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## Groups which are not properly 3-realizable

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**Abstract.** A group is properly 3-realizable if it is the fundamental group of a compact polyhedron whose universal covering is proper homotopically equivalent to some 3-manifold. We prove that when such a group is also quasi-simply filtered then it has *pro-(finitely generated free) fundamental group at infinity* and *semi-stable ends*. Conjecturally the quasi-simply filtration assumption is superfluous. Using these restrictions we provide the first examples of finitely presented groups which are not properly 3-realizable, for instance large families of Coxeter groups.

### 1. Introduction

The aim of this paper is to obtain necessary conditions for a finitely presented group to be properly 3-realizable, which lead conjecturally to a complete characterization. Lasheras introduced and studied this class of groups in [8], [9] and [21]. Recall that:

**Definition 1.1.** A finitely presented group  $\Gamma$  is said to be *properly 3-realizable* (abbreviated P3R from now on) if there exists a compact 2-dimensional polyhedron  $X$  with fundamental group  $\Gamma$  such that the universal covering  $\tilde{X}$  is proper homotopy equivalent to a 3-manifold  $W^3$ .

Hereafter we will consider only *infinite* groups  $\Gamma$  and thus the associated 3-manifolds  $W^3$  appearing in the definition above will be *non-compact*. Notice that, in general, the 3-manifolds  $W^3$  will also have *non-compact* boundary.

**Remark 1.1.** In the definition of a P3R group one does not claim that the universal covering of *any* compact 2-dimensional polyhedron  $X$  with fundamental group  $\Gamma$  is proper homotopy equivalent to a 3-manifold. However it was proved in Proposition 1.3 of [1] that given a P3R group  $G$  then for *any* 2-dimensional compact polyhedron  $X$  with fundamental group  $G$ , the universal covering of the wedge  $X \vee S^2$  is proper homotopy equivalent to a 3-manifold.

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Recall the following classical theorem about embeddings up to homotopy, due to Stallings. Let  $P$  be a finite CW-complex of dimension  $k$ , let  $M$  be a PL-manifold of dimension  $m$  and let  $f: P \rightarrow M$  be a  $c$ -connected map. If  $m - k \geq 3$  and if  $c \geq 2k - m + 1$  then there exist a compact subpolyhedron  $j: Q \hookrightarrow M$  and a homotopy equivalence  $h: P \rightarrow Q$  such that  $jh$  is homotopic to  $f$ . This was generalized to the non-compact situation in [7] by replacing the connectivity with the proper connectivity. Recall that a locally finite CW complex is said to be properly  $c$ -connected ( $c \geq 1$ ) if its proper homotopy type can be represented by a CW complex whose  $c$ -skeleton is reduced to an end-faithful tree (see [7] for details). Thus the proper homotopy type of a locally finite CW-complex  $X$  of dimension  $n$  is represented by a closed subpolyhedron of  $R^{2n-c}$  if  $X$  is properly  $c$ -connected.

In particular, the universal covering  $\tilde{X}$  of an *arbitrary* compact 2-polyhedron  $X^2$  is proper homotopy equivalent to a 4-manifold, because any 2-polyhedron embeds, up to *proper homotopy*, into  $\mathbb{R}^4$ . Therefore P3R groups are singled out among the set of all finitely presented groups by the fact that the universal covering  $\tilde{X}$  of some compact polyhedron  $X$  with given  $\pi_1(X)$  is proper homotopy equivalent to a particular 4-manifold, namely the product of a 3-manifold with an interval.

**Remark 1.2.** Fundamental groups of compact 3-manifolds are obviously P3R, but there also exist P3R groups which are *not* 3-manifold groups. For instance, any ascending HNN extension of a finitely presented group is P3R ([21], see also other explicit examples in [9]). Moreover, given any infinite finitely presented groups  $G$  and  $H$ , their direct product  $G \times H$  is P3R (according to [8]). Further amalgamated products of P3R groups (and HNN extensions) over finite groups yield P3R groups (see [10]).

Let us introduce very briefly, for the sake of completeness, some end invariants of non-compact spaces which will be used in the sequel. Standard references where these notions are studied in detail are [2] and [23].

Given the sequence of homomorphisms  $A_{i-1} \leftarrow A_i$ , called *bonding morphisms*, one builds the *tower of groups*  $A_0 \leftarrow A_1 \leftarrow \dots$ . A *pro-isomorphism* between the towers  $A_0 \leftarrow A_1 \leftarrow \dots$  and  $B_0 \leftarrow B_1 \leftarrow \dots$  is given by two sequences of morphisms  $B_{j_{2n+1}} \rightarrow A_{i_{2n+1}}$  and  $A_{i_{2n}} \rightarrow B_{j_{2n}}$  where  $0 = i_1 < j_1 < j_2 < i_2 < i_3 < j_3 < j_4 < i_4 < \dots$ , which commute with the respective compositions of bonding morphisms in the two towers. A pro-isomorphism class of towers of groups is called a *pro-group*.

**Definition 1.2.** A pro-group is said to be *pro-(finitely generated free)* if it has a representative tower in which all groups involved are finitely generated free groups.

It was shown in [21] that if a pro-group is pro-(finitely generated free) and has a representative tower with surjective bonding maps, then it has a representative telescopic tower (i.e., a tower in which both conditions hold simultaneously).

Pro-groups arise in topology by means of towers associated to exhaustions of non-compact spaces.

**Definition 1.3.** If  $X$  is a polyhedron then a proper map  $\omega : [0, \infty) \rightarrow X$  is called a *proper ray*. Two proper rays define the same *end* if their restrictions to the subset of natural numbers are properly homotopic.

An end is called *semi-stable* if every two proper rays defining this end are actually properly homotopic; one also says that the two rays define the same strong end.

A finitely presented group has *semi-stable ends* if there exists a compact polyhedron  $X$  with the given fundamental group whose universal covering has semi-stable ends.

Given now a proper base ray  $\omega$  in  $X$  and an exhaustion  $C_1 \subset C_2 \subset \dots \subset X = \bigcup_{i=1}^{\infty} C_i$  by compact subpolyhedra, we can associate a tower of groups

$$\pi_1(X, \omega(0)) \leftarrow \pi_1(X - C_1, \omega(1)) \leftarrow \dots$$

where the bonding morphisms are induced, on the one hand, by the inclusions of spaces and on the other hand, by the change of base points which are slid along the ray  $\omega$  restricted to integral intervals.

**Definition 1.4.** The (*fundamental*) *pro-group at infinity* of  $X$  based at  $\omega$ , denoted by  $\pi_1^\infty(X, \omega)$ , is the pro-group associated to the tower of groups

$$\pi_1(X, \omega(0)) \leftarrow \pi_1(X - C_1, \omega(1)) \leftarrow \dots$$

Two rays defining the same strong end yield isomorphic pro-groups. In particular, if the end is semi-stable, the pro-group at infinity is an invariant of the end, and called the (*fundamental*) *pro-group of the end*. The end is called *simply connected at infinity* (or  $\pi_1$ -trivial) if the associated pro-group is pro-isomorphic to a tower of trivial groups.

The (*fundamental*) *pro-group at infinity of a finitely presented group* is the pro-group at infinity of the universal covering of a compact polyhedron with the given fundamental group. This depends of course, on the base ray (and thus only on the end if it is semi-stable), but not on the particular compact polyhedron we chose.

**Remark 1.3.** There are alternative equivalent definitions of the semi-stability, in particular the one used in Siebenmann's thesis: an end is called semi-stable if its fundamental pro-group has a representative tower with *surjective* bonding morphisms (see also [20]). For the sake of completeness we recall that an end is called *stable* if there exist some representative tower in which all bonding morphisms are *isomorphisms*. Examples of Davis (see [12]) show that the ends of universal coverings of finite complexes might be *not stable*, although it is not known whether they should be always semi-stable. Notice that sometimes in the literature one uses the terms  $\pi_1$ -stable,  $\pi_1$ -semi-stable etc. for the corresponding notions introduced above. As already observed above, we can infer from [21] that a semi-stable end having pro-(finitely generated free) fundamental pro-group at infinity admits a representative *telescopic tower* for that fundamental pro-group at infinity.

If a group has semi-stable ends then the universal covering of *any* compact polyhedron with the given fundamental group has semi-stable ends. Although there exist spaces whose ends are not semi-stable, there are still no known examples of finitely presented groups (i.e., universal coverings of compact polyhedra) without semi-stable ends (see also [19], [24]).

The main source of examples of P3R groups is the paper of Lasheras ([21]), where it is proved that *a one-ended finitely presented group which is semi-stable and whose fundamental pro-group at infinity is pro-(finitely generated free) is P3R*. In particular, any one-ended finitely presented group  $\Gamma$  which is simply connected at infinity (and hence automatically semi-stable at infinity) is P3R.

We expect the following to be a complete characterization of this class of groups:

**Conjecture 1** (3-dimensional homotopy covering conjecture). *A finitely presented group is P3R iff each one of its ends is semi-stable and has pro-(finitely generated free) fundamental pro-group.*

**Remark 1.4.** In [22] the authors proved the sufficient part of the conjecture, namely that a finitely presented group whose ends are semi-stable and have pro-(finitely generated free) fundamental pro-groups is P3R.

In this paper we give evidence in the favor of this conjecture, by proving it in the case when the group under consideration satisfies an additional hypothesis related to the geometric simple connectivity. In order to explain this we have to introduce, following Brick–Mihalik ([5]) and Stallings ([33]), the following tameness condition for groups and spaces:

**Definition 1.5.** A space  $X$  is called *quasi-simply filtered* (abbreviated *qsf*) if for any compact  $C \subset X$  there exists a connected and simply connected compact  $K$  together with a map  $f : K \rightarrow X$  such that  $f(K) \supset C$  and  $f|_{f^{-1}(C)} : f^{-1}(C) \rightarrow C$  is a homeomorphism.

A finitely presented group  $\Gamma$  is called *qsf* if there exists a (equivalently, for every) compact polyhedron  $P$  with fundamental group  $\Gamma$  such that the universal covering  $\tilde{P}$  is *qsf*.

The condition *qsf* is a rather mild assumption on finitely presented groups. There are still no known examples of groups which do not have the *qsf* property and most classes of known groups, as hyperbolic, semi-hyperbolic, automatic, tame combable, etc., are *qsf* (see [16], [25]).

We can now state our main result:

**Theorem 1.1.** *If a finitely presented group is P3R and qsf then all its ends are semi-stable and have pro-(finitely generated free) fundamental group at infinity.*

**Remark 1.5.** We do not know whether all finitely presented groups which have semi-stable ends and pro-(finitely generated free) fundamental groups at each end are actually *qsf*. Notice that by a theorem of Wright (see Theorem 16.5.6 in [18]), one-ended groups with stable end having an element of infinite order must be either

simply connected at infinity or pro- $\mathbb{Z}$  at infinity. Thus they are P3R by the result of Lasheras cited above.

**Remark 1.6.** 1) The homotopy covering conjecture implies the well-known covering conjecture in dimension 3 which states that the universal covering of an irreducible closed 3-manifold  $M^3$  with infinite fundamental group is simply connected at infinity. In fact, the universal covering  $\widetilde{M}$  is an open contractible 3-manifold (thus one-ended) which is semi-stable and has pro-(finitely generated free) fundamental pro-group at infinity. This implies that there exists an exhaustion by compact submanifolds  $C_i$  such that  $\pi_1(\widetilde{M} - C_i)$  are finitely generated and free. Tucker's criterion from [34] implies that the manifold  $\widetilde{M}$  is a missing boundary manifold and thus it is homeomorphic to  $\text{int}(N^3)$ , for a suitable compact 3-manifold  $N^3$  with boundary. By the contractibility of the universal covering, each component of  $\partial N^3$  is homeomorphic to a 2-sphere and this implies that  $\text{int}(N^3)$  (and hence  $\widetilde{M}$ ) is simply connected at infinity.

2) Conversely, it is obvious that the universal covering conjecture implies the homotopy covering conjecture *for closed 3-manifold groups* because open 3-manifolds which are simply connected at infinity are semi-stable and have pro-(finitely generated free) pro-group at infinity (in fact a trivial pro-group!).

3) Notice that the universal covering  $\widetilde{X}$  of a compact 2-polyhedron  $X$  can never be proper homotopy equivalent to an open (simply connected) 3-manifold  $M^3$ . In fact, the Poincaré duality would give us that the third cohomology group with compact support  $H_c^3(\widetilde{X})$  is isomorphic to  $H_c^3(M) = H_0(M) = \mathbb{Z}$ , which is impossible, as  $\dim(\widetilde{X}) = 2$ .

**Remark 1.7.** Let us consider the universal covering  $\widetilde{M}^3$  of a 3-manifold  $M^3$  with boundary. If the boundary is a union of spheres then  $\widetilde{M}$  is obtained from the universal covering of a closed 3-manifold (obtained by capping off boundary spheres by balls) by deleting a collection of disjoint balls. Assume that the boundary is non-trivial i.e. not a union of 2-spheres. Then  $M^3$  is Haken and thus, by Thurston's theorem, it is a geometric 3-manifold. Let us moreover assume that  $M^3$  is *atoroidal*, i.e., there are no  $\mathbb{Z} \oplus \mathbb{Z}$  embedded in  $\pi_1(M)$  other than peripheral subgroups coming from the boundary torus components. Then Thurston's geometrization theorem tells us that  $M^3$  is hyperbolic. Therefore the universal covering  $\widetilde{M}$  is obtained geometrically by deleting a collection of horoballs from the hyperbolic 3-space. In particular the pro-group at infinity of  $\widetilde{M}$  is pro-(finitely generated free) and its ends are semi-stable. Thus the conjecture holds for fundamental groups of atoroidal 3-manifolds with non-trivial boundary. A similar but more involved discussion shows that it also holds for all 3-manifolds with non-trivial boundary (since they are geometric).

**Remark 1.8.** The homotopy covering conjecture implies that all 1-relator groups are P3R. This is already known for 1-relator finitely ended groups (see [11]). In fact, 1-relator groups are semi-stable at infinity (see [26]) and it was proved in Proposition 2.7 of [11] that their pro-groups at infinity are pro-(finitely generated free).

Notice that 1-relator groups are also qsf (see [25]). Recently, Lasheras and Roy (see [22]) have extended the results of [11] to a class of groups which contains all 1-relator groups.

It is presently unknown (but quite plausible) that any finitely presented group which is qsf and has pro-(finitely generated free) pro-groups at infinity is P3R.

As an application of Theorem 1.1 we will obtain explicit examples of groups which are not P3R, as follows:

**Theorem 1.2.** *Let  $\Gamma$  be one of the following:*

- 1) *The fundamental group of a finite non-positively curved complex which is a homology  $n$ -manifold ( $n \geq 3$ ), but not a topological manifold. We further assume that the link of every vertex is a topological manifold.*
- 2) *The right angled Coxeter group associated to a flag complex  $L$  whose geometric realization is a closed combinatorial  $n$ -manifold ( $n \geq 3$ ) and  $\pi_1(L)$  is not a free group.*

*Then  $\Gamma$  is not P3R.*

In particular many Coxeter groups are *not* P3R. Similar examples were announced by Cárdenas.

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## 2. Proofs

### 2.1. Tameness criterion for non-compact 3-manifolds

Recall that a polyhedron  $P$  is called *weakly geometrically simply connected (wgsc)* if it admits an exhaustion by compact connected subpolyhedra  $P_1 \subset P_2 \subset \dots$  such that  $\pi_1(P_n) = 0$ , for all  $n$ . The wgsc property for polyhedra is the piecewise-linear analogue of the geometric simple connectivity of open manifolds, namely the existence of a proper handlebody decomposition without index one handles.

It is proved in [14], [17] that an open 3-manifold proper homotopy equivalent to a weakly geometrically simply connected polyhedron is simply connected at infinity. In this section we will extend this result to non-compact 3-manifolds.

In the realm of manifolds with boundary the relevant tameness condition that will replace the simple connectivity at infinity is the following:

**Definition 2.1.** A manifold  $W$  is called a *missing boundary manifold* (also called *almost compact*) if there exists a compact manifold with boundary  $M$  and a closed subset  $A \subset \partial M$  of the boundary (not necessarily a subcomplex) such that  $W$  is homeomorphic to  $M - A$ .

Interesting examples of manifolds which are not missing boundary manifolds can be found in [32], [35].

We first introduce a family of 3-manifolds which is, in some sense, the smallest one containing the missing boundary 3-manifolds and allowing manifolds to have infinitely many boundary components. These manifolds will be the proper analog of the open manifolds which are simply connected at infinity in the non-compact case.

Before we proceed, let us recall that a compact 0-dimensional subset  $C$  is said to be *tame* (or tamely embedded) in  $\mathbb{R}^n$  if there exists a homeomorphism of  $\mathbb{R}^n$  sending  $C$  into a subset of  $\mathbb{R} \times \{0\} \subset \mathbb{R}^n$ . It is well-known that perfect (i.e., without isolated points) compact 0-dimensional separable topological spaces are homeomorphic to the Cantor space. Hence the tameness condition above is mostly relevant for Cantor subsets of  $\mathbb{R}^n$ . Notice that there exist wild Cantor sets in any  $\mathbb{R}^n$ , with  $n \geq 3$ , while Cantor sets in  $\mathbb{R}^2$  are tame, by a classical theorem of Bing ([3]).

**Definition 2.2.** A *standard model* is a 3-manifold with boundary  $V$  constructed as follows. Let  $\{B_i\}_{i \in I}$  be a collection of pairwise disjoint 3-balls in the interior  $\text{int}(B)$  of the 3-ball whose radii go to 0 and whose limit set  $L$  is a tame 0-dimensional subset disjoint from  $\partial B$ . Let  $X \supset L$  be a tame 0-dimensional subset of  $\text{int}(B)$  which is disjoint from  $\text{int}(B_i)$ , for all  $i \in I$ , and  $T \subset \partial B \cup \bigcup_{i \in I} \partial B_i$ . Then we put  $V = B - (X \cup T \bigcup_{i \in I} \text{int}(B_i))$ . Manifolds of this form, where  $T \cap \partial B = \emptyset$ , were called *ragged cells* by Brin and Thickstun in [6] (see pages 9–10).

In order to simplify some arguments we will use in the sequel the fact that there are no fake homotopy disks in dimension 3, as the Poincaré conjecture has been settled by Perelman in [29], [30] (see a detailed and self-contained exposition of Perelman's proof in [27]).

**Remark 2.1.** 1) Open simply connected 3-manifolds  $V$  which are simply connected at infinity can be described as the manifolds of the form  $S^3 - X$ , where  $X$  is a tame 0-dimensional compact subset of  $B^3$ . Alternatively,  $V$  can be written as an ascending union of compact simply connected submanifolds, i.e., disks-with-holes, by the Poincaré Conjecture (see [17], [36]).

2) A simply connected missing boundary 3-manifold  $V$  is homeomorphic to  $M - T$ , where  $M$  is a simply connected compact 3-manifold and  $T$  is a closed subset of  $\partial M$  (see e.g. [36]). By the Poincaré Conjecture there is a finite set of pairwise disjoint balls  $B_i$ ,  $i \in I$  such that  $V = B - (\bigcup_{i \in I} \text{int}(B_i) \cup T)$  and  $T$  is a closed subset of  $\partial B \cup \bigcup_{i \in I} \partial B_i$ . Thus standard models  $V$  with finite  $I$  correspond precisely to simply connected missing boundary manifolds. Actually, any standard model can be obtained by making connected sums of (possibly infinitely many) simply connected missing boundary manifolds.

**Remark 2.2.** 1) Another characterization of standard models was given by Brin and Thickstun (see Full End Description Theorem (b), page 10 of [6]), as follows: Modulo the Poincaré Conjecture, the set of simply connected end 1-movable 3-manifolds coincides with that of standard models. In particular, 3-manifolds with semi-stable ends are homeomorphic to standard models.



2) Cárdenas announced, as an application of the Brin–Thickstun structure theorem ([6]), that 1-ended groups which are P3R and semi-stable have actually pro-(finitely generated free) pro-group at infinity.

**Remark 2.3.** The boundary of a standard model consists of 2-spheres and open planar surfaces. Each end has pro-(finitely generated free) fundamental group at infinity. In fact, the complement of an unknotted ball in a 1-ended standard model is homotopy equivalent to the complement of a finite graph, namely a holed handlebody. Thus its fundamental group is a finitely generated free group. Moreover, each end of a standard model is semi-stable.

The homotopy covering conjecture admits an (*a priori* stronger) restatement as follows:

**Conjecture 2.** *Given a finitely presented P3R group, the universal covering of some compact 2-dimensional polyhedron with this fundamental group is proper homotopy equivalent to a standard model.*

**Remark 2.4.** The equivalence between the two conjectures stated in this paper is a consequence of the Brin–Thickstun structure theorem ([6]). Details are left to the reader.

The wgsc property is not so useful anymore if we consider the tameness of 3-manifolds *with boundary*.

The antecedent of the papers [14], [17] is the paper [31] of Poénaru in which the geometric simple connectivity is already defined and used for non-compact manifolds with boundary.

Poénaru proved in [31] that an open 3-manifold is simply connected at infinity if the product with a closed  $n$ -ball (for some  $n \geq 2$ ) is a geometrically simply connected manifold with boundary. One might therefore expect that the analogous statement is true for the more general case of non-compact 3-manifolds. However, we will have to consider products of non-compact manifolds and disks, namely manifolds with corners. It is thus natural to look for the piecewise-linear analogue of the geometric simple connectivity of manifolds with boundary. Specifically, we set:

**Definition 2.3.** A polyhedron  $P$  is said to be *pl-gsc* if it admits an exhaustion by compact connected subpolyhedra  $P_1 \subset P_2 \subset \dots$  such that  $\pi_1(P_n) = 0$  and  $\pi_1(A, A \cap P_n) = 0$ , for every connected component  $A$  of  $\overline{P_{n+1} - P_n}$  and all  $n$ . Equivalently, the map induced by inclusion  $\pi_1(A \cap P_n) \rightarrow \pi_1(A)$  is surjective for all  $A$ , as above and all  $n$ .

This definition is consistent with the previous ones since, by using Smale’s theorem, a non-compact manifold of dimension  $n \geq 6$  is pl-gsc iff it is gsc. Moreover the gsc and pl-gsc are equivalent for open manifolds without any dimensional restrictions. The pl-gsc is stronger than the wgsc for 3-manifolds with boundary.

We start by recalling the following tameness condition for topological spaces, which is directly related to the qsf:



**Definition 2.4.** A non-compact PL space  $X$  is called *Tucker* if the fundamental group of each component of  $X - K$  is finitely generated, for any finite subcomplex  $K \subset X$ .

This definition was motivated by Tucker's work [34] on 3-manifolds, who proved that a  $P^2$ -irreducible connected 3-manifold is a missing boundary 3-manifold if and only if it is Tucker.

The principal result of this section is the following extension of the result of [14] and [17] to arbitrary non-compact 3-manifolds, as follows:

**Proposition 2.1.** *A non-compact 3-manifold which has the proper homotopy type of a pl-gsc polyhedron is homeomorphic to a standard model. In particular, each end is semi-stable and its fundamental pro-group at infinity is pro-(finitely generated free).*

*Proof.* According to [14] and [17], the interior  $\text{int}(W^3)$  is homeomorphic to a sphere minus a tame 0-dimensional subspace and in particular, it is simply connected at infinity.

Observe that a pl-gsc polyhedron is Tucker, by an easy application of the van Kampen theorem. Furthermore, if  $W^3$  is proper homotopically equivalent to a Tucker polyhedron then  $W^3$  is also Tucker (see [28]).

Let  $e$  denote one of the countably many ends of  $\text{int}(W^3)$  which in  $W^3$  has the boundary  $\partial_e W^3$  associated to it. Consider a partial (Freudenthal) end-point compactification of  $\text{int}(W^3)$  which closes off all its ends but  $e$ . Recall that the end-point compactification of  $X$  is a connected space  $\widehat{X}$  containing  $X$  as an open subset, with  $\widehat{X} - X$  totally disconnected, such that for each  $p \in \widehat{X} - X$ ,  $U$  a connected open neighborhood of  $p$  in  $\widehat{X}$  the set  $U - (\widehat{X} - X)$  is connected. In other words, we have one compactification point for each end of  $X$ .

The end-point compactification is a manifold at some end if and only if the end is simply connected at infinity (see [13], [36]). We therefore obtain a simply connected 3-manifold  $Z_e^3$  with boundary  $\partial_e W^3$  whose interior  $\text{int}(Z_e^3)$  has only one end. Therefore  $Z_e^3$  is irreducible. Observe that  $Z_e^3$  has also the Tucker property. Using Tucker's criterion from [34] we deduce that  $Z_e^3$  is a missing boundary manifold, and thus of the form  $B^3 - T_e$ , where  $T_e$  is a closed subset of  $\partial B^3$  and  $B^3$  is a 3-ball.

Use this method for each end  $e$  of  $\text{int}(W^3)$  having a boundary associated to it. We can recover  $W^3$  as the intersection of all  $Z_e^3$  punctured along a tame Cantor subset corresponding to those ends having no boundary associated to them. Alternatively,  $W^3$  is an infinite connected sum of all  $Z_e^3$  punctured along a tame Cantor subset. Therefore  $W^3$  is homeomorphic to a standard model.  $\square$

**Remark 2.5.** We say that the polyhedron  $M$  is properly homotopically dominated by the polyhedron  $X$  if there exists a PL map  $f : M \rightarrow X$  whose mapping cylinder properly retracts on  $M$ . Then a manifold  $W^3$  which is properly homotopically dominated by a Tucker polyhedron is also Tucker (see [28]). This shows that Proposition 2.1 extends to 3-manifolds which are properly homotopically dominated by a pl-gsc polyhedron.

## 2.2. Proof of Theorem 1.1

We first prove:

**Proposition 2.2.** *If the finitely presented group  $G$  is P3R and qsf then there exists a 2-polyhedron  $X$  with fundamental group  $G$  such that  $\widetilde{X}$  is pl-gsc and proper homotopy equivalent to a 3-manifold  $W^3$ .*

*Proof.* Since  $G$  is qsf it follows that for *any* polyhedron  $Y$  with fundamental group  $G$ , the universal covering  $\widetilde{Y}$  is qsf (see [5]). Take  $Y$  to be a closed 5-manifold with fundamental group  $G$ . Then  $\widetilde{Y}$  is an open 5-manifold. It was proved in Proposition 3.2 of [16] that any open simply connected manifold of dimension at least 5 which is qsf is actually gsc, as a consequence of general transversality results. It follows that  $\widetilde{Y}$  is gsc. We triangulate  $Y$  and get an equivariant triangulation of  $\widetilde{Y}$ . Then the triangulated  $\widetilde{Y}$  is a pl-gsc polyhedron. The pl-gsc property is preserved when passing to the 2-skeleton. This means that the 2-skeleton  $Z$  of the triangulation of  $Y$  has the property that  $\widetilde{Z}$  is pl-gsc.

It was proved in Proposition 1.3 of [1], as an application of Whitehead's theorem, that given a P3R group  $G$ , for *any* 2-dimensional compact polyhedron  $X$  of fundamental group  $G$ , the universal covering of the wedge  $X \vee S^2$  is proper homotopy equivalent to a 3-manifold. In particular, this holds when taking the 2-polyhedron  $Z$  from above and thus  $Z \vee S^2$  is proper homotopy equivalent to a 3-manifold. Moreover,  $\widetilde{Z \vee S^2}$  is made of one copy of  $\widetilde{Z}$  with infinitely many  $S^2$ 's attached on it. In particular, if  $\widetilde{Z}$  is pl-gsc then it is immediate that  $\widetilde{Z \vee S^2}$  is also pl-gsc. Therefore  $X = Z \vee S^2$  has the required properties.  $\square$

*End of the proof of Theorem 1.1.* Let assume that we have a group  $G$  which is both P3R and qsf. The previous proposition shows that there exists some 2-polyhedron  $X$  such that  $\widetilde{X}$  is pl-gsc and also proper homotopy equivalent to some 3-manifold  $W^3$ . Looking the other way around we can apply Proposition 2.1 to the 3-manifold  $W^3$  (since it is proper homotopy equivalent to a pl-gsc polyhedron) and obtain that  $W^3$  is homeomorphic to the standard model. In particular,  $W^3$  has semi-stable ends and its pro-groups at infinity are pro-finitely generated free, as claimed. By the proper homotopy invariance of these end invariants  $\widetilde{X}$  has the same properties. This proves Theorem 1.1.  $\square$

**Remark 2.6.** It follows by Remark 2.5 that Theorem 1.1 holds for the qsf groups  $G$  for which there exists a finite complex  $X$  such that  $\widetilde{X}$  properly homotopically dominates some 3-manifold. In particular, these groups are P3R.

## 2.3. Proof of Theorem 1.2

First, recall that groups acting properly cellularly and co-compactly on a CAT(0)-complex are wpsc and qsf (see [16], [25]). Thus Coxeter groups and fundamental groups of finite non-positively curved complexes are qsf.

Let us consider a finite non-positively curved complex  $X$ . We will use the criterion for the semi-stability given in [4], which also provides a way to understand the pro-group at infinity. The link of a vertex in  $X$  can be given a piecewise spherical metric. Let  $p$  be a point of the link of some vertex. The set of points of the link which are at distance at least  $\frac{\pi}{2}$  from  $p$  is called the punctured link. The punctured link deformation retracts onto the maximal subcomplex of the link that it contains. The main theorem of [4] states that if the links and the punctured links of  $X$  are connected then  $\tilde{X}$  has a semi-stable end.

If  $X$  is a homology  $n$ -manifold both the links and the punctured links have the same  $k$ -homology as the  $(n - 1)$ -sphere, for  $k \leq n - 2$ . In particular they are connected. On the other hand, there is at least one vertex  $v$  of  $x$  whose link is not simply connected, since the complex  $X$  is not a topological manifold. The fundamental group of the link is then perfect and non-trivial and thus it cannot be a free group. The complement of a punctured link within the link is the set of points of distance at most  $\frac{\pi}{2}$  from the puncture. Since the metric structure of the link is CAT(1) this complement is convex. Since we assumed the links to be topological manifolds each complement is a topological ball.

In [4] the Morse subdivision of  $\tilde{X}$  was defined as a geodesic subdivision induced by adding the critical points of the distance to a fixed base point. Let  $\tilde{X}_{>r}$  be the maximal subcomplex contained in the complement of the ball of radius  $r$  in the Morse subdivision of  $\tilde{X}$ . Since the distance is a Morse function on a CAT(0)-complex and the links are connected it is proved in [4] that the inverse system

$$\pi_1(\tilde{X}_{>0}) \leftarrow \pi_1(\tilde{X}_{>1}) \leftarrow \pi_1(\tilde{X}_{>2}) \leftarrow \dots$$

has surjective bonding maps, i.e., the end is semi-stable. Take the base point to be a lift of the vertex  $v$ . Then  $\tilde{X}_{>0}$  deformation retracts onto the link of  $v$ , and thus the first term of the inverse system is a non-free group. Further  $\tilde{X}_{>r}$  deformation retracts (along geodesics) onto its boundary  $\partial\tilde{X}_{>r}$ . On the other hand, one gets  $\partial\tilde{X}_{>r+1}$  from  $\partial\tilde{X}_{>r}$  by iterative use of the following procedure: replace the complement of a punctured link embedded in  $\partial\tilde{X}_{>r}$  by the respective punctured link. Our hypothesis implies that the boundary of the complement of a punctured link is simply connected and hence, by induction and use of Van Kampen we find that  $\pi_1(\partial\tilde{X}_{>r})$  is a free product of fundamental groups of links. Moreover, the non-free group  $\pi_1(\tilde{X}_{>0})$  is a factor of the free product decomposition of every  $\pi_1(\partial\tilde{X}_{>r})$ . The previous description shows also that the bonding maps correspond to forgetting a number of factors of the free product. Therefore  $\tilde{X}$  has a semi-stable end which is not pro-free. Since  $\pi_1(X)$  is qsf it follows from the main theorem that it cannot be P3R.

The second part follows along the same lines. The topology at infinity of Coxeter groups was described in [12]. Recall that the right angled Coxeter group  $W_L$  associated to the flag complex  $L$  is generated by the vertices of  $L$  and the relations correspond to commutativity of adjacent vertices and the fact that these generators are of order two. Moreover  $W_L$  acts on the Davis complex properly and cellularly. The Davis complex is a flag cubical complex and thus a CAT(0) complex. Thus  $W_L$  is qsf (see also [25]).

There is a natural filtration of the end defined by iterated neighborhoods of some vertex (see [12]). If  $L$  is a closed connected combinatorial manifold, then  $W_L$  has one semi-stable end and the inverse sequence of fundamental groups is as follows (see also Theorem 16.6.1 in [18]):

$$G \leftarrow G * G \leftarrow G * G * G \leftarrow \dots$$

where  $G = \pi_1(L)$  and each bonding map is a projection annihilating the last factor. Thus if  $L$  has dimension at least 2 and  $G$  is not free then the fundamental group at infinity is not pro-free. The main theorem implies then that  $W_L$  cannot be P3R. This settles Theorem 1.2.

**Remark 2.7.** We can infer from Remark 2.3 that the higher homotopy groups at infinity  $\pi_k^\infty(W)$  vanish for any standard model  $W$  and  $k \geq 3$ . In particular, this furnishes another practical tool for proving that a qsf finitely presented group  $G$  is not P3R. Notice however, that this is a consequence of the fact that ends are semi-stable and pro-(finitely generated free).

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