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DIMENSION OF PRODUCTS WITH CONTINUA

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ABSTRACT. We construct a subset $W \subset \mathbb{R}^3$ and a continuum Y with the dimension of the product $\dim(W \times Y) = \dim W = 2$. This solves negatively a long standing problem in dimension theory.

0. INTRODUCTION

It has been known ever since the 1930's that the logarithmic law for dimension, $\dim(X \times Y) = \dim X + \dim Y$, fails to hold for arbitrary compact metric spaces. The first known counterexamples are due to L. S. Pontryagin (see e.g. [8]). His compacta, now called *Pontryagin surfaces*, lie in $\mathbb{R}^4$ and are 2-dimensional whereas the dimension of their product is equal to three.

The ingredients of Pontryagin's construction come from algebraic (rather than point-set) topology. Note that it follows from a classical theorem of P. S. Aleksandrov [8] that there are no such counterexamples in $\mathbb{R}^3$.

It is well known that the product inequality $\dim(X \times Y) \leq \dim X + \dim Y$ always holds. Also, for compact spaces X and Y of dimension ≥ 1 it is also known that $\dim(X \times Y) \geq \dim X + 1$. On the other hand, as it was shown in [2], for any fixed $n = \dim X$ and $m = \dim Y$ this inequality cannot be improved any further.

Approximately 40 years ago, K. Morita [10] proved that for every X (not necessarily compact), multiplication of X by the

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interval I increases dimension by one, $\dim(X \times I) \geq \dim X + 1$. A natural question arose whether the inequality $\dim(X \times Y) \geq \dim X + 1$ holds for an arbitrary compactum Y with $\dim Y \geq 1$ (see [8], [11; Problem (42.5)]).

The purpose of this paper is to give a negative answer to this question. Namely, we construct a 2-dimensional subset $W \subset \mathbb{R}^3$ and a 1-dimensional metric continuum Y such that $\dim(W \times Y) = 2$. Although this solves a problem in general topology, this paper, like in Pontryagin's case [8], belongs essentially to algebraic topology.

1. Supersolenoids

Every sequence of numbers $\{m_i > 1\}_{i \in \mathbb{N}}$ defines a *solenoid* as the limit space of the inverse system $\{S^1; p_i^{i+1}\}_{i \in \mathbb{N}}$ where each projection p_i^{i+1} is an m_i times winding of the circle S^1 onto itself. When $m_i = p$ for all i, the solenoid is called the p-adic solenoid and it's denoted by Σ_p.

Let $(C, c^\pm)$ be a continuum with a fixed pair of points $c^+, c^- \in C$. Attach an arc I to C at the points $c^\pm$ and denote such a continuum by $\bar{C}$. The exact sequence of the pair $(\bar{C}, C)$ produces the short exact sequence

$$0 \to \mathbb{Z} \to \check{H}^1(\bar{C}) \to \check{H}^1(C) \to 0 \qquad (*)$$

for the Čech cohomology with integer coefficients. Note that the pair $(C, \{c^+, c^-\})$ produces exactly the same sequence. The problem of splitting this exact sequence has a direct relation to the Generalized homotopy problem and was considered in [1], [12]. In the case when C is a solenoid we give the following splitting criterion: *Let $(C, c^\pm)$ be a solenoid. Then the sequence $(*)$ can be split if and only if c^+ and c^- can be connected by a path in C.* For the p-adic solenoid Σ_p this criterion claims, in algebraic terms, that $c^\pm$ generate a splittable sequence $(*)$ if and only if the pair $c^\pm$ is homotopic to a pair $a^\pm$ with $a^+ - a^- \in \mathbb{Z} \subset \mathbb{A}_p \subset \Sigma_p$. Here $\mathbb{A}_p$ denotes the group of p- adic integers and $\subset$ means 'is a subgroup of'. Note that every pair $c^\pm$ in Σ_p is homotopic to a pair in $a^\pm \in \mathbb{A}_p$.

Let $\mathbb{Z}_{(p)}$ denote the localization of $\mathbb{Z}$ in p. Then there exist the inclusions $\mathbb{Z} \subset \mathbb{Z}_{(p)} \subset \mathbb{A}_p$.

Proposition 1.1. Let C be a p-adic solenoid. Then there exist $c^{\pm} \in C$ such that $Hom(\pi, \mathbb{Z}) = 0$, where $\pi = \check{H}^1(\bar{C})$.

Proof: We will consider the Steenrod-Sitnikov homology. Whenever we omit the coefficient group we mean the integers. By [9] $\mathrm{Hom}(\pi, \mathbb{Z}) = H_1(\bar{C})$. Since $\bar{C}$ is one-dimensional, the Steenrod homology $H_1(\bar{C})$ coincides with the Čech homology $\check{H}_1(\bar{C})$ [13]. So it suffices to prove that the one-dimensional Čech homology group of $\bar{C}$ is trivial.

We do that here for any $c^{\pm}$ with $c^{+} - c^{-} \in \mathbb{A}_p - \mathbb{Z}_{(p)}$. Actually, we can prove a criterion which claims that a pair $c^{\pm}$ produces the nontrivial $\mathrm{Hom}(\pi, \mathbb{Z})$ if and only if it is homotopic to a pair $a^{\pm}$ such that $a^{+} - a^{-} \in \mathbb{Z}_{(p)}$.

Since $\bar{C} = \varprojlim\{S^1 \cup I\}$, where each bonding map sends S^1 onto S^1, winding p times around, and sends I onto I homeomorphically, it follows that $\check{H}_1(\bar{C}) = \varprojlim\{H_1(S^1 \cup I), \varphi_i^{i+1}\}_{i \in \mathbb{N}}$. We are going to describe the bonding maps $\varphi_i^{i+1} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. Note that $\mathbb{A}_p$ is identified with a fiber of the projection $\Sigma_p \to S^1$. Without loss of generality, we may assume that $c^{-} = 0$. Let c^{+} be represented as an element of $\mathbb{A}_p$ in the following way: $c^{+} = n_0 + n_1 p + \cdots + n_k p^k + \cdots$ [7]. To choose a basis in $H_1(S^1 \cup I)$, fix an orientation on the circle S^1 and on the interval I and consider this oriented circle as the first basis element, and the cycle generated by the interval I and a part of the circle with proper orientation as the second basis element. Then a homomorphism φ_i^{i+1} is defined by the matrix $A_i = \begin{pmatrix} p & n_i \\ 0 & 1 \end{pmatrix}$.

Claim. If $c^{+} \notin \mathbb{Z}_{(p)}$ then $\varprojlim\{\mathbb{Z} \oplus \mathbb{Z}; A_i\} = 0$.

Indeed, we may consider $A_i^{-1} = \begin{pmatrix} p^{-1} & -n_i p^{-1} \\ 0 & 1 \end{pmatrix}$ over $\mathbb{Q}$. Let c_k denote the truncated c^{+}: $c_k = n_0 + n_1 p + \cdots + n_k p^k$.

Then

$$p^k A_k^{-1} \circ \cdots \circ A_2^{-1} \circ A_1^{-1} = \begin{pmatrix} 1 & -c_k \\ 0 & p^k \end{pmatrix}.$$

First, show that the projection of the limit group on the first level is trivial. Choose an arbitrary $(n, m) \in \mathbb{Z} \oplus \mathbb{Z}$. If there is an element in the limit group which is projected to (n, m) then for each i, the number $n - c_k m$ is divisible by p^k. Let us consider a p-adic number $\beta = n - c^+ m$. Then the p-adic norm of β is zero hence $\beta = 0$ and $mc^+ \in \mathbb{Z}$. Therefore $c^+ = \frac{n}{m} \in \mathbb{Q} \cap \mathbb{A}_p = \mathbb{Z}_{(p)}$ so we get a contradiction.

Thus, by the above argument we can prove that the projection on the second level is trivial, and so on. This proves the claim and also the proposition. $\square$

Proposition 1.2. In the p-adic solenoid C there are points $c^\pm$ for which the inclusion-induced homomorphism $\tilde{H}_0(\{c^-, c^+\}) \to \tilde{H}_0(C)$ is a monomorphism.

Proof: Consider the exact sequence of the pair $(C, c^\pm)$ for the points $c^\pm$ from Proposition 1.1. It suffices to show that $H_1(C/c^\pm) = 0$. This was proved above. $\square$

For convenience, instead of the triple $(C, c^\pm)$ we shall consider sometimes a continuum *with hands*, i.e. a continuum C with two arcs $[b^-, c^-]$ and $[c^+, b^+]$ attached to the marked points. We denote a continuum with hands obtained from $(C, c^\pm)$ by $(C', b^\pm)$.

Definition. Let $(C', b^\pm)$ be a continuum with hands. A compactum X with the property

$(**)$for every closed subset $A \subset X$ and every continuous map $\varphi : A \to \{b^-, b^+\}$ this is an extension $\psi : X \to C'$

is called a $(C, c^\pm)$-compactum. We call X a $(C, c^\pm)$-continuum if it is in addition a continuum. (Note that hands are inessential here.) A $(C, c^\pm)$-continuum for solenoid C we shall call a *supersolenoid*.

Proposition 1.3. Let X be a $(C, c^{\pm})$-compactum and let $A \subset X$ be a closed subset. Then
(a) A is a $(C, c^{\pm})$-compactum; and
(b) X/A is a $(C, c^{\pm})$-compactum.

The proof easily follows from the definition.

Proposition 1.4. Suppose that X and Y are $(C, c^{\pm})$-compacta and that $\dim(X \cap Y) = 0$. Then $X \cup Y$ is a $(C, c^{\pm})$-compactum.

Proof: For arbitrary $\varphi : A \to \{c^{\pm}\}$ first extend φ over $X \cap Y$ to get $\psi : A \cup (X \cap Y) \to \{c^{\pm}\}$. Then extend ψ separately over X and over Y. $\square$

Proposition 1.5. Let $\pi = \check{H}^1(\bar{C})$. Then for every $(C, c^{\pm})$-compactum X there exists an epimorphism $\bigoplus\limits_i \pi \to \check{H}^1(X)$.

Proof: There is a natural projection $\omega : \bar{C} \to S^1$ with one non-trivial preimage. Since X has the property $(**)$ it follows that for every map $f : X \to S^1$ there is a homotopy lifting $f' : X \to \bar{C}$. Let $\{f_i\}_{i \in \mathbb{N}}$ be a countable family of maps to the circle, representing all cohomologies of X, and let $\{f_i'\}_{i \in \mathbb{N}}$ be a family of liftings. Consider the diagonal product $\Delta f_i' : X \to \prod\limits_i \bar{C}$. It induces an epimorphism for the 1-dimensional cohomologies. It remains to note that $\check{H}^1(\prod\limits_i C) = \bigoplus\limits_i \pi$. $\square$

Theorem 1.6. 1) For every triple $(C, c^{\pm})$ there exists a $(C, c^{\pm})$-continuum.

2) Suppose that a cohomology theory $\tilde{h}^*$ is trivial on a one-dimensional continuum C. Then for every n, there exists an n-dimensional $(C, c^{\pm})$-continuum.

Proof: We prove 2) so that the construction for 2) is valid also for 1).

We construct an n-dimensional $(C, c^{\pm})$-continuum X as the limit space of an inverse system $\{X_i, p_i^{i+1}\}_{i \in \mathbb{N}}$. The system will be constructed by induction.

Define $X_0 \cong S^n$. Note that $h^*(X_0)$ is a nontrivial group.

For each i, we define a finite covering $\mathcal{U}_i$ of a compact space X_i by closed sets A of diameter $\leq 1/i$ and moreover with diameters of projections $p_k^i(A)$ less than $1/i$, for all $k < i$. Denote by $\mathcal{B}_i$ the set of all disjoint pairs (B^-, B^+) consisting of the unions of elements of $\mathcal{U}_i$. For every element $\beta = (B^-, B^+) \in \mathcal{B}_i$ fix a map $\varphi_\beta : B^- \cup B^+ \to \{b^-, b^+\}$, by setting $\varphi_\beta(B^-) = b^-$ and $\varphi_\beta(B^+) = b^+$.

Now we can describe a step of the induction from k to $k+1$. We suppose the set $\bigcup\limits_{i=0}^{k} \mathcal{B}_i$ has a numeration: $\{\beta_1, \beta_2, ..., \beta_m\}$. Choose $\beta = \beta_k$. We have $\beta = (B^-, B^+) \in \mathcal{B}_i$ for some $i \leq k$. The map φ_β produces a map $\psi : (p_i^k)^{-1}(B^- \cup B^+) \to \{b^\pm\}$.

Let $\pi : C' \to [-1, 1]$ be a projection which sends $[b^-, c^-]$ onto $[-1, 0]$ and $[c^+, b^+]$ onto $[0, 1]$ and C in 0. There is an extension $\bar{\psi}$ of the composition map $\pi \circ \psi$ with $\dim(\bar{\psi}^{-1}(0)) \leq n-1$ (see for instance [5]). Define X_{k+1} as the pull-back of the following diagram:

$$\begin{tikzcd}
X_{k+1} \arrow[r, dashed, "\bar{\psi}\,'"'] \arrow[d, dashed] & C' \arrow[d] \\
X_k \arrow[r, "\bar{\psi}"'] & {[-1,1]}
\end{tikzcd}$$

The projection p_k^{k+1} is defined as a projection of the pull-back onto X_k. Note that:

(a) A homomorphism $(p_k^{k+1})^*$ is an isomorphism for h^* by virtue of the Vietoris-Begle theorem.

(b) Dimension of X_{k+1} is $\leq n$ because X_{k+1} consists of an open subset which is homeomorphic to a subset of X_k and a closed set $\bar{\psi}^{-1}(0) \times C$ which is n-dimensional.

(c) The map φ_β has an extension as a map to C' on the $k+1$ level. Indeed, $\psi' = \varphi_\beta \circ p_i^{k+1}$ has an extension $\bar{\psi}'$.

Choose a covering $\mathcal{U}_{k+1}$ and define $\mathcal{B}_{k+1}$ and add it to the union $\bigcup\limits_{i\leq k}\mathcal{B}_i$ with the corresponding numbering.

Properties a) and b) will imply the n-dimensionality of the limit space. Since all X_i are continua the limit space is also a continuum.

The property c) and the construction guarantee the condition $(**)$. Indeed, if $\varphi : A \to \{b^{\pm}\}$ is a map, there exists $\beta = (B^-, B^+) \in \bigcup\limits_{i=0}^{\infty}\mathcal{B}_i$ such that $(p_i^{\infty})^{-1}(B^- \cup B^+) \supset A$ and $\varphi_\beta \circ p_i^{\infty}{}_{|A} = \varphi$. By the construction there is an extension in C' of φ_β onto some level $k \geq i$. Hence φ has an extension. □

Corollary 1.7. . For any family of primes ℓ and for every pair $x^{\pm} \in \Sigma_\ell$ there exist the ℓ-adic supersolenoid of arbitrary dimension $n > 0$.

Proof: Let $p \in \ell$. Then $\tilde{\check{H}}^*(\Sigma_\ell; \mathbb{Z}_p) = 0$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. □

2. Connectedness with respect to a group

We call a space Y connected with respect to an abelian group G if its reduced Steenrod-Sitnikov 0-dimensional homology group with the coefficients in G is trivial. For example, Proposition 1.2 implies that a p-adic solenoid is disconnected with respect to the integers. This is also true for the corresponding supersolenoid.

Proposition 2.1. . Suppose that the inclusion $c^{\pm} \subset C$ induces a monomorphism of homology groups. Then for any $(C, c^{\pm})$-compactum X and for arbitrary pair $x^{\pm} \subset X$, the inclusion induces a monomorphism.

Proof: Extend the map $\{x^{\pm}\} \to \{c^{\pm}\}$ to a map $X \to C$. Then our homomorphism is a left divisor of a monomorphism. □

Proposition 2.2. Let a one-dimensional continuum X be the limit space of an inverse system $\{X_i, r_i^{i+1}\}_{i\in\mathbb{N}}$, all projection of which are retractions. Then $\varprojlim_i{}^1\{\mathrm{Hom}(\check{H}^1(X_i),\pi)\} = 0$ for an arbitrary group π.

Proof: Let β_i be a left inverse to $(r_i^{i+1})^*$, i.e. $\beta_i \circ (r_i^{i+1})^* = \mathrm{id}$. Show that every homomorphism $h_i : \mathrm{Hom}(\check{H}^1(X_{i+1}),\pi) \to \mathrm{Hom}(\check{H}^1(X_i),\pi)$ is an epimorphism. Let $f : \check{H}^1(X_i) \to \pi$ be an arbitrary homomorphism. Note that $h_i(f\circ\beta_i) = (f\circ\beta_i)\circ(r_i^{i+1})^* = f\circ(\beta_i\circ(r_i^{i+1})^*) = f$. □

Proposition 2.3. Let $(X,D) = \varprojlim\{(X_i,D_i); r_i^{i+1}\}$ where X is a 1-dimensional continuum, $D_i \cong D$ are two-point sets and r_i^{i+1} are retractions. Suppose that for all i, the boundary homomorphism $H_1(X_i/D_i;\pi) \to H_0(D_i,\pi)$ is an epimorphism. Then the boundary homomorphism $\partial : H_1(X/D;\pi) \to H_0(D;\pi)$ is also an epimorphism.

Proof: First, we show that the limit homomorphism

$$\varprojlim H_1(X_i/D_i;\pi) \to \varprojlim H_0(D_i;\pi)$$

is an epimorphism. We have the functor $\varprojlim$ applied to the short exact sequence:

$$0 \to H_1(X_i;\pi) \to H_1(X_i/D_i;\pi) \to H_0(D_i;\pi) \to 0$$

hence by [9] we have an exact sequence

$$\varprojlim H_1(X_i/D_i;\pi) \to \varprojlim H_0(D_i;\pi) \to \varprojlim{}^1 H_1(X_i;\pi)\,.$$

Since X_i are one-dimensional, $H_1(X_i;\pi) = \mathrm{Hom}(\check{H}^1(X_i),\pi)$. Apply Proposition 2.2 to obtain the required epimorphism. Since X is 1-dimensional, in dimension one Steenrod homologies coincide with the Čech homologies and hence $\varprojlim H_1(X_i/D_i;\pi) = H_1(X/D;\pi)$. It is easy to check that $H_0(D;\pi) = \varprojlim H_0(D_i;\pi)$ and our epimorphism coincides with ∂. □

Lemma 2.4. Let X be a $(C, c^{\pm})$-compactum and suppose that $\dim C = 1$. Then the inclusion-induced homomorphism $H_0(c^{\pm}; \check{H}^1(X)) \to H_0(C; \check{H}^1(X))$ is trivial (the points c^- and c^+ are $\check{H}^1(X)$-connected in C.

Proof: It is sufficient to show that the boundary homomorphism is an epimorphism. The boundary homomorphism is generated by the functor Hom(, $\check{H}^1(X)$) from the co-boundary homomorphism $\delta : \tilde{\check{H}}^0(\{c^{\pm}\}) \to \check{H}^1(C/c^{\pm})$. Choose an arbitrary homomorphism $f : \check{H}^0(\{c^{\pm}\}) \to \check{H}^1(X)$ and consider the extension problem. This extension problem diagram

$$\begin{tikzcd} \bar{C} \arrow[r] & S^1 \\ & X \arrow[ul] \arrow[u, "g"'] \end{tikzcd}$$

can be obtained from the diagram by applying cohomologies $\check{H}^1$. Here g represents $f(1)$ and the horizontal arrow is the collapsing of C in $\bar{C}$ to the point (see §1).

Since X is a $(C, c^{\pm})$-compactum there exists a homotopy lifting g' of g. $\square$

Proposition 2.5. For any one-dimensional compactum X there is a map of the Cantor discontinuum $f : K \to X$ which induces an epimorphism $f_* : H_0(K; G) \to H_0(X; G)$ for every group G.

Proof: We define a sequence of finite tilings $\mathcal{H}_i = \{H_i^j\}$ of X by closed subsets with nonempty interiors such that
a) the diameter of H_i^j is less than $1/i$;
2) $\dim(H_i^j \cap H_i^k) \leq 0$ for all i, j, k;
3) $\mathcal{H}_{i+1}$ is a refinement of $\mathcal{H}_i$; and
4) each $\mathcal{H}_i$ has an one-dimensional nerve.

This sequence defines an inverse system $\{X_i, p_i^{i+1}\}_{i \in \mathbb{N}}$ with $X_1 \cong X$ and with the limit space homeomorphic to the Cantor set K. Denote by $E_i = \bigcup\limits_{j,k}(H_i^j \cap H_i^k)$. Fix embeddings $X_i \subset \mathbb{R}^3$

and $X_{i+1} \subset \mathbb{R}^3$ and consider a graph of p_i^{i+1} in $\mathbb{R}^3 \times \mathbb{R}^3$. For every $x \in E_i$ we join the points in $(p_i^{i+1})^{-1}(x)$ by a straight interval in $\{x\} \times \mathbb{R}^3$. The resulting space we shall denote by $\bar{X}_{i+1}$. Since the projection of $\bar{X}_{i+1}$ on X_i is a cell-like map, the inclusion-induced homomorphism $H_0(X_{i+1}; G) \to H_0(\bar{X}_{i+1}; G)$ coincides with the bonding homeomorphism $(p_i^{i+1})_*$.

In order to prove that every bonding homomorphism is an epimorphism it is sufficient to show that $H_0(\bar{X}_i, X_i; G) = 0$ for every i. Note that $H_0(\bar{X}_i, X_i; G)$ $=\mathrm{Ext}(\check{H}^1(\bar{X}_i, X_i), G)$. This Ext group is trivial because of $\check{H}^1(\bar{X}_i, X_i) = \check{H}^1(S^1 \times E_{i-1}, \{pt\} \times E_{i-1}) = \check{H}^1(S^1 \times E_{i-1}) = \check{H}^0(E_{i-1}) = \oplus\mathbb{Z}$ is a free abelian group. $\square$

Proposition 2.6. Let X be a separable metrizable space and G be an abelian group. Suppose that X is G-connected and locally G-connected, i.e. for every two-points subset $D \subset X$ the inclusion-induced homomorphism $\tilde{H}_0(D; G) \to \tilde{H}_0(X; G)$ is trivial and if diameter of D is small enough then the inclusion-induced homomorphism is trivial in a small neighbourhood. Then $\tilde{H}_0(X; G) = 0$.

Proof: We show that for every compact $Y \subset X$, the inclusion-induced homomorphism i_* is trivial. Choose an arbitrary $\alpha \in H_0(Y; G)$. By Proposition 2.5, there exist a map $f : K \to Y$ of the Cantor set and an element $\beta \in H_0(K; G)$ such that $f_*(\beta) = \alpha$. There are maps $p_n : K \to D^n$ and $q_n : D^n \to K$ such that $\lim q_n \circ p_n = \mathrm{id}_K$. Here D^n is a 2^n-point set. Since X is locally G-connected, any two close enough maps of K in Y send a given element of the 0-dimensional homology of K into the same element of $H_0(X; G)$. Therefore for some n, we have that $i_*(\alpha) = i_* f_*(\beta) = i_* f_*(q_n)_*(p_n)_*(\beta)$. The right hand side of this equality is trivial because the cycle $(p_n)_*(\beta)$ has a finite support. $\square$

3. Continua nets and their complements in $\mathbb{R}^3$.

Let $\mathbb{N}^3 \subset \mathbb{R}^3$ be the integer lattice and let $\mathcal{N}_k = (\frac{1}{2^k}\mathbb{N})^3$ denote the corresponding subdivision of $\mathbb{N}^3$. Two points in $\mathcal{N}_k$ are called *neighbor points* if they agree in two coordinates and they differ in the third by $\frac{1}{2^k}$. Let $(X, x^\pm)$ be a one-dimensional continuum. We construct a 1-dimensional net T_k by attaching to every neighbor points a copy of X at the points x^- and x^+.

Proposition 3.1. For every 1-dimensional continuum $(X, x^\pm)$ there exists a sequence of nets T_k with the following properties:

(a) all examples X in T_k intersect each other only in the vertices of $\mathcal{N}_k$ at their marked points;
(b) for every $n > k$, $T_k \cap T_n = \mathcal{N}_k$; and
(c) every example X of T_k has diameter $\leq \frac{1}{2^k}$.

The proof easily follows by general position property in $\mathbb{R}^3$. $\square$

Denote by T the union of all T_k.

Proposition 3.2. Let $(C, c^\pm)$ be a 1-dimensional continuum with $\pi = \check{H}^1(\bar{C})$ such that $Hom(\pi, \mathbb{Z}) = 0$ and let the net T be constructed by means of $(C, c^\pm)$-continuum $(X, x^\pm)$. Then for any compactum $Y \subset T$ and for any two-point subset $D \subset Y$ there exists a proper subcompactum $Y' \subset Y$, $D \subset Y'$, such that the inclusion-induced homomorphism $H_1(Y'/D) \to H_1(Y/D)$ is an epimorphism.

Proof: It follows by the Baire Category theorem that there exists an open set $V \subset Y - D$ such that $V \subset T_k$ for some k. Define $Y' = Y - V$ and consider the exact sequence of the pair $(Y/D, Y'/D)$:

$$H_2(V) \to H_1(Y'/D) \to H_1(Y/D) \to H_1(V)\,.$$

First, note that $H_2(V) = 0$ by dimension reasons, and $H_1(V) = \mathrm{Hom}(H^1_c(V), \mathbb{Z}) = \mathrm{Hom}(\check{H}^1(Z), \mathbb{Z})$, where $Z = \mathrm{Cl}V/\partial V$. By Propositions 1.3 and 1.4, Z is a $(C, c^\pm)$-compactum. By Proposition 1.5, there is an epimorphism $\bigoplus_i \pi \to \check{H}^1(Z)$. The functor

Hom gives a monomorphism $\mathrm{Hom}(\check{H}^1(Z), \mathbb{Z}) \to \mathrm{Hom}(\oplus_i \pi, \mathbb{Z})$. The target is zero by the assumption, therefore $H_1(V) = 0$. $\square$

Lemma 3.3. Let T be as in Proposition 3.2. Then for every open subset $U \subset T$, $H_0(U) \neq 0$.

Proof: Suppose to the contrary that $H_0(U) = 0$. Let $D \subset U$ be a two-points set. Then there is a compactum $Y \supset D$ such that the inclusion-induced homomorphism $H_0(D) \to H_0(Y)$ is trivial. This means that $H_1(Y/D) \neq 0$. By the transfinite induction construct a decreasing sequence of compacta $Y_1 \supset Y_2 \supset \cdots \supset Y_\alpha \supset Y_{\alpha+1} \cdots$ such that
a) $D \subset Y_\alpha$ for every α;
b) $Y_1 = Y$; and
3) the inclusion $Y_\alpha \subset Y$ induces an isomorphism $H_1(Y_\alpha/D) \to H_1(Y/D)$.

We can do every non-limit step of the induction due to Proposition 3.2. Let us consider a limit step, $\alpha = \lim\limits_{\beta<\alpha} \beta$. We define in that case that $Y_\alpha = \bigcap\limits_\beta Y_\beta$. Since Y_α/D is one-dimensional, $H_1(Y_\alpha/D) = \varprojlim H_1(Y_\beta/D)$ and the property 3) holds. Properties 1)–2) hold by trivial reasons. Any decreasing sequence of distinct closed subsets of a metric compact space can not be more than countable. But we have constructed such a sequence of the length ω_1. This contradiction completes the proof. $\square$

By the definition, a paracompact space Y has the cohomological dimension $\leq n$ with respect to abelian group G (we write $\text{c-dim}_G(Y) \leq n$) if for every closed subset $A \subset Y$ and every map $\varphi : A \to K(G, n)$ to the Eilenberg-MacLane complex $K(G, n)$ has an extension. It is well known (see e.g. [8]) that this definition is equivalent to the property that $H^{n+1}(Y, A; G) = 0$, for every closed subset $A \subset Y$ (here we consider the Alexander-Spanier cohomologies).

Let us consider the net T as in Proposition 3.2. Such a net exists by virtue of Propositions 1.1 and 3.1. Additionally, we may assume the property of $(C, c^{\pm})$ from Proposition 1.2. Denote by $W(C, c^{\pm})$ the complement of T in $\mathbb{R}^3$.

Theorem 3.4. Under the above conditions the space $W(C, c^{\pm})$ is two-dimensional.

Proof: Let B be a 3-dimensional ball in $\mathbb{R}^3$. Sitnikov duality implies $H_0(\mathrm{Int}B \cap T) = H^2(W(C, c^{\pm}) \cap B, (C, c^{\pm}) \cap \partial B)$. By Lemma 3.3, this group is nontrivial, hence the integral cohomological dimension of $W(C, c^{\pm})$ is greater than or equal to 2. It is easy to see that it is less than 3. $\square$

Definition [8]. A system of open subsets $\{U_\alpha\}$ is called a *big basis for* X if for every closed subset $A \subset X$ and for every neighborhood $V \supset A$ there exists a locally finite covering of A by elements of $\{U_\alpha\}$ lying in V.

Example [8]. For $X \subset \mathbb{R}^n$ the set $U(a, r) = \{x : d(x, a) < r\} \cap X$ is a big basis for X.

Lemma 3.5. *[8]* Suppose that X is a paracompact space and $\{U_\alpha\}$ is a big basis for X. Assume that $H^{n+1}(X, X - U_\alpha; G) = 0$ for all α. Then $\text{c-dim}_G X \leq n$.

Theorem 3.6. Let $W(C, c^{\pm})$ be as above and suppose that the net T is constructed by means of $(C, c^{\pm})$-continuum $(X, x^{\pm})$. Then for every $(X, x^{\pm})$-compactum Y, $\text{c-dim}_{\check{H}^1(Y)} W(C, c^{\pm}) = 1$.

Proof: Consider a big basis for $W(C, c^{\pm})$ from the above example. For every regular open ball $V \subset \mathbb{R}^3$ we prove that $V \cap T$ is connected and locally connected with respect to the coefficient group $\check{H}^1(Y)$. We prove the connectedness of $V \cap T$. For every two-point set $D = \{a, b\} \subset V \cap T$ there are two sequences $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$ converging to a and b respectively, with the following properties:

(1) a_1 and b_1 are neighbor points for some $\mathcal{N}_k$ and the continuum X, joining a and b, lies in V; and

(2) for every i, points a_i and a_{i+1} (also b_i and b_{i+1}) are neighbor points for some $\mathcal{N}_k$ and the corresponding example of continuum X joining those points lies in V.

The union of all those continua X defines a compactum Z. We may assume that Z consists of an infinite chain of continua, homeomorphic to X, between a and b. Hence the continuum Z can be represented as the limit space of an inverse system of continua Z_i, consisting of the parts of that chain from a_i to b_i. The bonding maps in this system are retractions defined by collapsing the ends to the end points. Lemma 2.4 implies that for each space Z_i, the inclusion $D_i = \{a_i, b_i\} \subset Z_i$ induces trivial homomorphism of the 0-dimensional homology groups with $\check{H}^1(Y)$ as coefficients. Apply Proposition 2.3 to obtain that the inclusion $D \subset Z$ induces a trivial homomorphism in the dimension 0.

By Proposition 2.6, $\tilde{H}_0(V \cap T; \check{H}^1(Y)) = 0$. The Sitnikov duality for the n- sphere S^n says that $H^q(X;G) \cong \tilde{H}^c_{n-q-1}(S^n - X;G)$, for every nonempty subset $X \subset S^n$ (c.f. [9; Corollary (11.21)]). Let us consider the quotient space $V/\partial V \simeq S^3$ and let us apply the Sitnikov duality to $U/\partial U \subset V/\partial V$, where $U = V \cap W$ is an element of our big basis for $W = W(C, c^{\pm})$. We obtain that

$$\begin{aligned} H^2(U/\partial U; \check{H}^1(Y)) &\cong \tilde{H}_0(V - W; \check{H}^1(Y)) \\ &\cong \tilde{H}_0(V \cap T; \check{H}^1(Y)) = 0 \end{aligned}$$

Note also that $H^2(W, W-U; \check{H}^1(Y)) \cong H^2(U/\partial U; \check{H}^1(Y))$. $\square$

4. The main result.

The following fact we leave without a proof because it is an elementary exercise in general topology.

Lemma 4.1. Let $\{U_\alpha\}$ be a big basis for a paracompact space W and let $\{V_\beta\}$ be a basis for compact space Y. Then $\{U_\alpha \times V_\beta\}$ forms a big basis for the product $W \times Y$.

Theorem 4.2. There exist a 2-dimensional subset $W \subset \mathbb{R}^3$ and a 1-dimensional continuum Y with $\dim(W \times Y) = 2$.

Proof: We consider $W = W(C, c^{\pm})$, where $C \cong \Sigma_p$ and $c^{\pm}$ are as in Proposition 1.2 and the net T is constructed by using a $(C, c^{\pm}$-continuum $(X, x^{\pm})$. Let Y be a 1-dimensional $(X, x^{\pm})$-continuum. For every open subset $V \subset X$, the space $\mathrm{Cl}(V)/\partial V$ is a $(X, x^{\pm})$-compactum by virtue of Proposition 1.3. By Lemma 4.1 and Lemma 3.5, it suffices to show that $H^3(W \times Y, W \times Y - U \times V)) = 0$ for every element U of big basis for W, described in §3, and every open set $V \subset Y$.

Note that

$$\begin{aligned} H^3(W \times Y,\ \ & W \times Y - U \times V) \\ &= H^3((W, W-U) \times (Y, Y-V)) \\ &= H^2((W, W-U); \check{H}^1(Y, Y-V)) \\ &= H^2((W, W-U); \check{H}^1(\mathrm{Cl}(V)/\partial V)) = 0 \end{aligned}$$

The last equality is due to Theorem 3.6.

The space W is 2-dimensional according to Theorem 3.4. □

Lemma 4.3. Let Y be a continuum and $D \subset Y$ a two-point subset. Then for every prime p, the localization $\mathbb{Z}_{(p)}$ belongs to the Bockstein family $\sigma(\check{H}^1(Y/D))$.

Proof: By the definition of the Bockstein family it suffices to show that $\mathbb{Z}_{p^\infty} \otimes \check{H}^1(Y/D) \neq 0$ [4]. Since $\mathrm{Tor}\check{H}^1(Y) = 0$, the multiplication of the short exact sequence $0 \to \mathbb{Z} \to \check{H}^1(Y/D) \to \check{H}^1(Y) \to 0$ by $\mathbb{Z}_{p^\infty}$ produces a monomorphism $\mathbb{Z} \otimes \mathbb{Z}_{p^\infty} \to \check{H}^1(Y/D)$. □

Theorem 4.4. There exists a space W such that $\dim_{\mathbb{Z}} W = 2$ and $\sup\{\dim_H W; h \in \sigma(\mathbb{Z})\} = 1$. In particular, the Bockstein theorem asserting that $\text{c-dim}_G X = \sup\{\text{c-dim}_H X; H \in \sigma(G)\}$ does not generalize to the class of noncompact spaces.

Proof: Suppose that Bockstein theorem were correct. Consider a space W from Theorem 4.2. Then by Lemma 4.3 and Theorem 3.6, it would follow that $\text{c-dim}_{\mathbb{Z}_{(p)}} W \leq 1$. Since $\sigma(\mathbb{Z}) = \{\mathbb{Z}_{(p)}; p \text{ runs over all primes}\}$, Bockstein theorem would then imply that $\text{c-dim}_{\mathbb{Z}} W \leq 1$ which would contradict Theorem 3.4. $\square$

Remark. It is possible to construct such a space W as above with the dimensions $= 1$ with respect to all localization $\mathbb{Z}_{(p)}$. This solves a problem from [8].

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