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On (co)homology locally connected spaces

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Abstract

We prove that there exists a cohomology locally connected compact metrizable space which is not homology locally connected. In the category of compact Hausdorff spaces a similar result was proved earlier by G.E. Bredon. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is well known that the concept of (co)homology local connectedness plays an important role in the isomorphism theorems of homology and cohomology theories. If a compact metrizable space X is homology locally connected with respect to the singular homology (abbreviated as *HLC*), then the Borel–Moore, the Čech, the Vietoris, the Steenrod–Sitnikov, and the singular homology groups with integer coefficients of X are all naturally isomorphic (cf. [2,7,9,10]).

The concepts of (co)homology local connectedness with respect to different homology and cohomology theories are very closely related. For example, the *HLC* property implies the cohomology local connectedness with respect to the Čech cohomology (*clc*) (cf. [2, p. 195]).

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Bredon observed that there exists a compact Hausdorff space, which is a *clc*-space, but not an *HLC*-space (cf. [2, pp. 130, 131]). Griffiths proved in [4,5] that there exists a compact metric, homology locally connected in Vietoris homology, topological space which is not *HLC*.

The purpose of the present note is to prove the following theorem which extends the Bredon result [2] and which could be considered as an alternative proof of some Griffiths' results (cf. [5, p. 477]):

Theorem 1.1. *There exists a two-dimensional compact metrizable space* $X_{\mathcal{P}}$ *such that*:
(1) $X_{\mathcal{P}}$ *is acyclic in Čech cohomology*;
(2) $X_{\mathcal{P}}$ *is a clc-space*; *and*
(3) $X_{\mathcal{P}}$ *is not an HLC-space.*

2. Preliminaries

Let $H_*^s(X)$ (respectively $\check{H}^*(X)$) denote singular homology (respectively Čech cohomology) groups of a topological space X with integer coefficients. A finite-dimensional space X is said to be *homology* (respectively *cohomology*) *locally connected*, and called an *HLC*-space (respectively, *clc*-space), if for every point $x \in X$ and every neighborhood $U_x \subset X$ of x there exists a neighborhood $V_x \subset U_x$ of x such that the inclusion-induced homomorphism $H_*^s(V_x, \{x\}) \to H_*^s(U_x, \{x\})$ (respectively $\check{H}^*(U_x, \{x\}) \to \check{H}^*(V_x, \{x\})$) is zero.

Let $g \in G$ be an arbitrary element of a group G. By the commutator length $\mathrm{cl}(g)$ of g we shall denote the *minimal* number of commutators of the group G whose product is equal to g, i.e.,

$$\mathrm{cl}(g) = \min\bigl\{n \in \mathbb{N} \mid g = [g_1, g_2] \circ \cdots \circ [g_{2n-1}, g_{2n}],\ \text{for some } g_i \in G\bigr\}.$$

If such a number does not exist then we set $\mathrm{cl}(g) = \infty$ (cf. [3, Definition 4.15]).

Since clearly, for every $g_i, h \in G$, one has that $[g_1, g_2]^{-1} = [g_2, g_1]$ and

$$h \circ [g_1, g_2] \circ h^{-1} = \bigl[h \circ g_1 \circ h^{-1}, h \circ g_2 \circ h^{-1}\bigr],$$

it follows that

$$\mathrm{cl}(g) = \infty \quad \text{if and only if} \quad g \notin G', \tag{1}$$

where $G' = [G, G]$ is the *commutator* subgroup of G.

If $\varphi : G_1 \to G_2$ is any homomorphism between groups G_1 and G_2, then for every $g \in G_1$ clearly,

$$\mathrm{cl}\bigl(\varphi(g)\bigr) \leqslant \mathrm{cl}(g). \tag{2}$$

For every path connected space X, its fundamental group $\pi_1(X)$ does not depend on the choice of the base point and

$$H_1^s(X) = \pi_1(X)/\bigl[\pi_1(X), \pi_1(X)\bigr]. \tag{3}$$

The *compact bouquet* of topological spaces X_i rel points $x_i \in X_i$, $i \in \mathbb{N}$, is defined as the quotient space of the topological sum $\bigsqcup_{i=1}^{\infty} X_i$ by the subset $\bigsqcup_{i=1}^{\infty}\{x_i\}$, equipped with the *strong* topology, i.e., a set $U \subset \bigsqcup_{i=1}^{\infty} X_i / \bigsqcup_{i=1}^{\infty}\{x_i\}$ is open if and only if:

(1) for every $i \in \mathbb{N}$, the set $\Pi^{-1}(U) \cap X_i$ is open in X_i, where

$$\Pi : \bigsqcup_{i=1}^{\infty} X_i \to \bigsqcup_{i=1}^{\infty} X_i / \bigsqcup_{i=1}^{\infty}\{x_i\}$$

is the canonical projection onto the quotient space; and

(2) if U contains the point $\bar{x} \in \bigsqcup_{i=1}^{\infty} X_i / \bigsqcup_{i=1}^{\infty}\{x_i\}$ which corresponds to the points x_i, then there exists an index $n_0 \in \mathbb{N}$ such that $\Pi(X_j) \subset U$, for all $j \geqslant n_0$.

The point $\bar{x}$ is called the *base point* of the bouquet X. We shall denote the compact bouquet of spaces X_i rel the points $x_i \in X_i$, $i \in \mathbb{N}$, by $(X, \bar{x}) = \bigvee_{i=1}^{\infty}(X_i, x_i)$.

3. Proof of Theorem 1.1

Let for every integer $i \in \mathbb{N}$, P_i be any finite acyclic 2-dimensional polyhedron with a nontrivial fundamental group—for example, take the 2-polyhedron constructed by any of the presentations (cf., e.g., [1]):

$$\{a, b \mid b^{-2} \circ a \circ b \circ a,\ b^{-3k} \circ a^{6k-1}\}$$

or

$$\{a_1, \ldots, a_r \mid a_1 \circ a_2 \circ a_1^{-1} \circ a_2^{-2},\ a_2 \circ a_3 \circ a_2^{-1} \circ a_3^{-2},\ \ldots,\ a_r \circ a_1 \circ a_r^{-1} \circ a_1^{-2}\},$$

where $r > 3$ (cf. [6]).

Let furthermore X_i denote the bouquet $P_i \vee P_i$. For every $i \in \mathbb{N}$, choose a point $x_i \in X_i$ and let X be the compact bouquet of X_i's rel the points $\{x_i\}$, i.e.,

$$(X, \bar{x}) = \bigvee_{i=1}^{\infty}(X_i, x_i).$$

By the continuity property of Čech cohomology we can conclude that $\check{H}^*(X, \bar{x}) = 0$. Indeed, X is the inverse limit of the following spectrum:

$$\left\{ \bigvee_{i=1}^{i=n} X_i \leftarrow \bigvee_{i=1}^{i=n+1} X_i \right\}_{n \in \mathbb{N}}.$$

Since the polyhedra X_i are acyclic, X is acyclic with respect to Čech cohomology, i.e.,

$$\check{H}^*(X, \bar{x}) = 0. \tag{4}$$

Since every point of $X \setminus \{\bar{x}\}$ has a closed polyhedral neighborhood and the point $\bar{x}$ has arbitrary small closed neighborhoods in X which are all homotopy equivalent to X, the space X is a *clc*-space.

Since clearly, for every $i \in \mathbb{N}$,

$$\pi_1(X_i) = \pi_1(P_i) * \pi_1(P_i),$$

there exists, by [3, Lemma 4.17], for every $i \in \mathbb{N}$, an element $q_i \in \pi_1(X_i)$ such that:

$$i < \mathrm{cl}(q_i) < \infty, \quad \text{for every } i \in \mathbb{N}. \tag{5}$$

Consider now the unit circle $S^1 \subset \mathbb{C}$ and its closed subset

$$A = \{ \mathrm{e}^{\mathrm{i}\varphi} \in S^1 \mid \varphi = 2\pi/k,\ k \in \mathbb{N} \}.$$

The quotient space S^1/A is homeomorphic to the *Hawaiian earring* H, i.e., to the compact bouquet of a countable number of circles $\{S_i^1\}_{i\in\mathbb{N}}$. Let $p : S^1 \to H$ be the canonical projection.

For every $i \in \mathbb{N}$, let $f_i : S_i^1 \to X_i$ be a representative of q_i, which maps the base point to the base point. Next, let $f : H \to X$ be a continuous mapping such that for every $i \in \mathbb{N}$, its restriction onto S_i^1 is f_i. Finally, let $g = f \circ p : S \to X$.

Suppose that $H_1^s(X) = 0$. Then by (1) and (3),

$$[g] \in \pi_1'(X, \bar{x}) \quad \text{and} \quad \mathrm{cl}([g]) < n_0 < \infty, \quad \text{for some } n_0 \in \mathbb{N}.$$

Let $p_{n_0} : X \to X_{n_0}$ be the canonical projection and $p_{n_0^\#} : \pi_1(X) \to \pi_1(X_{n_0})$ the induced homomorphism of fundamental groups. Then it follows by (2) that

$$\mathrm{cl}(p_{n_0^\#}[g]) < \mathrm{cl}([g]) < n_0.$$

However, by (5) we can conclude that

$$n_0 < \mathrm{cl}(q_{n_0}) \quad \text{and} \quad q_{n_0} = p_{n_0^\#}[g].$$

This is a contradiction. Therefore $H_1^s(X) \neq 0$. It now follows by the Universal Coefficient Theorem, that $H_s^*(X, \bar{x}) \neq 0$. Since by (4), $\check{H}^*(X, \bar{x}) = 0$, it follows that the compactum X cannot be an *HLC*-space (for *HLC*-spaces, the Čech cohomology groups are naturally isomorphic to the singular cohomology groups—cf. [2]). □

Remark 3.1. It can be shown [3, Theorem 4.14] that the group $H_1^s(X)$ of the space X constructed in the proof above contains a torsion-free divisible group of the cardinality of the continuum. On the other hand, the group $H_s^1(X)$ is trivial. In order to verify this, it suffices to show that every homomorphism from $\pi_1(X)$ to $\mathbb{Z}$ is trivial.

Indeed, by [3, Theorem A.1], $\pi_1(X)$ is the free σ-product of the groups $\pi_1(P_i \vee P_i)$. Next, by [3, Proposition 3.5 and Corollary 3.7], any homomorphism from $\pi_1(X)$ to $\mathbb{Z}$ factors through the free product of finitely many groups $\pi_1(P_i \vee P_i)$. Hence the assertion follows.

Remark 3.2. Since $H_s^*(X) \neq 0$, it follows by the Mayer–Vietoris exact sequence that none of the kth suspensions $\Sigma^k(X)$ is acyclic and hence none of them is contractible. On the other hand, it follows by the Seifert–van Kampen and the Hurevicz Theorems that all suspensions

$$\Sigma^k\left(\bigvee_{i=1}^{i=n} X_i\right)$$

are contractible spaces (cf. [8]).

Question 3.3. *Let G be one of the following two types of groups*:

$$\{a, b \mid b^{-2} \circ a \circ b \circ a,\ b^{-3k} \circ a^{6k-1}\}$$

or

$$\{a_1, \ldots, a_r \mid a_1 \circ a_2 \circ a_1^{-1} \circ a_2^{-2},\ a_2 \circ a_3 \circ a_2^{-1} \circ a_3^{-2},\ \ldots,\ a_r \circ a_1 \circ a_r^{-1} \circ a_1^{-2}\},$$

where $r > 3$ (cf. [6]*). Does it then follow that*: $\sup\{\mathrm{cl}(g)\colon\ g \in G\} = \infty$?

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