HOMOGENEITY GROUPS OF ENDS OF OPEN 3-MANIFOLDS

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For every finitely generated abelian group G, we construct an irreducible open 3-manifold M_G whose end set is homeomorphic to a Cantor set and whose homogeneity group is isomorphic to G. The end homogeneity group is the group of self-homeomorphisms of the end set that extend to homeomorphisms of the 3-manifold. The techniques involve computing the embedding homogeneity groups of carefully constructed Antoine-type Cantor sets made up of rigid pieces. In addition, a generalization of an Antoine Cantor set using infinite chains is needed to construct an example with integer homogeneity group. Results about the local genus of points in Cantor sets and about the geometric index are also used.

1. Introduction

Each Cantor set C in S^3 has for complement an open 3-manifold M^3 with end set C. Properties of the embedding of the Cantor set give rise to properties of the corresponding complementary 3-manifold M^3. See [Souto and Stover 2013], [Garity and Repovš 2013], and [Garity et al. 2014] for examples of this.

We investigate possible group actions on the end set C of the open 3-manifold M^3 in the following sense: the *homogeneity group of the end set* is the group of homeomorphisms of the end set C that extend to homeomorphisms of the open 3-manifold M^3. Referring specifically to the embedding of the Cantor set, this group can also be called the *embedding homogeneity group of the Cantor set*. See [Dijkstra 2010] and [van Mill 2011] for some other types of homogeneity.

The standardly embedded Cantor set is at one extreme here. The embedding homogeneity group is the full group of self-homeomorphisms of the Cantor set, an extremely rich group (there is such a homeomorphism taking any countable dense set to any other). Cantor sets with this full embedding homogeneity group are called *strongly homogeneously embedded*. See [Daverman 1979] for an example of a nonstandard Cantor set with this property.

At the other extreme are *rigidly embedded* Cantor sets, those Cantor sets for

which only the identity homeomorphism extends. Shilepsky [1974] constructed
Antoine-type [1920] rigid Cantor sets. Their rigidity is a consequence of Sher's
result [1968] that if two Antoine Cantor sets are equivalently embedded, the stages
of their defining sequences must match up exactly. In the last decade, new examples
[Garity et al. 2006; 2011] of nonstandard Cantor sets were constructed that were
both rigidly embedded and had simply connected complement. See [Wright 1986]
for additional examples of rigidity.

These examples naturally lead to the question of which types of groups can
arise as end homogeneity groups between the two extremes mentioned above. In
this paper we show that for each finitely generated abelian group G, there is an
irreducible open 3-manifold with end set homeomorphic to a Cantor set and end
homogeneity group isomorphic to G. (See Corollary 6.3.)

The Cantor sets produced are *unsplittable*, in the sense that for each such C, no
2-sphere in the complement of C separates points of C. We produce these examples
by constructing, for each natural number m greater than one, 3-manifolds with
end homogeneity groups $\mathbb{Z}_m$, and by separately constructing 3-manifolds with
end homogeneity group $\mathbb{Z}$. We then link the Cantor sets needed for a given abelian
group in an unsplittable manner.

In Section 2, we give definitions and the basic results needed for the rest of
the paper. In Section 3, we review the needed results about Antoine Cantor sets.
In Section 4 we produce Cantor sets with embedding homogeneity group $\mathbb{Z}_m$. In
Section 5 we produce Cantor sets with embedding homogeneity group $\mathbb{Z}$. Section 6
ties together the previous results and lists and proves the main theorems. Section 7
lists some remaining questions.

2. Preliminaries

Background. Refer to [Garity et al. 2005; 2006; 2014] for a discussion of Cantor
sets in general and of rigid Cantor sets, and to [Željko 2005] for results about
the local genus of points in Cantor sets and defining sequences for Cantor sets.
The bibliographies in these papers contain additional references to results about
Cantor sets. Two Cantor sets X and Y in S^3 are said to be *equivalent* if there is
a self-homeomorphism of S^3 taking X to Y; otherwise they are *inequivalent*, or
inequivalently embedded. A Cantor set C is *rigidly embedded* in S^3 if the only
self-homeomorphism of C that extends to a homeomorphism of S^3 is the identity.

Geometric index. We list the results we need on geometric index. See [Schubert
1953] and [Garity et al. 2011] for more details.

If K is a link in the interior of a solid torus T, the *geometric index* of K in T,
denoted by $N(K, T)$, is defined as the minimum of $|K \cap D|$ over all meridional
disks D of T intersecting K transversely. If T is a solid torus and M is a finite

union of disjoint solid tori such that $M \subset \text{Int}(T)$, then the geometric index $\text{N}(M, T)$ of M in T is $\text{N}(K, T)$, where K is a core of M.

Theorem 2.1 [Schubert 1953; Garity et al. 2011, Theorem 3.1]. *Let T_0 and T_1 be unknotted solid tori in S^3 with $T_0 \subset \text{Int}(T_1)$ and $\text{N}(T_0, T_1) = 1$. Then ∂T_0 and ∂T_1 are parallel; i.e., the manifold $T_1 - \text{Int}(T_0)$ is homeomorphic to $\partial T_0 \times I$, where I is the closed unit interval $[0, 1]$.*

Theorem 2.2 [Schubert 1953; Garity et al. 2011, Theorem 3.2]. *Let T_0 be a finite union of disjoint solid tori. Let T_1 and T_2 be solid tori such that $T_0 \subset \text{Int}(T_1)$ and $T_1 \subset \text{Int}(T_2)$. Then $\text{N}(T_0, T_2) = \text{N}(T_0, T_1) \cdot \text{N}(T_1, T_2)$.*

There is one additional result we will need:

Theorem 2.3 [Schubert 1953; Garity et al. 2011, Theorem 3.3]. *Let T be a solid torus in S^3 and let $T_1, \ldots, T_n$ be unknotted pairwise disjoint solid tori in T, each of geometric index 0 in T. Then the geometric index of $\bigcup_{i=1}^n T_i$ in T is even.*

Defining sequences and local genus. We review the definition and some basic facts from [Željko 2005] about the local genus of points in a Cantor set. See that work for a discussion of defining sequences.

Let $\mathcal{D}(X)$ be the set of all defining sequences for a Cantor set $X \subset S^3$. Let $(M_i) \in \mathcal{D}(X)$ be a specific defining sequence for an X. For $A \subset X$, denote by M_i^A the union of those components of M_i which intersect A. The *genus* $g(M_i^A)$ of M_i^A is the maximum genus of a component of M_i^A. Define

$$g_A(X; (M_i)) = \sup\{g(M_i^A) : i \geq 0\}, \quad g_A(X) = \inf\{g_A(X; (M_i)) : (M_i) \in \mathcal{D}(X)\}.$$

The number $g_A(X)$ is called *the genus of the Cantor set X with respect to the subset A.* For $A = \{x\}$ we call the number $g_{\{x\}}(X)$ *the local genus of the Cantor set X at the point x* and denote it by $g_x(X)$.

Let x be an arbitrary point of a Cantor set X and $h \colon S^3 \to S^3$ a homeomorphism. Then the local genus $g_x(X)$ is the same as the local genus $g_{h(x)}(h(X))$. Also note that if $x \in C \subset C'$, then the local genus of x in C is less than or equal to the local genus of x in C'. See [Željko 2005, Theorem 2.4].

The following result is needed to show that certain points in our examples have local genus 2.

Theorem 2.4 [Željko 2005]. *Let $X, Y \subset S^3$ be Cantor sets and $p \in X \cap Y$. Suppose there exists a 3-ball B and a 2-disk $D \subset B$ such that*

(1) $p \in \text{Int}(B)$, $\partial D = D \cap \partial B$, $D \cap (X \cup Y) = \{p\}$; *and*

(2) $X \cap B \subset B_X \cup \{p\}$ *and* $Y \cap B \subset B_Y \cup \{p\}$, *where B_X and B_Y are the components of $B - D$.*

Then $g_p(X \cup Y) = g_p(X) + g_p(Y)$.

Discussion and examples of ends and homogeneity groups. For background on Freudenthal compactifications and theory of ends, see [Dickman 1968; Freudenthal 1942; Siebenmann 1965]. For an alternate proof using defining sequences of the result that every homeomorphism of the open 3-manifold extends to a homeomorphism of its Freudenthal compactification, see [Garity and Repovš 2013].

At the end of the next section, we will discuss elements of the homogeneity group of a standard self-similar Antoine Cantor set. Note that removing n points from S^3 yields a reducible open 3-manifold with end homogeneity group the symmetric group on n elements. It is not immediately obvious how to produce examples that are irreducible, have a rich end structure (for example a Cantor set), and at the same time have specified abelian end homogeneity groups.

3. Properties of the Antoine Cantor set

An Antoine Cantor set is described by a defining sequence (M_i) as follows: Let M_0 be an unknotted solid torus in S^3. Let M_1 be a chain of at least four linked, pairwise disjoint, unknotted solid tori in M_0, as in Figure 1. Inductively, M_i consists of pairwise disjoint solid tori in S^3 and M_{i+1} is obtained from M_i by placing a chain of at least four linked, pairwise disjoint, unknotted solid tori in each component of M_i. If the diameter of the components goes to 0, the Antoine Cantor set is $C = \bigcap_{i=0}^{\infty} M_i$.
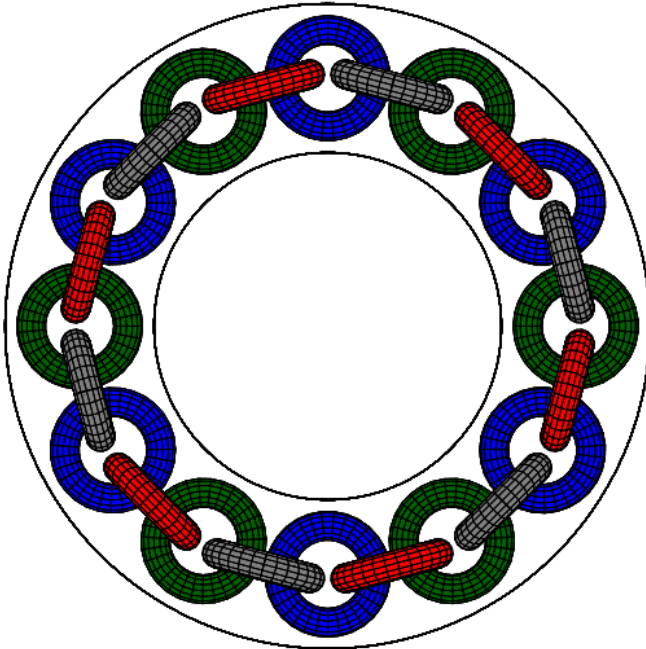

Figure 1. Antoine chain with $\mathbb{Z}_6$ group action.

We refer to [Sher 1968] for basic results and description of Antoine Cantor sets. The key result we shall need is the following:

Theorem 3.1 [Sher 1968, Theorems 1 and 2]. *Two Antoine Cantor sets in S^3, with defining sequences (M_i) and (N_i), respectively, are equivalently embedded if and only if there is a self-homeomorphism h of S^3 with $h(M_i) = N_i$ for each i.*

In particular, the number and adjacency of links in the chains must match up at each stage. Because we need a modification of this result for infinite chains in Section 5, we outline an alternative proof of the forward implication.

Proof of forward implication of Theorem 3.1. It suffices to show that if C has two Antoine defining sequences (M_i) and (N_i), then there is a homeomorphism h as in the theorem.

Step 1: There is a general position homeomorphism h_1, fixed on C, such that $h_1(\partial(M_1) \cup \partial(M_2))$ is in general position with $\partial(N_1) \cup \partial(N_2)$. The curves of intersection of $h_1(\partial(M_1) \cup \partial(M_2)) \cap (\partial(N_1) \cup \partial(N_2))$ can be eliminated by a homeomorphism h_2 also fixed on C, by a standard argument and the facts that any nontrivial curve on $\partial(M_i)$ does not bound a disk in the complement of C and that no 2-sphere separates the points of C. For details on the type of argument in this step, see [Sher 1968] or [Garity et al. 2011].

Step 2: Let T be a component of $h_2 \circ h_1(M_1)$ and assume T intersects a component S of N_1. Either $T \subset \text{Int}(S)$ or $S \subset \text{Int}(T)$. First assume $T \subset \text{Int}(S)$. If the geometric index of T in S is 0, then since the other components of $h_2 \circ h_1(M_1)$ are linked to T by a finite chain, all components of $h_2 \circ h_1(M_1)$ are in the interior of S. This is a contradiction since there are points of C not in S. So the geometric index of T in S is greater than or equal to 1.

Note that T cannot be contained in any component of N_2 that is in S since these have geometric index 0 in S. So T contains all the components of N_2 that are in S. Each of these components has geometric index 0 in T, so the union of these components has an even geometric index in T by Theorem 2.3. This geometric index must then be 2 and the geometric index of T in S must be 1. Now there is a homeomorphism h_3, fixed on C and the complement of S, that takes T to S.

If instead $S \subset \text{Int}(T)$, a similar argument shows there is a homeomorphism h_3, fixed on C and the complement of T, taking S to T. The net result is that it is possible to construct a homeomorphism h_3' taking the components of $h_2 \circ h_1(M_1)$ to the components of S. One now proceeds inductively, matching up further stages in the constructions, obtaining the desired homeomorphism h as a limit. □

Remark 3.2. A standard argument shows that an Antoine Cantor set cannot be separated by a 2-sphere. This is also true if the construction starts with a finite open chain of linked tori as in Figure 3.

Remark 3.3. Also note that the homeomorphism of Theorem 3.1 can be realized as the final stage of an isotopy since each of the homeomorphisms in the argument can be realized by an isotopy.

Homogeneity groups of Antoine Cantor sets. Let C be obtained by a standard Antoine construction where the same number of tori are used in tori of previous stages in each stage of the construction. For example, the Antoine pattern in Figure 1 with 24 smaller tori, each geometrically similar to the outer torus, can be repeated in each component at each stage of the construction.

We now consider some elements of the embedding homogeneity group of C. There is an obvious $\mathbb{Z}_{24}$ group action on the resulting Cantor set obtained by rotating and twisting the large torus. There is also a $\mathbb{Z}_{24} \oplus \mathbb{Z}_{24}$ action on C obtained by considering the first two stages, where we require each torus in the second stage to rotate the same amount. If we allow the tori in the second stage to rotate different amounts, we get an even larger group action by a wreath product of $\mathbb{Z}_{24}$ with itself. Considering more stages results in even more complicated group actions.

In addition to these group actions arising from rotating and twisting, there are also orientation-reversing $\mathbb{Z}_2$ actions that arise from reflecting through a horizontal plane (containing the core of the large torus) or through a vertical plane (containing meridians of the large torus).

From this we see that even for a simple self-similar Antoine Cantor set, the embedding homogeneity group is more complex than just the group of obvious rotations from the linking structure. In the next section we shall carefully combine certain Antoine constructions to produce a more rigid example with nontrivial end homogeneity group, in such a way that these kinds of orientation-reversing homeomorphisms are not possible, and that also restricts the possible rotations.

4. A Cantor set with embedding homogeneity group $\mathbb{Z}_m$

Fix an integer $m > 1$. We describe how to construct a Cantor set in S^3 with embedding homogeneity group $\mathbb{Z}_m$.

Construction 4.1. As in the previous section, let S_0 be an unknotted solid torus in S^3. Let $\{S_{(1,i)} : 1 \leq i \leq 4m\}$ be an Antoine chain of $4m$ pairwise disjoint linked solid tori in the interior of S_0 and let

$$S_1 = \bigcup_{i=1}^{4m} S_{(1,i)}.$$

See Figure 1 for the case when $m = 6$. Let C_j, $1 \leq j \leq 4$, be a rigid Antoine Cantor set with first stage $S_{(1,j)}$. Choose these four rigid Antoine Cantor sets so that they are inequivalently embedded in S^3. Let h be a homeomorphism of S^3,

fixed on the complement of the interior of S_0, that takes $S_{(1,j)}$ to $S_{(1,j+4 \bmod 4m)}$ for $1 \leq j \leq 4m$. Require that h^m is the identity on each $S_{(1,i)}$.

For $4k < i \leq 4k+4$, let C_i be the rigid Cantor set in $S_{(1,i)}$ given by $h^k(C_{i-4k})$. Note that this produces m copies of each of the rigid Cantor sets C_1, C_2, C_3, and C_4. Again, see Figure 1, where the coloring indicates the four classes of rigid Cantor sets. The Cantor set we are looking for is

$$C = \bigcup_{i=1}^{4m} C_i.$$

Theorem 4.2. *The Cantor set C from the previous construction has embedding homogeneity group $\mathbb{Z}_m$ and is unsplittable.*

Proof. Let $\ell : S^3 \to S^3$ be a homeomorphism taking C to C. We show that $\ell|_C = h^k|_C$ for some k, $1 \leq k \leq m$. By [Sher 1968], we may assume that ℓ takes each $S_{(1,i)}$ to some $S_{(1,j)}$, and so $\ell(C_i) = C_j$. Because of the distinct rigid Cantor sets involved, this is only possible if $j - i \equiv 0 \bmod 4$.

So assume that $\ell(S_{(1,1)}) = S_{(1,4k+1)}$. Then $\ell(S_{(1,2)})$ must be one of the two tori linked with $S_{(1,4k+1)}$, namely $S_{(1,4k)}$ or $S_{(1,4k+2)}$. Since $(4k-2) \not\equiv 0 \bmod 4$, $\ell(S_{(1,2)})$ must be $S_{(1,4k+2)}$. Continuing inductively, one sees that $\ell(S_{(1,i)}) = S_{(1,4k+i \bmod 4m)}$. Thus $\ell(C_i) = C_{4k+i \bmod 4m}$. But $h^k(C_i)$ is also $C_{4k+i \bmod 4m}$. Since these are rigid Cantor sets, $\ell|_{C_i} = h^k|_{C_i}$ for each i.

So the embedding homogeneity group of C is $\{h^k : 1 \leq k \leq p\} \simeq \mathbb{Z}_m$. By Remark 3.2, C is unsplittable. The assertion follows. $\qquad\square$

5. A Cantor set with embedding homogeneity group $\mathbb{Z}$

We now construct a Cantor set in S^3 with embedding homogeneity group $\mathbb{Z}$. This requires careful analysis of an infinite chain analogue of the Antoine construction.

Construction 5.1. Let S_0 be a pinched solid torus in S^3, i.e., the quotient of a solid torus with a meridional disk collapsed to a single point w. Let T_i, $i \in \mathbb{Z}$, be a countable collection of unknotted pairwise disjoint solid tori in S_0 such that each T_i is of simple linking type with both T_{i-1} and T_{i+1}, and is not linked with T_j, $j \neq i - 1$ or $i + 1$. Place the tori T_i so that the T_i, $i > 0$, and the T_i, $i < 0$, have w as a limit point as in Figure 2.

For $1 \leq j \leq 3$, let C_j be a rigid Antoine Cantor set with first stage T_j. Choose these three rigid Antoine Cantor sets so that they are inequivalently embedded in S^3. Let α be a homeomorphism of S^3, fixed on the complement of the interior of S_0, that takes T_j to T_{j+3} for $j \in \mathbb{Z}$.

For $3k < i \leq 3k + 3$, let C_i be the rigid Cantor set in T_i given by $\alpha^k(C_{i-3k})$. Note that this produces a countable number of copies of each of the rigid Cantor
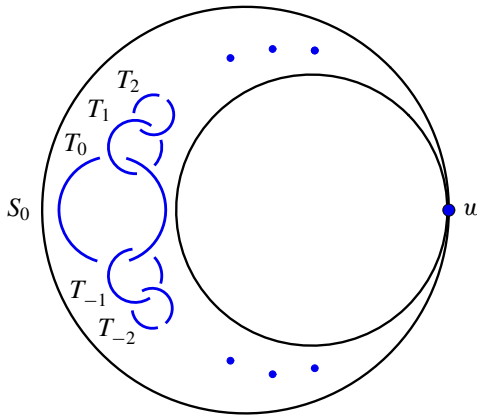
Figure 2. Infinite Antoine chain.

sets C_1, C_2, and C_3. Again, see Figure 2. The Cantor set we are looking for is

$$C = \overline{\bigcup_{i \in \mathbb{Z}} C_i} = \bigcup_{i \in \mathbb{Z}} C_i \cup \{w\}.$$

Note that C is a Cantor set since it is perfect, compact, and totally disconnected.

Theorem 5.2. *The Cantor set C from the previous construction has embedding homogeneity group $\mathbb{Z}$ and is unsplittable.*

Proof. It is clear from the construction that each point of $C - \{w\}$ has local genus 1. Theorem 2.4, applied to w and the Cantor sets $C_+ = \overline{\bigcup_{i>0} C_i}$ and $C_- = \overline{\bigcup_{i<0} C_i}$, shows that w has local genus 2 in C. Thus, any homeomorphism of S^3 that takes C to C must fix w.

Let h be such a homeomorphism of S^3 taking C to C. Let T_i' be the union of the linked tori in the Antoine chain at the second stage of the construction of C_i. Let

$$\Lambda_N = \bigcup_{i=-N}^{N} T_i, \quad \Gamma_N = \bigcup_{i=-N}^{N} C_i, \quad \text{and} \quad \Lambda_N' = \bigcup_{i=-N}^{N} T_i'.$$

Fix an integer $n \in \mathbb{Z}$. Since $h(T_n)$ does not contain w, there is a positive integer $N_1 > |n|$ such that $h(C_n) \subset \Gamma_{N_1}$. Similarly, there is a positive integer $N_2 > N_1$ such that $h^{-1}(\Gamma_{N_1}) \subset \Gamma_{N_2}$.

As in Step 1 in the proof of Theorem 3.1, there is a homeomorphism k of S^3 to itself, fixed on C, such that

$$k\big(h(\partial(\Lambda_{N_2+1}) \cup \partial(\Lambda_{N_2+1}'))\big) \cap \big(\partial(\Lambda_{N_2+1}) \cup \partial(\Lambda_{N_2+1}')\big) = \varnothing.$$

Fix a point p of C_n and let $k(h(p)) = h(p) = q \in C_m$. We will show that $k(h(C_n)) = h(C_n) = C_m$. Let $\ell = k \circ h$. Since $\ell(T_n) \cap T_m \neq \varnothing$, and since the boundaries do

not intersect, either $\ell(T_n) \subset \text{Int}(T_m)$ or $\text{Int}(\ell(T_n)) \supset T_m$. We consider these cases separately.

<u>Case I</u>: $\ell(T_n) \subset \text{Int}(T_m)$. If $\ell(T_n)$ has geometric index 0 in T_m, then $\ell(T_n)$ is contained in a cell in T_m and so it contracts in T_m. Since a contraction of $\ell(T_n)$ meets the boundary of the linked $\ell(T_{n+1})$, and since the boundary of $\ell(T_{n+1})$ is disjoint from the boundary of T_m, $\ell(T_{n+1}) \subset \text{Int}(T_m)$. Continuing inductively, one finds that one of the following two situations occur when $\ell(T_n)$ has geometric index 0 in T_m:

<u>Case Ia</u>: Each $\ell(T_j)$, for $n \le j \le N_2$, is contained in T_m and has geometric index 0 there. It follows that $\ell(T_{N_2+1}) \subset \text{Int}(T_m)$. But then $C_m \subset \Gamma_{N_1}$ and $h^{-1}(C_m) \cap C_{N_2+1} \ne \varnothing$, contradicting the choice of N_2.

<u>Case Ib</u>: There exists j with $n < j \le N_2$ and such that $\ell(T_j)$ is contained in $\text{Int}(T_m)$ and geometric index k in T_m, where $k > 1$. Then, by Theorem 2.2, $\ell(T_j)$ cannot be contained in any component of the next stage of the construction contained in T_m, since these have geometric index 0 in T_m. So some component of the next stage in T_m is contained in $\ell(T_j)$ and has geometric index 0 there by Theorem 2.2. Since the components of the next stage are linked, all components of the next stage in T_m are contained in $\ell(T_j)$. The geometric index of the union of the next stages of in T_m in $\ell(T_n)$ is even by Theorem 2.3 and cannot be equal to 0. Otherwise, by Theorem 2.2 the union of the next stages of T_m would have index 0 in T_m, which is a contradiction. So the geometric index of the union of the next stages of in T_m in $\ell(T_n)$ is at least 2. Then by Theorem 2.2, the geometric index of the union of the next stages of in T_m in T_m is at least 4, contradicting the fact that this geometric index is 2.

It follows that $\ell(T_j)$ has geometric index 1 in T_m and contains the union of the next stages contained in T_m. Since ℓ is a homeomorphism that takes C to C, it follows from the construction of C that $\ell(C_j) = C_m$. Since $\ell(p) \in C_m$, $\ell(T_n) \cap \ell(C_j) \ne \varnothing$, contradicting the fact that ℓ is a homeomorphism.

Thus, neither Case Ia nor Case Ib can occur. So the geometric index of $\ell(T_n)$ in T_m must be at least 1. Repeating the argument from Case Ib above with T_j replaced by T_n, we see that $\ell(T_n)$ has geometric index 1 in T_m and contains the union of the next stages contained in T_m. Since ℓ is a homeomorphism that takes C to C, it follows from the construction of C that $\ell(C_n) = C_m$ as desired.

<u>Case II</u>: $\text{Int}(\ell(T_n)) \supset T_m$. Then $\ell^{-1}(T_m) \subset \text{Int}(T_n)$. The argument from Case I can now be repeated, replacing ℓ by ℓ^{-1} and interchanging T_n and T_m. It follows that $\ell^{-1}(C_m) = C_n$ and so $\ell(C_n) = C_m$ as desired.

Since $\ell(C_n) = h(C_n) = C_m$, it must be the case that $(m - n) \equiv 0 \mod 3$. Continuing as in the proof of the $\mathbb{Z}_m$ result (Theorem 4.2), we have that for each i,

$h(C_i) = C_{i+(m-n)}$. Recall that for the homeomorphism α from the construction of C, it is also the case that $\alpha^{(m-n)/3}(C_i) = C_{i+(m-n)}$. By the rigidity of these Cantor sets, it follows that $\alpha^{(m-n)/3}|_{C_i} = h|_{C_i}$. Thus the embedding homogeneity group of C is $\{\alpha^k : k \in \mathbb{Z}\} \simeq \mathbb{Z}$.

We now show that C is unsplittable. Assume that Σ is a 2-sphere in S^3 that separates C. Choose $\epsilon > 0$ so that the distance from Σ to C is greater than ϵ. Choose N so that each T_i, $|i| \geq N$, has diameter less than $\epsilon/6$ and is within $\epsilon/6$ of w. Since Σ separates C, $w \cup \bigcup_{|i| \geq N} T_i$ must be in one component of $S^3 - \Sigma$ and there must be points of C in the other component of $S^3 - \Sigma$. So $\bigcup_{|i| \leq N} T_i$ contains points in both components of $S^3 - \Sigma$.

Form an Antoine Cantor set C' related to C as follows. Use $\bigcup_{|i| \leq N} T_i$ as a part of the first stage of the construction. Complete the first stage of the construction by adding an unknotted solid torus T, linked to T_N and T_{-N}, that is within the $\epsilon/3$-neighborhood of w. For successive stages of the Antoine Cantor set C' in T_i, $|i| \leq N$, use the successive stages in forming the Cantor set $C_i \subset C$. For successive stages of the Antoine Cantor set C' in T, use any Antoine construction.

By construction and the properties of Σ, the 2-sphere Σ separates the Antoine Cantor set C', contradicting Remark 3.2.   □

6. Main results

Given a finitely generated abelian group G, we use the results from the previous two sections to construct an unsplittable Cantor set C_G in S^3 with embedding homogeneity group G.

Construction 6.1. Let $G \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k}$ be any finitely generated abelian group. Form a simple chain of $n + k$ pairwise disjoint unknotted solid tori. Figure 3 illustrates the case $n + k = 6$. Label the tori as $T_1, T_2, \ldots, T_{n+k}$ so that T_1 is only linked with T_2, T_{n+k} is only linked with T_{n+k-1}, and each T_i, for $2 \leq i \leq n + k - 1$, is linked with T_{i-1} and T_{i+1}.

For $1 \leq i \leq n$, perform Construction 5.1 in T_i, treating a pinched version of T_i in the interior of T_i as the torus S_0 in Construction 5.1. Let w_i be the limit point corresponding to w in Construction 5.1. This yields a Cantor set C_i in T_i with embedding homogeneity group $\mathbb{Z}$. See Figure 4.
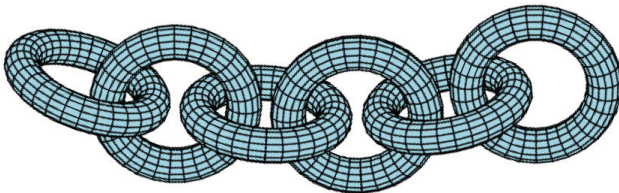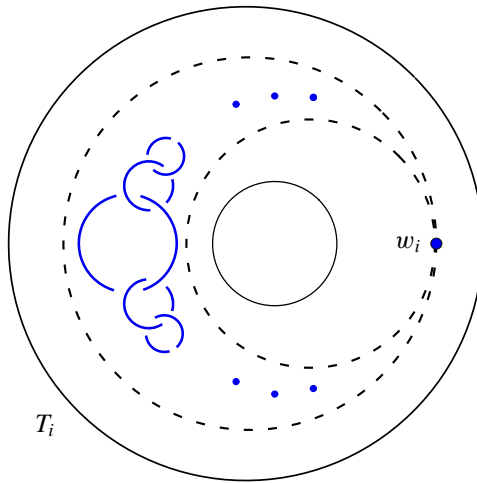

Figure 3. An Antoine chain containing C_G.

Figure 4. Pinched torus in T_i.

For $n + 1 \leq i \leq n + k$, perform Construction 4.1 for the group $\mathbb{Z}_{m_{i-n}}$ in T_i. This yields a Cantor set C_i in T_i with embedding homogeneity group $\mathbb{Z}_{m_{i-n}}$. Choose all the rigid Cantor sets from Constructions 5.1 and 4.1 to be inequivalent.

Let

$$C_G = \bigcup_{i=1}^{n+k} C_i.$$

Theorem 6.2. *The Cantor set C_G constructed above has embedding homogeneity group G and is unsplittable.*

Proof. For $1 \leq i \leq n + k$, let h_i be a self-homeomorphism of S^3, fixed on the complement of T_i, such that $h_i|_{C_i}$ generates the embedding homeomorphism group of C_i ($\mathbb{Z}$ for $1 \leq i \leq n$ and $\mathbb{Z}_{m_{i-n}}$ for $n + 1 \leq i \leq n + k$). Then

$$\left\{ (h_1^{j_1} \circ h_2^{j_2} \circ \cdots \circ h_{n+k}^{j_{n+k}})\big|_{C_G} \right\} \simeq G \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k}.$$

Let h be a homeomorphism of S^3 to itself taking C_G to C_G. We will show that $h|_{C_G} = (h_1^{j_1} \circ h_2^{j_2} \circ \cdots \circ h_{n+k}^{j_{n+k}})\big|_{C_G}$ for some choice of j_i.

Step 1: The homeomorphism h must take each C_i to itself. As in the proof of Theorem 5.2, there are exactly n points of genus 2 in C_G, one in each C_i, $1 \leq i \leq n$. These are the points $\{w_1, w_2, \ldots, w_n\}$. The homeomorphism must take this set of genus 2 points to itself.

Let T be one of the solid torus components of the first stage of the Antoine construction for some C_i, $1 \leq i \leq n + k$. As in the proofs of Theorems 4.2 and 5.2, after a general position adjustment, either $h(T)$ must lie in the interior of some solid torus component T' of the first stage of the Antoine construction for some C_j,

or T' must lie in the interior of $h(T)$. A similar argument to that in Theorem 5.2 shows that $N(h(T), T') = 1$ or $N(T', h(T)) = 1$, and that $h(C_i \cap T) = C_j \cap T'$.

This same argument can be applied to all first stage tori in C_i, resulting in the fact that $h(C_i) = C_j$. Because of the inequivalence of the rigid Cantor sets used in the construction, $i = j$ and $h(C_i) = C_i$.

<u>Step 2</u>: For each i, $h|_{C_i} = h_i^{k(i)}|_{C_i}$ for some $k(i)$. By Step 1, we have that $h(C_i) = C_i$. It follows from the construction that $h|_{C_i} = h_i^{k(i)}$ for some $k(i)$. From this, it follows that $h|_{C_G} = (h_1^{j_1} \circ h_2^{j_2} \circ \cdots \circ h_{n+k}^{j_{n+k}})|_{C_G}$ for some choice of j_i.

Thus, the embedding homeomorphism group of C_G is isomorphic to G.

<u>Step 3</u>: C_G is unsplittable. Let Σ be a 2-sphere in S^3 separating C_G. As in the proof of Theorem 5.2, an Antoine Cantor set with first stage $\bigcup_{i=1}^{n+k} T_i$ can be formed so that Σ separates this Antoine Cantor set. This is a contradiction. (See Remark 3.2.) □

Corollary 6.3. *Let* $G \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k}$ *be any finitely generated abelian group. There is a irreducible open 3-manifold* M_G *with the following properties*:

(a) *The Freudenthal compactification of* M_G *is* S^3.

(b) *The end set of* M_G *is homeomorphic to a Cantor set.*

(c) *The end homogeneity group of* M_G *is isomorphic to* G.

(d) *The genus of* M_G *at infinity is 2 at the n points corresponding to* $\mathbb{Z}^n$ *and is* 1 *otherwise.*

Proof. Let M_G be $S^3 - C_G$, where C_G is as in Construction 6.1. The end set of M_G is C_G and the end homogeneity group of M_G is isomorphic to the embedding homogeneity group of C_G. M_G is irreducible because C_G is unsplittable. Claims (b) and (c) now follow from Theorem 6.2, while (d) follows from the proof of that theorem. □

Remark 6.4. For each finitely generated abelian group G as above, there are uncountably many nonhomeomorphic 3-manifolds as in the corollary. This follows from varying the rigid Cantor sets used in the construction.

7. Questions

Question 7.1. If a finitely generated abelian group is infinite, is there an open 3-manifold with end homogeneity group G that is genus 1 at infinity?

Question 7.2. Given a finitely generated abelian group G, are there simply connected open 3-manifolds with end homogeneity group G?

Question 7.3. Is the mapping class group of the open 3-manifold M_G isomorphic to G?

Question 7.4. If G is a finitely generated nonabelian group, is there an open 3-manifold with end homogeneity group G?

Question 7.5. If G is a nonfinitely generated group, is there an open 3-manifold with end homogeneity group G?

Acknowledgments

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