq 0\} \quad \text{and}$$
$$g_x(C) = \inf\{g_x(C; (M_i)); \ (M_i) \in \mathcal{D}(C)\}.$$

The number $g_x(C)$ is called the local genus of the Cantor set C at the point x.

Remark 2.1. This definition immediately implies that $g_x(D) \leq g_x(C)$ if D is a Cantor set contained in C.

Determining $g_x(C)$ using the definition can be difficult. If a defining sequence for C is given, one can easily determine an upper bound. The idea of slicing discs introduced in [3] can be used to derive the following addition theorem for local genus. This can then be used for establishing the exact local genus. See [25, Theorem 14] for details.

Theorem 2.2. *Let $X, Y \subset S^3$ be Cantor sets and p a point in $X \cap Y$. Suppose there exist a 3-ball B and a 2-disc $D \subset B$ such that*

1. *$p \in \operatorname{Int} B$, $\operatorname{Fr} D = D \cap \operatorname{Fr} B$, $D \cap (X \cup Y) = \{p\}$; and*
2. *$X \cap B \subset B_X \cup \{p\}$ and $Y \cap B \subset B_Y \cup \{p\}$ where B_X and B_Y are the components of $B \setminus D$.*

Then, $g_p(X \cup Y) = g_p(X) + g_p(Y)$.

The 2-disc D in the above theorem is called a slicing disc for the Cantor set $X \cup Y$.

The lemma below points out the relationship between genus of an end and local genus of points in a Cantor set in the context we are discussing.

Lemma 2.3. *Let $C \subset S^3$ be a Cantor set. Let $M = S^3 \setminus C$. Then, the endpoint compactification of M is S^3 with the ends of M corresponding to the points in C. The genus $g(e)$ of an end e of M is equal to the local genus $g_e(C)$ of e in C.*

The proof of this lemma is just an exercise in the definitions. This follows since every defining sequence (M_i) for C corresponds to an exhaustion $C_1 \subset C_2 \subset \ldots$ of M where $C_i = \overline{S^3 \setminus M_i}$.

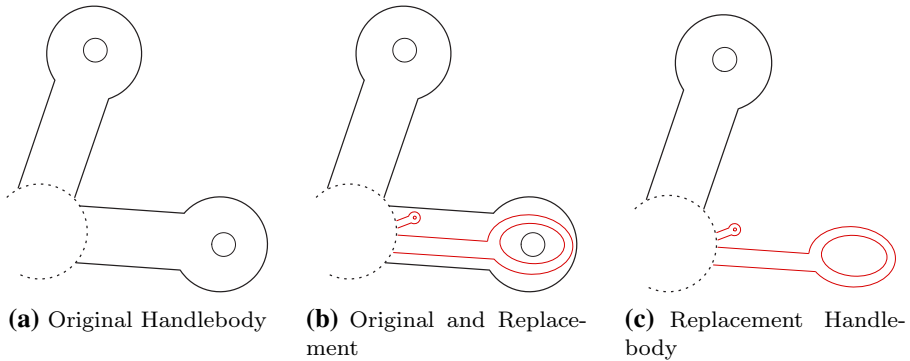**(a)** Original Handlebody **(b)** Original and Replacement **(c)** Replacement Handlebody

Figure 1. Genus Replacement

3. Replacement Constructions

In this section, given a solid handlebody $H \subset S^3$, we define two possible constructions that replace H by one or more handlebodies contained in the interior of H. These replacement constructions will be used in the next section in constructing a defining sequence for a certain Cantor set with the properties needed for the proof of the main theorem.

3.1. Genus Replacement

(See Fig. 1.) Let H be a unknotted genus k solid handlebody in S^3. View H as a solid ball with k 1-handles attached as in Fig. 1a.

  The genus replacement construction replaces H by a handlebody $H' \subset H$ of genus $k+1$. One of the handles of H is replaced by two smaller handles. One of the new handles is a parallel interior copy of the replaced handle. The other new handle is contractible in H and lies in the interior of the replaced handle as in Fig. 1. As a final step, remove a small neighborhood of the boundary of the new handlebody to guarantee that H' is in the interior of H as needed for a defining sequence.

3.2. Size Replacement

(See Fig. 2). Let H be an unknotted genus k solid handlebody in S^3. View H as a solid ball with k 1-handles attached as in Fig. 1a. Replace H by a single smaller genus k handlebody H' contained in the solid ball (see Fig. 2a), and by chains of smaller unknotted genus 2 handlebodies in each handle of H as in Fig. 2b. This replacement can be done so the diameter of each of the new handlebodies in H is less than half the diameter of H.

Remark 3.1. Let B be a 3-cell containing H, and let $H_1, H_2, \ldots H_k$ be the new handlebodies in H. Then, there are pairwise disjoint 3-cells $B_1, B_2, \ldots B_k$ in B so that $H_i \subset B_i$. This follows from the fact that the genus 2 handlebodies around each handle are not linked in B even though they are linked in H. See [13,20] for more details on this.

(a) General Pattern   **(b)** Linking in Each Handle
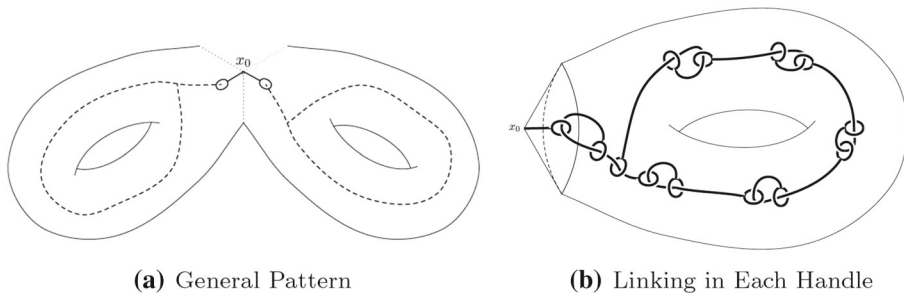
Figure 2. Size Replacement

Note: The inductive hypotheses imply that there is a nested sequence of unions of pairwise disjoint 3-cells associated with the Cantor set constructed in the next section. This collection of 3-cells, however, does not imply that the constructed Cantor set is tame, since the diameters of the 3-cells do not go to 0.

4. The Construction

Fix a sequence $\mathfrak{S} = (n_1, n_2, \ldots)$ where each n_i is either an integer greater than 1 or is ∞. We will construct inductively a defining sequence S_1, S_2, $\ldots$ for a Cantor set $X = X(\mathfrak{S})$ in S^3. Equivalently, we specify an increasing union of open 3$-$manifolds N_i where each $N_i = S^3 \setminus S_i$. At the same time, we associate with each component $M_{(i,j)}$ of S_i the specific term n_j of the sequence $\mathfrak{S}$ above.

Inductive hypotheses:

IH 1: The components $M_{(i,1)}, \ldots M_{(i,m(i))}$ of S_i are unknotted handlebodies of genus 2 or greater which are contained in pairwise disjoint 3-cells in S^3.

IH 2: The genus of $M_{(i,j)}$ is less than or equal to n_j.

The construction will show that the components of S_{2k+1} are obtained from S_{2k} by a suitable replacement construction performed on the components of S_{2k}. The components of S_{2k} will be obtained by replacing each component of S_{2k-1} by an appropriate chain of linked handlebodies. All except possibly one handlebody in this chain will have genus 2.

Stage 1: To begin the construction, let S_1 consist of a single component $M_{(1,1)}$, where $M_{(1,1)}$ is an unknotted genus 2 handlebody in S^3. Note that n_1 is greater than or equal to the genus of $M_{(1,1)}$. The inductive hypotheses are clearly satisfied since $n_1 \geq 2$.

Stage $\mathbf{k+1}$ if $\mathbf{k}$ is odd:

By the inductive hypothesis, every component of S_k is an unknotted handlebody of genus 2 or greater. Let $M_{(k,i)}$ be a component of S_k. Again, by the inductive hypothesis, the genus of $M_{(k,i)}$ is less than or equal to n_i. If the genus of $M_{(k,i)}$ is less than n_i, perform a genus replacement on $M_{(k,i)}$. Note
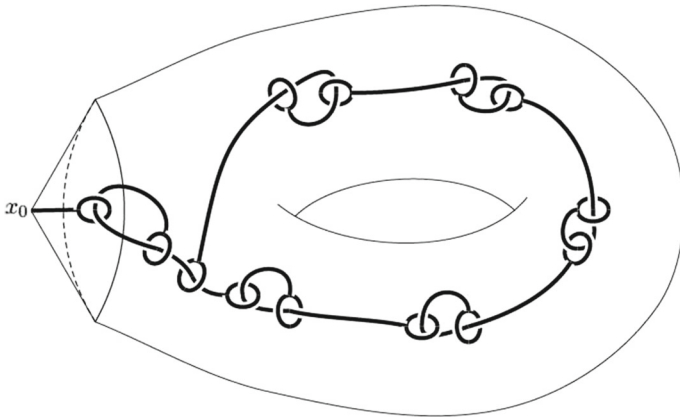
Figure 3. Linking Along the Spine of Some Handle of N

that this replaces a component $M_{(k,i)}$ at stage k by a component $M_{(k+1,i)}$ at stage $k + 1$. This genus of $M_{(k+1,i)}$ is then (genus of $M_{(k,i)}$)+1. So the genus of $M_{(k+1,i)}$ is less than or equal to n_i.

If the genus of $M_{(k,i)}$ is equal to n_i, let $M_{(k+1,i)}$ be $M_{(k,i)}$ with a small open neighborhood of the boundary removed. Since the genus of $M_{(k+1,i)}$ is the genus of $M_{(k,i)}$, this genus is still less than or equal to n_i. In either case, inductive hypothesis **IH 2** is satisfied at stage $k + 1$.

By construction, each component $M_{(k+1,i)}$ is an unknotted handlebody of genus at least 2. Since each new component is in the interior of a component from S_n, the inductive hypothesis **IH 1** at stage k guarantees that the inductive hypothesis **IH 1** is still satisfied at stage $k + 1$.

Stage **k + 1** if **k** is even:

By the inductive hypothesis, every component of S_k is an unknotted handlebody of genus 2 or greater. Let $M_{(k,i)}$ be a component of S_k of genus g. Again, by the inductive hypothesis, g is less than or equal to n_i. Perform a size replacement on $M_{(k,i)}$, replacing $M_{(k,i)}$ by a single genus k smaller copy (labelled $M_{(k+1,i)}$) of $M_{(k,i)}$, and by 6g unknotted genus 2 handlebodies as in Fig. 2.

This procedure performed on each component of S_k produces smaller copies of $M_{(k,i)}, 1 \leq i \leq m(n)$, labeled $M_{(n+1,i)}$, and a collection of unlabeled genus 2 handlebodies. Arbitrarily label these new genus 2 handlebodies as $M_{(k+1,m(k)+1)} \cdots M_{(n+1,m(k+1))}$.

By **IH 1** at stage k, by the construction above, and by Remark 3.1, **IH 1** is satisfied at stage $k + 1$.

If $1 \leq i \leq m(k)$, the genus of $M_{(k+1,i)}$ is equal to the genus of $M_{(k,i)}$ which is less than or equal to n_i, by **IH 2** at stage k. If $i \geq m(k)+1$, the genus of $M_{(k+1,i)}$ is equal to 2 which is less than or equal to n_i by the assumption about the terms of $\mathfrak{S}$. In either case, **IH 2** is satisfied at stage $k + 1$.
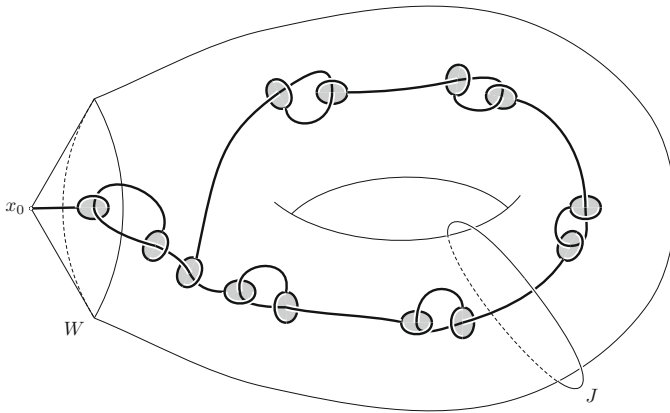
Figure 4. Simple Connectivity of the Complement

This completes the inductive description of the defining sequence. Define the Cantor set $C = C_{\mathfrak{S}}$ associated with the sequence $\mathfrak{S}$ to be $C = C_{\mathfrak{S}} = \bigcap_i M_i$.

In the next section, we will show that $C = C_{\mathfrak{S}}$ is the required Cantor set needed for proving Theorem 1.2.

5. Proof of the Main Theorems

For a sequence $\mathfrak{S} = (n_1, n_2, \ldots)$ where each n_i is either an integer greater than 1 or is ∞, let $C = C_{\mathfrak{S}}$ be the Cantor set constructed in the previous section. Note that C is indeed a Cantor set since each component of M_n contains more than one component of M_{n+2} and since the diameter of the components goes to 0 as n goes to ∞. We now verify that C satisfies the conditions listed in Theorem 1.2.

5.1. Simple Connectivity of the Complement

Let $\gamma \colon S^1 \to S^3 \setminus C$. The set $\gamma(S^1)$ is compact and misses C so there exists an even n large enough such that $\gamma(S^1) \cap M_n = \emptyset$.

By **IH 1**, the components $M_{(n,1)}, \ldots M_{(n,m(n))}$ of M_n are unknotted handlebodies of genus 2 or greater which are contained in pairwise disjoint 3-cells in S^3. Since the components are cubes with unknotted handles and lie in disjoint 3-cells, the fundamental group of the complement of the components is generated by the meridional curves on the components. It, therefore, suffices to show how any meridional loop (say J) of some component N can be shrunk to a point in the complement of the components at the next stage contained in N.

By construction, it is clear that J can be moved in $N \setminus M_{n+1}$ to the loop W on BdN (see Fig. 4) and then moved off N. Hence, $[J] = 0 \in \pi_1(S^3 \setminus C)$. $\qquad \square$

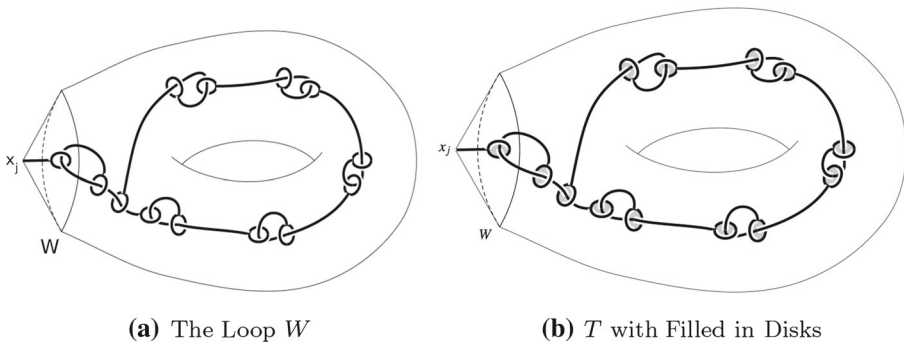**(a)** The Loop W **(b)** T with Filled in Disks

Figure 5. The Handlebody T

5.2. The Countable Dense Subset

Fix a point $x \in C$ and let $M(j, x(j))$ be the component of M_j containing x. Then

$$M(1, x(1)) \supset M(2, x(2)) \supset \ldots \supset M(n, x(n)) \supset \ldots, \text{ and}$$
$$x = \bigcap_n M(n, x(n)).$$

For a fixed $i > 0$, define the point $x_i \in C$ as follows. Choose n so that the number $m(n)$ of components in M_n is greater than i. Then, $M(k, i)$ is a component of M_k for each $k \geq n$ and by construction $M(k+1, i) \subset M(k, i)$. Let $x_i = \bigcap_{k \geq n} M(k, i)$. The countable dense subset of C that we are looking for is the subset $D = \{x_1, x_2 \ldots\}$. The set D is dense in C since each component $M(k, j)$ of M_k contains a point of D, namely x_j.

5.3. Genus at x_j When n_j is Finite

Note that $x_j = \bigcap_{k \geq n} M(k, j)$ where n is chosen so that $m(n) \geq j$. The genus of $M(n, j)$ is less than or equal to n_j, and the genus modification at subsequent stages guarantees that there is an $n' \geq n$ so that the genus of $M(n', j)$ is equal to n_j. Then, for each $\ell > n'$, the genus of $M(\ell, j)$ is also equal to n_j. It follows from the definition of local genus that the local genus $g_{x_j}(C)$ of C at x_j is then less than or equal to n_j.

It remains to be shown that $g_{x_j}(C)$ is greater than or equal to n_j.

Choose the component $N = M(2i+1, j)$ of M_{2i+1} containing x_j, where $m(2i+1) > j$ and $2i+1$ is large enough so that the genus of N is equal to n_j. Then, N is a union of genus 1 handlebodies as in the previous section. Let T be one of these genus 1 handlebodies. By construction, we have that $\text{Bd}(T) \cap C$ is the singleton x_j. Let W be a loop in $\text{Bd}(T)$ that bounds a disc in $\text{Bd}(T)$ containing x_j in its interior as in Fig. 5a.

By Theorem 2.2, to show that the local genus of x_j in C is at least n_j, it suffices to show that the local genus of x_j in $C \cap T$ is at least 1. For this, we will need two technical lemmas.

Lemma 5.1. *If there exists a 2-disc $D \subset T$ such that $D \cap M_{r+1} = \emptyset$ for some $r > 2i + 1$, and $Bd(D) = W$, then there exists a 2-disc $D' \subset T$ such that $Bd(D') = Bd(D)$ and $D' \cap M_r = \emptyset$.*

Proof. This is essentially Lemma 5.1 in [13]. Complete details of the proof are provided there. The main idea is that if $D \cap M_{r+1} = \emptyset$, D can be adjusted so as to miss a core of each component of $M_r \cap T$. This allows the formation of a new disc D' with the same boundary so that $D' \cap M_r = \emptyset$.   □

Lemma 5.2. *Let W be a loop on $Bd(T)$ as in Fig. 5a and let x_j be the point in $C \cap Bd(T)$. If $g_{x_j}(C \cap T) = 0$, then W bounds a disc D in T missing C.*

Proof. This is essentially Lemma 5.2 in [13]. Complete details of the proof are provided there. The argument there is for a component N of genus greater than or equal to 3. Since the argument is actually applied separately to each 1-handle of N, the same argument works here where the genus of N is greater than or equal to 2.   □

Now assume that $g_{x_j}(C \cap T) = 0$. Then by Lemma 5.2, the loop W bounds a disc D missing C, and so missing $M_r \cap T$ for some large r. Repeated application of Lemma 5.1 then implies that W bounds a disc D' so that $D' \cap M_{2i+2} = \emptyset$. An argument similar to that used in Section 6.2 of [13] then shows that W bounds a disc also missing the filled in disks in Fig 5b. This disk can then be pushed to the boundary of T minus x_j. Since W is nontrivial in the boundary of T minus x_j, this cannot happen. It follows that $g_{x_j}(C \cap T) \geq 1$.

This completes the argument that the local genus of C at x_j is equal to n_j when n_j is finite.   □

5.4. Genus at x_j When n_j is Infinite

It suffices to show that $g_{x_j}(C)$ is greater than or equal to K for each positive integer K.

Fix a positive integer K and choose a stage M_n of the defining sequence for C so that $m(n) > j$ and so that the genus g of $N = M(n, j)$ is at least K. Then, N is a union of g genus 1 handlebodies. Let T be one of these genus 1 handlebodies. By construction, even though the genus of the handlebody containing x_j in M_r increases as r goes to infinity, we still have that $Bd(T) \cap C$ is the singleton x_j.

Exactly the same argument as above (in the case when n_j was finite) can now be applied to conclude that the local genus of x_j in $C \cap T$ is at least one. It follows that the local genus of x_j in $C \cap N$ is at least K which is what we needed to show. So the local genus of x_j in C is infinity.   □

5.5. Genus at Points Not in the Dense Set

If $x \in C \setminus D$, then for each $N \in \mathbb{Z}_+$, there is an $N' \geq N$ so that the genus of $M(N', x(N'))$ is two. To see this, consider the component $M(N, i)$ of M_N that contains x. If the genus of $M(N, i)$ is two, we are done. If not, there must be a first stage $N + k > N$ such that $x(N + k) \neq i$. If there is no such stage, then $x = x_i \in D$ which cannot happen. For this stage $N + k$,

$M(N + k, x(N + k))$ has genus two by the construction of C. It follows that in the sequence whose intersection is x,

$$M(1, x(1)) \supset M(2, x(2)) \supset \cdots \supset M(n, x(n)) \supset \dots \, ,$$

infinitely many of the terms have genus two. It then follows from the definition of local genus that $g_x(C) \leq 2$.

Remark 5.3. An inductive argument using techniques similar to those of Lemma 5.1 can be used to show the following. Let $M(k, i)$ be a component of M_k. Then, there are no 2-spheres missing C in the interior of $M(k, i)$ that separate $M(k, i) \cap C$. This in turn can be used to show that the local genus of each point in C is strictly greater than 0. So the local genus at each point of $C \setminus D$ is either 1 or 2.

   This completes the proof of Theorem 1.2. By Lemma 2.3, Theorem 1.1 (the main theorem) follows as an immediate corollary.   □

6. Questions

Both Theorems 1.1 and 1.2 contain the requirement that each term in the sequence $\mathcal{S}$ is greater than one. Additionally, the genus of each end not in the dense set D (Theorem 1.1) and of each point of C not in D (Theorem 1.2) are shown to be less than two. This leads to two interesting questions, listed below.

Question 6.1. *Can the restrictions that $n_i \geq 2$ in Theorem 1.1 and Theorem 1.2 be removed? That is, is it possible to construct simply connected examples with certain points in the dense set specified to have genus 1?*

Question 6.2. *By Remark 5.3, the genus of points not in the dense sets from Theorem 1.1 and Theorem 1.2 is either 1 or 2. Can it be shown that their genus is exactly equal to 2?*

   It seems that new construction techniques need to be developed to answer the first question since the specific genus 2 handlebodies used in the size replacement modification are essential to the argument that the complement of the Cantor set constructed is simply connected.

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