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On approximation and embedding problems for cohomological dimension

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Abstract

We prove that the following fundamental problems of geometric dimension theory are equivalent: (1) *The Mapping Intersection Problem.* Given compacta X and Y such that $\dim(X \times Y) < n$, can every pair of maps $f: X \to \mathbb{R}^n$ and $g: Y \to \mathbb{R}^n$ be approximated arbitrarily closely by maps $f': X \to \mathbb{R}^n$ and $g': Y \to \mathbb{R}^n$ such that $f'(X) \cap g'(Y) = \emptyset$? (2) *The Cohomological Dimension Approximation Problem.* Given a compactum X of dimension $\leqslant n-2$ and an Abelian group G, can every map $f: X \to \mathbb{R}^n$ be approximated arbitrarily closely by a map $f': X \to \mathbb{R}^n$ such that $\dim_G f'(X) = \dim_G X$? (3) *The Dimension Type Embedding Problem.* Given a compactum X of dimension $\leqslant n-2$, does there exist a compactum $X' \subset \mathbb{R}^n$ of the same *dimension type*, i.e., $\dim_G X' = \dim_G X$, for every Abelian group G? By using this equivalence, we obtain the main result of the paper: *Let X and Y be compacta such that* $\dim(X \times Y) < n$ *and* (codim X)(codim Y) $\geqslant n$. *Then every pair of maps $f: X \to \mathbb{R}^n$ and $g: Y \to \mathbb{R}^n$ can be approximated arbitrarily closely by maps $f': X \to \mathbb{R}^n$ and $g': Y \to \mathbb{R}^n$ such that $f'(X) \cap g'(Y) = \emptyset$, provided that* $\dim X \leqslant n-3$ *and* $\dim Y \leqslant n-3$.

Key words: Approximations of mappings of compacta; Embeddings of compacta in Euclidean spaces; Bockstein algebra; Test spaces for cohomological dimension; Mapping intersection property; Stable intersection of compacta; Dimension types of compacta

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1. Introduction

There are two approaches to define the *dimensional type* of a space X: algebraic and geometric. The algebraic or *cohomological dimension type* of a space X is defined by its cohomological dimensions $\dim_G X$ with respect to different coefficient groups G. Two spaces X and Y are said to be of the same *cohomological dimension type* if and only if $\dim_G X = \dim_G Y$ for every Abelian group G. For the geometric approach, based on dimensions of products, one defines X and Y to be of the same *geometric dimension type* if and only if $\dim(X \times Z) = \dim(Y \times Z)$ for every compactum Z. These approaches are known to be equivalent for the class of *compacta*, i.e., compact metric spaces – the only class of interest in this paper.

Bockstein defined a countable family of groups, the so-called *Bockstein basis* and proposed an effective algorithm to calculate the dimension of products via the cohomological dimension of factors over the basic groups [1]. Hence the coincidence of cohomological types of compacta implies the coincidence of geometrical dimension types. On the other hand, the theory of *test spaces* originated by Kodama [11] and Kuz'minov [12] and completed recently by Dranišnikov [2] allows one to prove the converse implication.

Given an Abelian group G, one says that a compactum X is a *G-testing space* for some class $\mathscr{C}$ of spaces if for all spaces $Y \in \mathscr{C}$ the following *testing equality* holds:

$$\dim_G Y = \dim(X \times Y) - \dim X.$$

Since Kuz'minov [12] constructed testing compacta $T_n(G)$ for the class of n-dimensional compacta and any Abelian group G, it follows that the geometric dimension type completely determines the cohomological dimension types. Kuz'minov test spaces $T_n(G)$ have dimension in general greater than n. His result was strengthened by Dranišnikov [2], who constructed test spaces of dimension $\dim T_n(G) = n$. For our purposes we need a little more exact result which is proved in Section 2 – on the Bockstein algebra.

Test Space Theorem 1.1. *For any Abelian group G and any natural number n, there exists an n-dimensional compactum $T_n(G)$ such that $T_n(G)$ is a G-testing space for the class of compacta Y which satisfy the inequality* $\dim Y - \dim_G Y < n$.

Compacta having the same cohomological (and hence geometric) type are said to be of the same *dimension type*. The dimension type of a space will be written as DIM X (in contrast with the *Lebesgue* dimension dim X). It was the problem stated below which stimulated us to introduce the notion of the dimension type. A pair of maps $f: X \to S$ and $g: Y \to S$ of compacta X and Y into a metric space S is said to have a *stable intersection* [5] is for some $\varepsilon > 0$, every pair of maps $f': X \to S$ and $g': Y \to S$ such that $d(f, f') < \varepsilon$ and $d(g, g') < \varepsilon$ has the property that $f'(X) \cap g'(Y) \neq \emptyset$. In the sequel, we shall assume that $S \in \mathrm{AR}$.

Mapping Intersection Problem 1.2. Is it true that compacta X and Y such that $\dim(X\times Y)<n$ fail to possess mappings into $\mathbb{R}^n$ with stable intersections?

Compacta X and Y having no stable intersections in space S are called *disjoinable in* S and denoted by $X\|Y$ *in* S. (Note the following two elementary properties of disjoinability: (1) If $X\|Y$ in S, $X'\subset X$ and $Y'\subset Y$ then also $X'\|Y'$ in S; and (2) if $X=\bigcup\{X_i|i\in\mathbb{N}\}$, X_i are closed and pairwise disjoint and $X_i\|Y$ in S then also $X\|Y$ in S.) It turns out that the property of disjoinability is a problem of dimension types:

Dimension Type Determination Theorem 1.3. *If* $\dim X\leqslant n-3$, $\dim Y\leqslant n-3$, $\mathrm{DIM}\ X=\mathrm{DIM}\ X'$, *and* $\mathrm{DIM}\ Y=\mathrm{DIM}\ Y'$ *then* $X\|Y$ *in* $\mathbb{R}^n$ *implies* $X'\|Y'$ *in* $\mathbb{R}^n$.

The proof of this theorem is based on the interplay of two dualities, one of them being the duality of negligibility and dimension, stated in the following theorem from [3,4]. (Note that the *tameness* of X is used *only* in the proof of the implication (1) $\Rightarrow$ (2).)

Duality of Negligibility and Dimension Theorem 1.4. *For any tame compactum* $X\subset\mathbb{R}^n$ *of dimension* $\leqslant n-3$, *and any other compactum* Y, *the following statements are equivalent*:

(1) $\dim(X\times Y)<n$; *and*

(2) *X is Y-negligible.*

A subset X of a space S is said to be *negligible* with respect to some compactum Y *(Y-negligible)* if mappings of Y into S are approximable by mappings whose images miss X. (Note the following two elementary properties of Y-negligibility: (1) Every subspace $X'\subset X$ of a Y-negligible set X is also Y-negligible; and (2) the union of any countable family $\{X_i\}_{i\in\mathbb{N}}$ of closed Y-negligible sets X_i is also Y-negligible.) The Duality of Negligibility and Dimension Theorem 1.4, together with the Test Space Theorem 1.1 allow us to determine the dimension type of X by its negligibility type.

The second duality, needed for the proof of Theorem 1.3, is rather elementary. Before stating it let us introduce some terminology. We will say that a compactum X is *approximable in space S with mappings of class $\mathscr{P}$ ($\mathscr{P}$-approximable)* if mappings of the class $\mathscr{P}$ form a dense subset in the space $\mathscr{C}(X, S)$ of all continuous mappings from X into S. We now state our second duality—for mutual neglibility— which easily follows from a lemma due to H. Toruńczyk (cf. [13]):

Mutual Negligibility Lemma 1.5. *For any pair of compacta X and Y, the following statements are equivalent*:

(1) $X\|Y$ *in* $\mathbb{R}^n$;

(2) *X is approximable in $\mathbb{R}^n$ with Y-negligible images*; *and*

(3) *Y is approximable in $\mathbb{R}^n$ with X-negligible images.*

The strongest result up until now, with respect to the Mapping Intersection Problem 1.2 has been the following theorem (cf. [3,6], and also [15]):

Metastable Disjoining Theorem 1.6. *If compacta X and Y satisfy the inequalities* $\dim(X \times Y) < n$ *and* $2 \dim X + \dim Y \leqslant 2n - 2$ *then* $X \| Y$ *in* $\mathbb{R}^n$.

This theorem is now an easy consequence of Theorem 1.4 and the following approximation theorem (and it also follows from Theorem 1.18).

Metastable Approximation Theorem 1.7. *Suppose that G is an Abelian group and X is a compactum such that* $\dim X \leqslant n-3$ *and* $2 \dim X - \dim_G X \leqslant n-1$. *Then* $\{f \in \mathscr{C}(X, \mathbb{R}^n) | \dim_G f(X) = \dim_G X\}$ *is a dense G_δ-subset of* $\mathscr{C}(X, \mathbb{R}^n)$.

If we consider X and a test space $T_n(G)$, the disjoining X and the test space $T_n(G)$ in $\mathbb{R}^n$ is equivalent, by the Mutual Negligibility Lemma 1.5 to $T_n(G)$-negligible approximability of X in $\mathbb{R}^n$. But $T_n(G)$-negligibility by Theorem 1.4 means simply restriction on the G-dimension of $f(X)$. Precise analysis of the above arguments produces the following result:

Disjoining Approximation Equivalence Theorem 1.8. *For every compactum X of dimension* $\leqslant n-3$, *the following statements are equivalent*:

(1) *X is approximable in $\mathbb{R}^n$ by mappings not changing its dimension type*; *and*

(2) *$X \| Y$ in $\mathbb{R}^n$ for any compactum Y such that* $\dim(X \times Y) < n$.

So we see that the Mapping Intersection Problem 1.2 turns out to be equivalent to the following general fundamental problem:

Cohomological Dimension Approximation Problem 1.9. Is every compactum X of dimension $\leqslant n-2$ approximable in $\mathbb{R}^n$ with mappings whose images have the same cohomological dimension as X?

In fact, it follows that the results in Section 4 that the Cohomological Dimension Approximation Problem 1.9 reduces to the case of *cyclic* groups. The described reduction was not surprising for the authors (cf. [2]). But the following reduction of both problems to a problem of realization of a given dimension type in some $\mathbb{R}^n$ was *very* surprising for us. This reduction is based on the following ingenuous geometric observation.

Approximation Lemma 1.10. *If X is a compact subspace of $\mathbb{R}^n$ then any mapping of X into $\mathbb{R}^n$ is approximable by mappings which do not change its dimension type.*

As an immediate corollary of the Approximating Lemma 1.10 and by virtue of the Disjoining Approximation Equivalence Theorem 1.8, one obtains the following result:

Subset Disjoining Theorem 1.11. *If $X \subset \mathbb{R}^n$ and Y are compacta such that* $\dim X \leqslant n-3$ *and* $\dim(X \times Y) < n$ *then* $X \| Y$ *in* $\mathbb{R}^n$.

Positive solution of the Cohomological Dimension Approximation Problem 1.9 for some X implies that for most mappings $f: X \to \mathbb{R}^n$, the images have the same dimension type as X. So DIM X is embeddable into $\mathbb{R}^n$.

Now we are ready to observe that embedding of X into $\mathbb{R}^n$ up to DIM-type immediately solves the Cohomological Dimension Approximation Problem 1.9. Indeed, Problem 1.9 for X is equivalent to the disjoining problem for dimension types, by the Dimension Type Determination Theorem 1.3. So the Approximation Lemma 1.10 solves Problem 1.9 for subsets of $\mathbb{R}^n$, and hence solves it for all dimension types which are representable in $\mathbb{R}^n$.

The above discussion can be summarized by the following result:

Reduction Theorem 1.12. *For any compactum X of dimension* $\leqslant n-3$, *the following statements are equivalent*:

(1) DIM $X =$ DIM X' *for some compactum* $X' \subset \mathbb{R}^n$;

(2) $X \| Y$ *in* $\mathbb{R}^n$ *for any Y with* $\dim(X \times Y) < n$; *and*

(3) *X is approximable in $\mathbb{R}^n$ with mappings $\varphi: X \to \mathbb{R}^n$ such that* DIM $\varphi(X) =$ DIM X.

Now one sees the third aspect of the Mapping Intersection Problem 1.2 which turns out to be equivalent to the following:

Dimension Types Embedding Problem 1.13. Given a compactum X of dimension $\leqslant n-2$, does there exist a compact subset $X' \subset \mathbb{R}^n$ of the same dimension type, DIM $X' =$ DIM X?

Having finished with reduction let us start to solve the problem. First, the Reduction Theorem 1.12, coupled with the Approximation Lemma 1.10 produces almost complete (codimension 2 excluding) solution of the Mapping Intersection Problem 1.2 for compact subsets of $\mathbb{R}^n$.

The main tool in the further developments provides the Splitting Theorem for Dimension Type 1.14. This theorem presents a decomposition of an arbitrary type into basic types. The complete description is presented in Section 2 (cf. Theorems 2.11 and 2.13). For our present purpose it suffices to state the following version of this theorem.

***G*-Splitting Theorem 1.14.** *For every Abelian group G and every compactum X there exists a pair of compacta X_1 and X_2 such that*

(1) DIM $X =$ DIM$(X_1 \sqcup X_2)$;

(2) $\dim_G X_1 = 1$; *and*

(3) $\dim X_2 \leqslant \dim_G X + 1$.

First, apply the G-Splitting Theorem 1.14 to solve the Cohomological Dimension Approximation Problem 1.9 for $\dim_G$. By DIM-invariance of approximability it is sufficient to solve this problem for the case $X = X_1 \cup X_2$, where X_1 and X_2 satisfy conditions of Theorem 1.14.

Now the problem breaks into two parts. It follows for X_2 by the Metastable Approximation Theorem 1.7. Concerning X_1, by using regularly branched maps and the G-analogue of the Hurewicz theorem [10] on dimension raising mappings we can prove the following result (by codim X we shall always denote $n - \dim X$, even when X doesn't lie in $\mathbb{R}^n$):

Dimension Bounding Lemma 1.15. *Let G be any Abelian group. If $f : X \to \mathbb{R}^n$ is a regularly branched map and if* $\dim_G X = 1$ *then*

$$\dim_G f(X) \leqslant \frac{n-1}{\operatorname{codim} X}.$$

Namely, X is approximable in $\mathbb{R}^n$ with regularly branched maps [5]. As an immediate application of the above lemma we obtain the following:

Approximation Theorem 1.16. *Let G be any Abelian group and X any compactum of dimension $\leqslant n-3$. Then every mapping of X into $\mathbb{R}^n$ can be approximated arbitrarily closely by mappings $f : X \to \mathbb{R}^n$ such that*

$$\dim_G f(X) \leqslant \max\left\{\dim_G X, \frac{n-1}{\operatorname{codim} X}\right\}.$$

If one denotes by $\underline{\dim}\, X$ the minimum of $\{\dim_G X | G\}$ the *inferior dimension* of X, one can formulate the following condition providing a positive solution of the approximation problem for X:

$$\operatorname{codim} X \cdot \underline{\dim}\, X \geqslant n-1.$$

Instead of introducing restrictions on the inferior dimension it is possible to change it by adding to X some k-dimensional cubes. Equality $\mathrm{DIM}(X \sqcup I^k) = \mathrm{DIM}\ X$ is the equivalent form for $\underline{\dim}\, X \geqslant k$. In such a way one deduces from Approximation Theorem 1.16 the following result:

Realization Theorem 1.17. *For any compactum X of dimension $\leqslant n-3$ there exists a compact subset $X' \subset \mathbb{R}^n$ of type* $\mathrm{DIM}\ X' = \mathrm{DIM}(X \sqcup I^k)$ *where* $k = [(n-1)/\operatorname{codim} X]$.

Apply now our Realization Theorem 1.17 to produce a new solution of the disjoining problem, thus attaining the main goal of this paper:

Disjoining Theorem 1.18. *Suppose that X and Y are compacta such that* $\dim X \leqslant n-3$, $\dim Y \leqslant n-3$. *If* $\dim(X \times Y) < n$ *and* $\operatorname{codim} X \cdot \operatorname{codim} Y \geqslant n$ *then $X \| Y$ in* $\mathbb{R}^n$.

It is not too difficult to show that the inequality $\operatorname{codim} X \cdot \operatorname{codim} Y \geqslant n$ is a consequence of the *metastable* case, i.e., when $2 \dim X + \dim Y \leqslant 2n - 2$. Indeed, observe first that the inequality dim X + dim $Y \leqslant n$ is equivalent to the inequality codim X + codim $Y \geqslant n$, whereas 2 dim X + dim $Y \leqslant 2n - 2$ is equivalent to 2 codim X + codim $Y \geqslant n + 2$. It then easily follows (under the hypothesis that codim $X > 2$ and codim $Y > 2$) that the inequality 2 codim X + codim $Y \geqslant n + 2$ implies the inequality codim $X \cdot$ codim $Y \geqslant n$. So the Disjoining Theorem 1.18 covers the metastable case, i.e., it implies the Metastable Disjoining Theorem 1.6.

On the other hand, note that the Disjoining Theorem 1.18 significantly improves the Metastable Disjoining Theorem 1.6. For example, every pair of compacta X and Y such that dim $X \leqslant n - \sqrt{n}$ and dim $Y \leqslant n - \sqrt{n}$ clearly satisfies the inequality codim $X \cdot$ codim $Y \geqslant n$ of the Disjoining Theorem 1.18, whereas it may clearly fail to satisfy the condition 2 dim X + dim $Y \leqslant n - 2$ of the Metastable Disjoining Theorem 1.6. It is reasonable to expect that Theorem 1.18 is true also in the *exceptional case* when codim X = codim $Y = 2$. However, the well-known result on nonembeddability of certain real projective spaces into $\mathbb{R}^n$ leads us to believe that Theorem 1.18 with just one condition, i.e., $\dim(X \times Y) < n$, probably fails to be true.

A compactum X is said to be the *Bockstein n-complement* of a compactum Y if $\dim(X \times Y) = n$ and X has the maximal dimension type among compacta with this property, i.e., if for some compactum X', $\dim(X' \times Y) = n$ then DIM $X' \leqslant$ DIM X, i.e., for every G, $\dim_G X' \leqslant \dim_G X$. In the Realization Theorem 1.17 we also have a duality:

Realization of the Complements Theorem 1.19. *Suppose that X is a compactum such that* dim $X \leqslant n - 3$ *and the* DIM X *can be realized in* $\mathbb{R}^n$. *Then* DIM X^* *can also be realized in* $\mathbb{R}^n$, *where* X^* *is the Bockstein* $(n-1)$*-complement of* X.

We shall prove that $(X^*)^*$ has the same DIM-type as X. The previous theorem allows us to give a new formulation of the Reduction Theorem 1.12:

Theorem 1.20. *If X and X^* are* DIM*-complementary compacta and* dim $X \leqslant n - 3 \geqslant$ dim X^* *then the following statements are equivalent*:

(1) DIM X *embeds in* $\mathbb{R}^n$;
(2) DIM X^* *embeds in* $\mathbb{R}^n$; *and*
(3) $X \| X^*$ *in* $\mathbb{R}^n$.

2. Bockstein algebra: testing and splitting

Let be the class of all Abelian groups. Bockstein defined a family of Abelian groups which is sufficient to calculate the cohomological dimension of compacta with respect to any group $G \in \mathscr{A}$ [1]. Bockstein's basis $\sigma \subset \mathscr{A}$ consists of:

(1) The rationals $\mathbb{Q}$;
(2) the p-cyclic groups $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ (p-prime);
(3) the p-localization of integers $\mathbb{Z}_{(p)} = \{m/n \mid n$ is not divisible by $p\}$; and
(4) the p-adic circle $\mathbb{Z}_{p^\infty} = \mathbb{Q}/\mathbb{Z}_{(p)}$.

To define a cohomological dimension with respect to an arbitrary group $G \in \mathscr{A}$ it is sufficient to know the cohomological dimension over a family $\sigma(G) \subset \sigma$ defined as follows

(1) $\mathbb{Q} \in \sigma(G)$ if and only if $\mathbb{Q} \otimes G \neq 0$;
(2) $\mathbb{Z}_p \in \sigma(G)$ if and only if $\mathbb{Z}_p \otimes G \neq 0$;
(3) $\mathbb{Z}_{(p)} \in \sigma(G)$ if and only if $\mathbb{Z}_{p^\infty} \otimes G \neq 0$; and
(4) $\mathbb{Z}_{p^\infty} \in \sigma(G)$ if and only if $\mathbb{Z}_p * G \neq 0$.

Theorem 2.1 (Bockstein Basis Theorem) [1]. *For every compactum X and for every Abelian group $G \in \mathscr{A}$, $\mathrm{d}im_G X = \max\{\dim_H X | H \in \sigma(G)\}$.*

Any compact space X defines a function $D_X : \mathscr{A} \to \mathbb{N} \cup \{0, \infty\}$ on the class $\mathscr{A}$ by the formula $D_X(G) = \dim_G X$, for every $G \in \mathscr{A}$. The second fundamental Bockstein theorem [1] establishes inequalities which are valid for all such functions. It is known that these inequalities are the only restrictions on the set of values of cohomological dimensions with respect to Bockstein groups [2].

Theorem 2.2 (Bockstein Inequalities) [1]. *For every compactum X the following conditions hold:*

(1) $D_X(\mathbb{Z}_{p^\infty}) \leqslant D_X(\mathbb{Z}_p)$;
(2) $D_X(\mathbb{Z}_p) \leqslant D_X(\mathbb{Z}_{p^\infty}) + 1$;
(3) $D_X(\mathbb{Z}_p) \leqslant D_X(\mathbb{Z}_{(p)})$;
(4) $D_X(\mathbb{Z}_{(p)}) \leqslant \max\{D_X(\mathbb{Q}), D_X(\mathbb{Z}_{p^\infty}) + 1\}$;
(5) $D_X(\mathbb{Z}_{p^\infty}) \leqslant \max\{D_X(\mathbb{Q}), D(\mathbb{Z}_{(p)}) - 1\}$;
(6) $D_X(\mathbb{Q}) \leqslant D_X(\mathbb{Z}_{(p)})$; *and*
(7) *if $D_X(G) = 0$ for some $G \in \sigma$ then $D(G) = 0$ for all $G \in \sigma$.*

The system of inequalities (1)–(6) above can be written in the following (simpler) equivalent form:

(i) $\mathrm{def}_p X \geqslant 0$;
(ii) $\Delta_p X \geqslant 0$;
(iii) $\Delta_p X \cdot (\mathrm{def}_p X - 1) = 0$; and
(iv) $\varepsilon_p X \cdot (\varepsilon_p X - 1) = 0$;

where $\mathrm{def}_p X = D_X(\mathbb{Z}_{(p)}) - D_X(\mathbb{Z}_{p^\infty})$ is the *p-defect* of X, $\Delta_p X = D_X(\mathbb{Z}_{(p)}) - D_X(\mathbb{Q})$ is the *p-variation* of X, and $\varepsilon_p X = D_X(\mathbb{Z}_p) - D_X(\mathbb{Z}_{p^\infty})$.

For an arbitrary prime p, the compactum X is said to be *p-regular* if $\dim_{\mathbb{Z}_p} X = \dim_{\mathbb{Z}_{p^\infty}} X = \dim_{\mathbb{Z}_{(p)}} X = \dim_{\mathbb{Q}} X$, otherwise X is said to be *p-singular*. It is easy to see that X is dimensionally full valued (i.e., for every $G \in \mathscr{A}$, $\dim_G X = \dim X$) if and only if X is p-regular for all primes p. Note that a compactum X is p-regular if and only if $\mathrm{def}_p X = 0$. Furthermore, if X is p-regular then the logarithmic law is valid for the dimension of the product $X \times Y$ with respect to the groups $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{p^\infty}$. Every p-singular compactum X has the following properties: (i) $\Delta_p X \geqslant 0$; (ii) $\mathrm{def}_p X > 0$; (iii) $\Delta_p X = 0$ if $\mathrm{def}_p X > 1$; and (iv) $\dim_{\mathbb{Z}_p} X \in \{\dim_{\mathbb{Z}_{p^\infty}} X, \dim_{\mathbb{Z}_{p^\infty}} X + 1\}$.

An integral function $D : \sigma \to \mathbb{N} \cup \{0, \infty\}$ defined over σ and satisfying conditions (1)–(7) above, will be called in *Bockstein function*. The following fundamental theorem is proved in [2].

Realization Theorem 2.3. *For any Bockstein function D there exists a compactum X such that $D = D_X$.*

To complete the reduction of the theory of dimension types to Bockstein algebra, we have to define *operations* for Bockstein functions which correspond to the *sum* and the *product* operation for spaces. The first one of them, denoted by $\vee$, and corresponding to the *union* of spaces, is very simple: $(D_1 \vee D_2)(G) = \max\{D_1(G), D_2(G)\}$. It is easy to see that $D_{X_1 \sqcup X_2} = D_{X_1} \vee D_{X_2}$. The definition of $\vee$ generalizes to arbitrary sums in a straightforward manner.

Proposition 2.4. *For every (possibly uncountable) family $\{D_\alpha\}_{\alpha \in \Lambda}$ of Bockstein functions there exists a countable subset $\Lambda' \subset \Lambda$ such that $\bigvee_{\alpha \in \Lambda'} D_\alpha = \bigvee_{\alpha \in \Lambda} D_\alpha$.*

Proof. Take an arbitrary group $G \in \sigma$ from the Bockstein basis σ. If the maximum of $D_\alpha(G)$ is attained on one of the Bockstein functions D_{α_G}, for some index $\alpha_G \in \Lambda$, then define $\Lambda_G = \{\alpha_G\}$. Otherwise, there exists a sequence $\{\alpha_G^i\}_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} D_{\alpha_G^i}(G) = \infty$. In this case we define $\Lambda_G = \{\alpha_G^i\}_{i \in \mathbb{N}}$. We do this for all groups $G \in \sigma$. Let $\Lambda' = \bigcup_{G \in \sigma} \Lambda_G$. Since σ is countable, and since card $\Lambda_G \leqslant \aleph_0$, it follows that card $\Lambda' = \aleph_0$, too. $\square$

Proposition 2.5. *For every (possibly uncountable) family $\{D_\alpha\}_{\alpha \in \Lambda}$ of Bockstein functions, $\bigvee_{\alpha \in \Lambda} D_\alpha$ is also a Bockstein function.*

Proof. Apply Proposition 2.4 to reduce to the countable case and then apply the Realization Theorem 2.3 (or use the *majorization* argument, i.e., $x_\alpha \leqslant y_\alpha$ for all $\alpha \in \Lambda$ implies that $\max\{x_\alpha | \alpha \in \Lambda\} \leqslant \max\{y_\alpha | \alpha \in \Lambda\}$). $\square$

The definition of the other operation, the *Bockstein product* $D_1 \boxplus D_2$, yields the equality $D_X \boxplus D_Y = D_{X \times Y}$ and it turns out to be rather complicated.

For any prime p, we call a Bockstein function *p-regular* if all three p-related dimensions coincide: $D(\mathbb{Z}_p) = D(\mathbb{Z}_{(p)}) = D(\mathbb{Z}_{p^\infty})$. Functions which are not p-regular are called *p-singular*. As it follows from the Bockstein inequalities for p-regular D we additionally have the following equalities (p-regular equality):

$$D(\mathbb{Q}) = D(\mathbb{Z}_{(p)}) = D(\mathbb{Z}_p) = D(\mathbb{Z}_{p^\infty}).$$

Now we are ready to define the *Bockstein product* $D_1 \boxplus D_2$ as follows:

(1) $(D_1 \boxplus D_2)(\mathbb{Q}) = D_1(\mathbb{Q}) + D_2(\mathbb{Q})$;
(2) $(D_1 \boxplus D_2)(\mathbb{Z}_p) = D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_p)$;
(3) $(D_1 \boxplus D_2)(\mathbb{Z}_{p^\infty}) = \max\{D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}), (D_1 \boxplus D_2)(\mathbb{Z}_p) - 1\}$;
(4R) $(D_1 \boxplus D_2)(\mathbb{Z}_{(p)}) = D_1(\mathbb{Z}_{(p)}) + D_2(\mathbb{Z}_p)$ if D_1 or D_2 is p-regular; and
(4S) $(D_1 \boxplus D_2)(\mathbb{Z}_{(p)}) = \max\{D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1, (D_1 \boxplus D_2)(\mathbb{Z}_p), (D_1 \boxplus D_2)(\mathbb{Q})\}$ if both D_i are p-singular.

The Bockstein theorem on products now takes the following form:

Bockstein Theorem on Products 2.6 [1]. *For every pair of compacta X and Y, $D_X \boxplus D_Y = D_{X \times Y}$.*

By the infinite sum $\coprod X_i$ in the category of compacta we mean the one-point compactification of $\coprod X_i$.

Theorem 2.7 (Distributivity Law). *For every family $\{D_\alpha\}_{\alpha \in \Lambda}$ of Bockstein functions and for every Bockstein function D, $D \boxplus (\bigvee_{\alpha \in \Lambda} D_\alpha) = \bigvee_{\alpha \in \Lambda}(D \boxplus D_\alpha)$.*

Proof. For the case when Λ is countable, the assertion follows by the Realization Theorem 2.3. Indeed, by Theorem 2.3, for very D_α there exists a compactum X_α such that $D_\alpha = D_{X_\alpha}$. Similarly, for D there exists a compactum X such that $D = D_X$. By Proposition 2.4 it suffices to prove the theorem for the case when Λ is countable. In this case, $X^* = \coprod_{\alpha \in \Lambda} X_\alpha$ realizes $\bigvee_{\alpha \in \Lambda} D_\alpha$ and $\dim(X^* \times X) = \max\{\dim(X_\alpha \times X) | \alpha \in \Lambda\}$, so $X^* \times X$ realizes $\bigvee_{\alpha \in \Lambda}(D \boxplus D_\alpha)$. Therefore the formula is the consequence of the distributivity law for spaces, i.e., $(\bigcup_{\alpha \in \Lambda} X_\alpha) \times Y = \bigcup_{\alpha \in \Lambda}(X_\alpha \times Y)$. □

We shall write $D_1 \leqslant D_2$ if $D_1(G) \leqslant D_2(G)$ for all Abelian groups $G \in \sigma$. Define the *superior* and *inferior norm* of function D as $\|D\| = \max\{D(G) | G \in \sigma\}$ and $|D| = \min\{D(G) | G \in \sigma\}$, respectively. The superior norm of D_X is known to coincide with the Lebesgue dimension of X, i.e., $\|D_X\| = \dim X$, for finite-dimensional compacta X [12].

Proposition 2.8 (Monotonicity Property of Multiplication). *For all triples of Bockstein functions D_1, D_2 and D such that $D_1 \geqslant D_2$, the inequality $D_1 \boxplus D \geqslant D_2 \boxplus D$ holds.*

Product Inequalities Lemma 2.9. *For all pairs of Bockstein functions D_1 and D_2 one has that $|D_1| + \|D_2\| \leqslant \|D_1 \boxplus D_2\| \leqslant \|D_1\| + \|D_2\|$.*

Proof. It easily follows from the Bockstein inequalities that $\|D\| = \max\{D(\mathbb{Z}_{(p)})\}$ and $|D| = \min\{D(\mathbb{Q}), D(\mathbb{Z}_{p^\infty})\}$. Apply now the formulae (4R) and (4S) to complete the proof. □

Kuz'minov [12] found a basis for the Bockstein algebra. Kuz'minov's basic functions are presented in Table 1.

Theorem 2.10 (Algebraic Splitting of Basis) [12]. *Every Bockstein function D can be represented in the form $D = \bigvee\{\Phi(G, k) | G \in \sigma\}$.*

Table 1

	$\mathbb{Z}_{(p)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^\infty}$	$\mathbb{Q}$	$\mathbb{Z}_{(q)}$	$\mathbb{Z}_q$	$\mathbb{Z}_{q^\infty}$
$\Phi(\mathbb{Q}, n)$	n	1	1	n	n	1	1
$\Phi(\mathbb{Z}_{(p)}, n)$	n	n	n	n	n	1	1
$\Phi(\mathbb{Z}_p, n)$	n	n	$n-1$	1	1	1	1
$\Phi(\mathbb{Z}_{p^\infty}, n)$	n	$n-1$	$n-1$	1	1	1	1

Compacta realizing the basic functions are called the *fundamental compacta* and denoted by $F(G, n)$, where $D_{F(G,n)} = \Phi(G, n)$. The Algebraic Splitting Theorem 2.10 plus the Realization Theorem 2.3 immediately imply the following:

Splitting Theorem 2.11. *For every compactum X there exists a sequence of fundamental compacta $\{F_i(G_i, k_i)\}_{i\in\mathbb{N}}$ such that* $\operatorname{DIM} X = \operatorname{DIM}(\coprod_{i\in\mathbb{N}} F_i(G_i, k_i))$, *where by the infinite sum in the category of compacta one understands the one-point compactification of the infinite union.*

Lemma 2.12. *For every Abelian group G, every integer n, and for every fundamental compactum $F(H, n)$, either* $\dim_G F(H, n) = 1$ *or* $\dim_G F(H, n) \geqslant n-1$.

Proof. Due to Theorem 2.1 it suffices to do the proof only for basic groups G. But in this case this property follows from Table 1. □

Now we are ready to prove the G-Splitting Theorem.

G-Splitting Theorem 2.13. *For every G, any compactum X with* $\dim X \geqslant 1$ *has dimension type of a union $X_1 \cup X_2$, such that* $\dim_G X_1 = 1$ *and* $\dim_G X \geqslant \dim X_2 - 1$.

Proof. By Splitting Theorem 2.11 we can work with the union $\bigcup F_i(G_i, n_i)$ of fundamental compacta. Let us denote by $X_1 = \coprod\{F_i(G_i, n_i) \mid \dim_G F_i(G_i, n_i) = 1\}$ and $X_2 = \coprod\{F_i(G_i, n_i) \mid \dim_G F_i(G_i, n_i) \neq 1\}$. Then $\dim_G X_1 = 1$ and $\dim_G X_2 \geqslant \dim X_2 - 1$ follows by Lemma 2.12. □

Now we have to perform calculations of dimensions of products of fundamental compacta ($n \geqslant m$):

The result of the calculations, presented in Table 2, can be summarized in the following formula ($n \geqslant m$):

$$\begin{aligned}\|\Phi(G, n) \boxplus \Phi(G', m)\| &= \max\{\Phi(G, n)(G') + m, n+1\} \\ &= \Phi(G', m)(G) + n.\end{aligned}$$

Lemma 2.14. *For any fundamental function $\Phi(G, n)$ and any other function D,*

$$\|D \boxplus \Phi(G, n)\| = \begin{cases} \max\{\|D\| + 1, n + D(G)\} & \text{if } \|D\| \geqslant n, \\ n + D(G) & \text{if } \|D\| \leqslant n. \end{cases}$$

Table 2

	$\Phi(\mathbb{Z}_{(p)}, n)$	$\Phi(\mathbb{Z}_p, n)$		$\Phi(\mathbb{Z}_{p^\infty}, n)$		$\Phi(\mathbb{Q}, n)$
		$p=q$	$p\neq q$	$p=q$	$p\neq q$	
$\Phi(\mathbb{Z}_{(q)}, m)$	$m+n$	$1+n$		$1+n$		$m+n$
$\Phi(\mathbb{Z}_q, m)$	$m+n$	$m+n$	$1+n$	$1+n$		$1+n$
$\Phi(\mathbb{Z}_{q^\infty}, m)$	$m+n$	$m+n-1$		$m+n-1$	$1+n$	$1+n$
$\Phi(\mathbb{Q}, m)$	$m+n$	$1+n$		$1+n$		$m+n$

Proof. If $D = \Phi(G', m)$ and G' is a basic group then the assertion follows from Table 2. If D is arbitrary, use the Splitting Principle 2.11 and the Distributivity Law 2.7 to complete the proof. □

Now we are ready to prove Test Space Theorem 1.1 from the Introduction. For any Abelian group G one defines $T(G, n)$ as the sum $\bigvee\{\Phi(H, n) | H \in \sigma(G)\}$. Let us consider a compactum X such that $|\dim_G X - \dim X| < n$. If $\| D_X \| < n$ then by Lemma 2.14 the assertion follows. On the other hand, for $\| D_X \| \geqslant n$ it follows from the above calculations that $\| \Phi(G, n) \boxplus D_X \| = \max\{\| D_X \| + 1, n + D_X(G)\}$. So if $\| D_X \| + 1 \leqslant n + D_X(G)$, i.e., if $\| D_X \| - D_X(G) < n$ then one obtains the test equality $\| \Phi(G, n) \boxplus D_X \| = n + D_G(G)$.

3. Approximation and disjoining: proofs

We begin by an observation that the Mutual Negligibility Lemma 1.5 easily follows from the following lemma, due to Toruńczyk [13], Lemma (A.4)]:

Lemma 3.1 [13]. *Let X and Y be compacta such that $X \| Y$ in $\mathbb{R}^n$. Then $K = \{f \in \mathscr{C}(X, \mathbb{R}^n) | f(X)$ is Y-negligible in $\mathbb{R}^n\}$ is a dense G_δ-set in $\mathscr{C}(X, \mathbb{R}^n)$.*

Indeed, Lemma 3.1 implies $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ of Lemma 1.5, whereas $(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$ is obvious.

Proof of the Dimension Type Determination Theorem 1.3. It clearly suffices to consider the case when $X = X'$. Since, the hypothesis, $X \| Y$, it follows by the Mutual Negligibility Lemma 1.5 and [16] that for every map $f : X \to \mathbb{R}^n$ there exists an approximation $f' : X \to \mathbb{R}^n$ such that $f'(X)$ is Y-negligible and tame. Therefore by the Duality of Negligibility and Dimension Theorem 1.4, $\dim(f'(X) \times Y) < n$. Apply now Proposition 3.3 below to conclude that $\dim(f'(X) \times Y') < n$. Again, by Theorem 1.4 we have that $f'(X)$ is Y'-negligible, so by Lemma 1.5, $X \| Y'$. □

Disjoining Monotonicity Theorem 3.2. *Let X, X', Y and Y' be compacta such that* $\dim X \leqslant n - 3$, $\dim Y \leqslant n - 3$, $\mathrm{DIM}\ X' \leqslant \mathrm{DIM}\ X$, $\mathrm{DIM}\ Y' \leqslant \mathrm{DIM}\ Y$, *and $X \| Y$ in $\mathbb{R}^n$. Then $X' \| Y'$ in $\mathbb{R}^n$, too.*

Proof. The hypotheses imply that $\mathrm{DIM}(X \sqcup X') = \mathrm{DIM}\ X$ and $\mathrm{DIM}(Y \sqcup Y') = \mathrm{DIM}\ Y$, so by Theorem 1.3, $(X \sqcup X') \| (Y \sqcup Y')$ in $\mathbb{R}^n$ which obviously implies that $X' \| Y'$ in $\mathbb{R}^n$, too. □

Proposition 3.3. *Let X and X' be arbitrary compacta. Then the following statements are equivalent*:

(1) $\mathrm{DIM}\ X = \mathrm{DIM}\ X'$; *and*

(2) *for every compactum Y*, $\dim(X \times Y) = \dim(X' \times Y)$.

Proof. ($\Rightarrow$) Suppose that DIM $X = $ DIM X'. Then by the definition of the cohomological dimension type, $\dim_G X = \dim_G X'$, for every Abelian group G. Now, by the Bockstein formula for dimension of the product (see [12, Chapter 2]), $\dim(X \times Y)$ (respectively $\dim(X' \times Y)$) is calculated via $\dim_G X$ and $\dim_G Y$ (respectively $\dim_G X'$ and $\dim_G Y$) for various Abelian groups G. Consequently, $\dim(X \times Y) = \dim(X' \times Y)$.

($\Leftarrow$) We must check that for any Abelian group G, $\dim_G X = \dim_G X'$. So pick G and let $n > \max\{\dim X, \dim X'\}$. By the Test Space Theorem 1.1, there exists the test space $T_n(G)$. If we now take Y to be the compactum $T_n(G)$, we obtain the following equalities: $\dim_G X + n = \dim_G X + \dim T_n(G) = \dim(X \times T_n(G)) = \dim(X' \times T_n(G)) = \dim_G X' + \dim T_n(G) = \dim_G X' + n$. Therefore $\dim_G X = \dim_G X'$, for any G, hence DIM $X = $ DIM X'. □

In order to prove the Metastable Approximation Theorem 1.7 we need the following lemma:

Lemma 3.4 [5]. *Let X and Y be compacta such that* $\dim(X \times Y) < n$ *and* $2 \dim X + \dim Y \leqslant 2n - 2$. *Then* $\{f \in \mathscr{C}(X, \mathbb{R}^n) | \dim(f(X) \times Y) \leqslant \dim(X \times Y)\}$ *is a dense G_δ-subset of* $\mathscr{C}(X, \mathbb{R}^n)$.

Proof of the Metastable Approximation Theorem 1.7. If $\dim_G X = \dim X$ the assertion follows trivially. So we may assume that $\dim_G X < \dim X$. Let $m = (n-1) - \dim_G X$ and $Y = T_m(G)$. Since by hypothesis, $\dim X - \dim_G X \leqslant (n-1) - \dim X = m + \dim_G X - \dim X < m$, Y is the G-test space for X. Therefore $\dim(X \times Y) = \dim_G X + \dim Y = \dim_G X + m = \dim_G X + (n-1) - \dim_G X = n - 1 < n$. Also, $2 \dim X + \dim Y = 2 \dim X + m = 2 \dim X - \dim_G X + (n-1) \leqslant (n-1) + (n-1) = 2n - 2$.

Therefore we can now apply Lemma 3.4 to conclude that $\mathscr{H}(G) = \{f \in \mathscr{C}(X, \mathbb{R}^n) | \dim(f(X) \times Y) \leqslant \dim(X \times Y)\}$ is a dense G_δ-subset of $\mathscr{C}(X, \mathbb{R}^n)$. Since $Y = T_m(G)$ we have that $\mathscr{H}(G) = \{f \in \mathscr{C}(X, \mathbb{R}^n) | \dim_G f(X) \leqslant \dim_G X\}$.

On the other hand, by the Hurewicz Light Mappings Theorem [10], $\mathscr{H}_L = \{f \in \mathscr{C}(X, \mathbb{R}^n) | f \text{ is light}\}$ is a dense G_δ-subset of $\mathscr{C}(X, \mathbb{R}^n)$. Therefore the intersection $\mathscr{H}(G) \cap \mathscr{H}_L$ is a dense G_δ-subset of $\mathscr{C}(X, \mathbb{R}^n)$, too. Since light maps do not lower dimension [12], we have that

$$\mathscr{H}(G) \cap \mathscr{H}_L \subset \{f \in \mathscr{C}(X, \mathbb{R}^n) | \dim_G f(X) = \dim_G X\}$$

so the assertion follows. □

Corollary 3.5. *Suppose that X is a compactum such that* $\dim X \leqslant n - 3$ *and* $2 \dim X - \dim_G X \leqslant n - 1$, *for every Abelian group G. Then* $\{f \in \mathscr{C}(X, \mathbb{R}^n) | \text{DIM } f(X) = \text{DIM } X\}$ *is a dense G_δ-subset of* $\mathscr{C}(X, \mathbb{R}^n)$.

Proof. By the proof of the Metastable Approximation Theorem 1.7 we have that for every Abelian group G, $\mathscr{H}(G) \cap \mathscr{H}_L$ is a dense G_δ-subset of $\mathscr{C}(X, \mathbb{R}^n)$. It suffices to consider only all $G \in \sigma$, where σ is the Bockstein basis. Therefore such is also the intersection $\bigcap\{\mathscr{H}(G) \cap \mathscr{H}_L | G \in \sigma\}$. □

Proof of the Disjoining Approximation Equivalence Theorem 1.8. ($\Rightarrow$) Suppose that $\dim(X \times Y) < n$ and take any pair of maps $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^n$. By hypothesis, we may assume that DIM $f(X) =$ DIM X. Therefore by Proposition 3.3, $\dim(f(X) \times Y) = \dim(X \times Y) < n$. By [16], we may assume that $f(X)$ is tamely embedded in $\mathbb{R}^n$. Apply the Duality of Negligibility and Dimension Theorem 1.4 to conclude that $f(X)$ is Y-negligible. This proves that $X \| Y$ in $\mathbb{R}^n$.

($\Leftarrow$) Suppose now that $X \| Y$ in $\mathbb{R}^n$ and that $\dim(X \times Y) < n$. It follows by the Mutual Negligibility Lemma 1.5 that X can be approximated by a compactum $X' \subset \mathbb{R}^n$ such that X' is Y-negligible, therefore by [2,3], $\dim(X' \times Y) < n$. This is true for all compacta Y which satisfy the hypotheses of (2), so in particular for the test space $Y = T_m(G)$, where $m = (n-1) - \dim_G X$. Thus $n > \dim(X' \times T_m(G)) = \dim_G X' + \dim\ T_m(G) = \dim_G X' + m = \dim_G X' + (n-1) - \dim_G X$ hence $\dim_G X' - \dim_G X \leqslant 0$. □

Remark 3.6. For every compactum X such that $\dim X \leqslant n-3$, the following statements are equivalent:
(1) X is approximable in $\mathbb{R}^n$ by maps which do not change $\dim_G X$; and
(2) $X \| T_m(G)$ in $\mathbb{R}^n$ for any G-test space $T_m(G)$ for X such that $\dim_G X + m < n$.

For the proof of the Approximation Lemma 1.10 we shall need the following result:

Proposition 3.7. *Let $X \subset \mathbb{R}^n$ be an arbitrary compactum. Then every map $f : X \to \mathbb{R}^n$ is approximable by a map $f' : X \to \mathbb{R}^n$ such that $d(f'(X)) = d(X)$ for all dimensional functions d such that* (1) *for every closed subset $Y_0 \subset Y$ of Y, $d(Y_0) \leqslant d(Y)$ and* (2) *for every pair of compacta Y_1 and Y_2, $d(Y_1 \cup Y_2) = \max\{d(Y_1), d(Y_2)\}$.*

Proof. Let $f : X \to \mathbb{R}^n$ be given. Take a compact, n-dimensional polyhedron $P \subset \mathbb{R}^n$ such that $X \subset P$ and extend f over P, i.e., get a map $\bar{f} : P \to \mathbb{R}^n$ such that $\bar{f}|X = f$ (this is possible since $\mathbb{R}^n$ is an absolute retract). Approximate $\bar{f}$ by a simplicial, general position map $F : P \to \mathbb{R}^n$, i.e., $F|_\Delta \to F(\sigma)$ is an embedding for every simplex Δ in P. Let $f' : X \to \mathbb{R}^n$ be the map $f' = F|X$. Since $X = \bigcup\{X \cap \Delta | \Delta \in P\}$ it follows that $f'(X) = \bigcup\{f'(X \cap \Delta) | \Delta \in P\}$. By hypotheses, $df'(X \cap \Delta) = d(X \cap \Delta) < d(\Delta)$ for every $\Delta \in P$ and so $d(f'(X)) \leqslant \max\{d(\Delta) | \Delta \in P\} \leqslant d(X)$. □

Proof of the Approximation Lemma 1.10. Let $f : X \to \mathbb{R}^n$ be any map. Apply Proposition 3.7 for $d = \dim_G$, where G is any Abelian group, to get an approximation $f' : X \to \mathbb{R}^n$ such that $\dim_G f'(X) = \dim_G X$, for all Abelian groups G. Therefore DIM $f'(X) =$ DIM X. □

Proof of the Reduction Theorem 1.12. First note that the equivalence (2) $\Leftrightarrow$ (3) follows by the Disjoining Approximation Equivalence Theorem 1.8 while (3) $\Rightarrow$ (1) is obvious. It therefore remains to prove that (1) $\Rightarrow$ (2). So suppose that $X' \subset \mathbb{R}^n$ is a compactum in $\mathbb{R}^n$ such that DIM $X' =$ DIM X. Let Y be a compactum such that $\dim(X \times Y) < n$. Then by Proposition 3.3 $\dim(X' \times Y) < n$. Consider a map $g : X' \to \mathbb{R}^n$. Apply the Approximation Lemma 1.10 and [16] to approximate g by a map $g' : X' \to \mathbb{R}^n$, such that $g'(X')$ is tame, DIM $g'(X) =$ DIM X', and dim $g'(X') \leqslant n - 3$. By the Duality of Negligibility and Dimension Theorem 1.4, $g'(X') \| Y$ in $\mathbb{R}^n$, hence by the Dimension Type Determination Theorem 1.3, $X \| Y$ in $\mathbb{R}^n$. $\square$

Proof of the Dimension Bounding Lemma 1.15. Let G be any Abelian group and $f : X \to \mathbb{R}^n$ any regularly branched map of a compactum X such that $\dim_G X = 1$. Then by the cohomological analogue of the Hurewicz theorem on $(k+1)$-to-1 maps [12, Theorem (14.1), p. 27], applied for $Y = f(X)$, we get that $1 = \dim_G X \geqslant \dim_G f(X) - k$, where $k - 1$ is the multiplicity of f. Hence $\dim_G f(X) \leqslant k + 1$. Let $B_k = \{x \in \mathbb{R}^n \mid |f^{-1}(x)| \geqslant k\}$. Since f is regularly branched, it follows that dim $B_{k+1} \leqslant n - (k+1) \cdot$ codim X, thus $k + 1 \leqslant (n - \dim B_{k+1})/\text{codim } X$, hence $k + 1 \leqslant [(n - \dim B_{k+1})/\text{codim } X] \leqslant (n-1)/\text{codim } X$ or dim $B_{k+1} = 0$.

In the first case we are done. In the second case we apply the cohomological analogue of the Hurewicz theorem on $(k+1)$-to-1 maps [14,17] to conclude that $\dim_G f(X) \leqslant \max_{1 \leqslant i \leqslant k+1}\{\dim_G \hat{B}_i + i - 1\}$, where $\hat{B}_i = f^{-1}(B_i)$. Since $\dim_G X = 1$, we have that $\max_{1 \leqslant i \leqslant k}\{\dim_G \hat{B}_i + i - 1\} \leqslant k$. Since dim $B_{k+1} = 0$ it follows that dim $\hat{B}_{k+1} = 0$ hence $\dim_G \hat{B}_{k+1} + k = k$, thus $\dim_G f(X) \leqslant k \leqslant (n-1)/\text{codim } X$. $\square$

Proof of the Approximation Theorem 1.16. Let $f : X \to \mathbb{R}^n$ be any map and G any Abelian group. By the G-Splitting Theorem 1.14, there exist compacta X_1 and X_2 such that DIM $X = \text{DIM}(X_1 \sqcup X_2)$, $\dim_G X_1 = 1$ and dim $X_2 \leqslant \dim_G X + 1$. Apply for X_1 and $f_1 : X_1 \to \mathbb{R}^n$ the Dimension Bounding Lemma 1.15: It follows that f_1 can be approximated by $f_1' : X_1 \to \mathbb{R}^n$ with the property that $\dim_G f_1'(X_1) \leqslant (n-1)/\text{codim } X_1$. Apply for X_2 and $f_2 : X \to \mathbb{R}^n$ the Metastable Approximation Theorem 1.7. Since dim $X_2 \leqslant n - 2$, it remains to check that 2 dim $X_2 - \dim_G X_2 \leqslant n - 1$ which, in turn, is an easy exercise.

So apply Theorem 1.7 to conclude that $f_2 : X_2 \to \mathbb{R}^n$ can be approximated by a map $f_2' : X_2 \to \mathbb{R}^n$ such that $\dim_G f_2'(X_2) = \dim_G X_2$. So now $\dim_G(f_1 \sqcup f_2)(X_1 \sqcup X_2) \leqslant \max\{\dim_G X_2, (n-1)/\text{codim } X_1\}$.

We have that DIM $X = \text{DIM}(X_1 \sqcup X_2)$. Hence $\text{DIM}(X \sqcup I^k) = \text{DIM}(X_1 \sqcup X_2 \sqcup I^k)$. Choose $k = [(n-1)/\text{codim } X_1]$. Then every map g of $Y = X_1 \sqcup X_2 \sqcup I^k$ into $\mathbb{R}^n$ can be approximated by a map $g' : Y \to \mathbb{R}^n$ such that $\dim_G g'(Y) \leqslant \max\{\dim_G X_2, k\} = \dim_G Y$.

Remark 3.6 implies that $Y \| T_G$ for every testing space T_G with $\|T_G\| + \dim_G Y < n$. By Theorem 1.3 we have that $X \sqcup I^k \| T_G$. By Remark 3.6, $X \sqcup I^k$ is approximable by a map into $\mathbb{R}^n$ not changing $\dim_G$. Therefore $\dim_G f'(X) \leqslant \dim_G f(X \sqcup I^k) \leqslant \max\{\dim_G X, k\}$. $\square$

Proof of the Realization Theorem 1.17. Suppose that X is a compactum of dimension $\leqslant n - 3$. For every Abelian group $G \in \sigma$ from the Bockstein basis σ, the set $F_G = \{f: X \to \mathbb{R}^n | \dim_G f(X) \leqslant \max\{\dim_G X, (n-1)/\text{codim } X\}\}$ is a dense G_δ-subset of $\mathscr{C}(X, \mathbb{R}^n)$, by the Approximation Theorem 1.16. Therefore the set $\mathscr{F} = (\bigcap\{F_G | G \in \sigma\}) \cap \{f \in \mathscr{C}(X, \mathbb{R}^n) | f \text{ light}\}$ is also dense in $\mathscr{C}(X, \mathbb{R}^n)$. Let $f \in \mathscr{F}$ and set $X' = f(X) \sqcup I^k$, where $k = (n-1)/\text{codim } X$. Then $\text{DIM } X' = \text{DIM}(f(X) \sqcup I^k) = \text{DIM } X \sqcup I^k$. □

Proof of the Disjoining Theorem 1.18. Consider $X' = X \sqcup I^{[(n-1)/\text{codim } X]}$. Since DIM X' is realizable in $\mathbb{R}^n$, the disjoining of X' and Y will be provided by the inequality $\dim(X' \times Y) < n$. Now,

$$\dim(X' \times Y) = \max\{\dim(X \times Y), \dim I^{[(n-1)/\text{codim } X]} \times Y\}.$$

So the condition $X' \| Y$ follows by the inequality $[(n-1)/\text{codim } X] + \dim Y < n$ which after a simple transformation reduces to $\text{codim } X \cdot \text{codim } Y \geqslant n$. Therefore the conditions of our theorem imply $X' \| Y$. But X being a subspace of X' is thus also disjoinable with Y. This completes the proof. □

4. Complementary dimensional functions

For any Bockstein function D with $\|D\| \leqslant n$ we define a function $[n-D] = \vee\{D' | \|D' \boxplus D\| \leqslant n\}$ which is called the *n-complementary* function of D. The following theorem summarizes the basic properties of the n-complementary functions:

Theorem 4.1.
(1) $\|[n-D] \boxplus D\| = n$;
(2) (*maximality*) $[n-D] \geqslant D'$, *for every* D' *with* $\|D' \boxplus D\| \leqslant n$;
(3) $[n-[n-D]] = D$; *and*
(4) $\|[n-D]\| = n - |D|$.

Proof. (1) By the definition, $\|[n-D] \boxplus D\| \leqslant n$. Hence $\|[n-D] \boxplus D\| = \|(\vee\{D' | \|D' \boxplus D\| \leqslant n\}) \boxplus D\| = \|\vee\{D' \boxplus D | \|D' \boxplus D\| \leqslant n\}\| \leqslant n$, by distributivity. But $D_1 = n - \|D\|$ satisfies the property that $\|D_1 \boxplus D\| = n$ and $[n-D] \geqslant D_1$ so $\|[n-D] \boxplus D\| \geqslant n$, by Theorem 2.10.

(2) This property is a straightforward consequence of the definition.

(3) $D \boxplus [n-D] \leqslant n$, hence by (2), i.e., the maximality of $[n-[n-D]]$, $D \leqslant [n-[n-D]]$. For any basic group G, consider the fundamental function $\Phi(G, n-D(G))$. Since $\|D\| - D(G) < n - D(G)$ it follows that $\|\Phi(G, n-D(G)) \boxplus D\| = n - D(G) + D(G) = n$, by Section 2. Hence, due to the maximality of $[n-D]$, one obtains that $[n-D] \geqslant \Phi(G, n-D(G))$. Therefore by Theorem 2.10, $n = \|[n-D] \boxplus [n-[n-D]]\| \geqslant \|[n-[n-D]] \boxplus \Phi(G, n-D(G))\|$. Since $\|[n-[n-D]] \boxplus [n-D]\| = n$ it follows by Lemma 2.9 that $n \geqslant |[n-D]| + \|[n-[n-D]]\| > \|[n-[n-D]]\|$. This, plus the inequality $D \leqslant [n-[n-D]]$ from above, imply that $n - \|[n-[n-D]]\| > D(G) - [n-[n-D]](G)$, hence $\|[n-[n-D]]\| - [n-[n-D]](G) < n - D(G)$. Therefore $\Phi(G, n-D(G))$ is a G-test function for the function $[n-[n-D]]$ by virtue of the Realization Theorem 2.3 and the Testing Space Theorem 1.1. This means that $\|[n-[n-D]] \boxplus$

$\|\Phi(G, n-D(G))\| = [n-[n-D]](G) + n - D(G)$. Hence $[n-[n-D]](G) \leqslant D(G)$. Since G is arbitrary, we have that $[n-[n-D]] \leqslant D$ so $[n-[n-D]] = D$.

(4) By Lemma 2.9, $|D_1| + \|D_2\| \leqslant \|D_1 \boxplus D_2\| \leqslant \|D_1\| + \|D_2\|$ and hence $|D| + \|[n-D]\| \leqslant \|[n-D] \boxplus D\| = n$, so $\|[n-D]\| \leqslant n - |D|$. Consider any basic group G such that $D(G) = |D|$. Then $\|\Phi(G, n-D(G)) \boxplus D\| = n$ hence $\Phi(G, n-D(G)) \leqslant [n-D]$. Now, $n - |D| = n - D(G) = \|\Phi(G, n-D(G))\| \leqslant \|[n-D]\|$. □

Proof of the Realization of the Complements Theorem 1.19. We shall use the notation DIM $X = D_X$. Suppose that $X \subset \mathbb{R}^n$ is a compactum of codimension $\geqslant 3$. We must show that there exists a compactum $Y \subset \mathbb{R}^n$ of complementary DIM-type, such that $\dim(X \times Y) = n-1$. Let us consider a compactum Y' which is the Bockstein $(n-1)$-complement of X. By the Approximation Lemma 1.10, X is DIM-approximable and by Štanko's theorem [16], we may assume that the image is tame in $\mathbb{R}^n$. By the Duality of Negligibility and Dimension Theorem 1.4 it follows that X is Y'-negligibly approximable. Therefore by the Mutual Negligibility Lemma 1.5, Y' is X-negligibly approximable. Consider the space $\mathscr{C}(Y', \mathbb{R}^n)$ of all maps of Y' to $\mathbb{R}^n$. Then there are two dense G_δ-subsets of $\mathscr{C}(Y', \mathbb{R}^n)$, the first one consists of the maps $f: Y' \to \mathbb{R}^n$ whose images are X-negligible, whereas the second set consists of the zero-dimensional maps (in fact, they are even regularly branched) which do not decrease *any* kind of dimension. Taking their intersection, we get a map $f: Y' \to \mathbb{R}^n$ with both of these properties, so DIM $f(Y') \geqslant$ DIM Y' and $f(Y')$ is X-negligible. By [8], the latter condition implies that $\dim(f(Y') \times X) < n$, so by the maximality property of the complement it follows that DIM $f(Y') \leqslant [(n-1) -$ DIM $X] =$ DIM Y', hence DIM $f(Y') =$ DIM Y'. □

Corollary 4.2. *For every compactum X such that* $\dim X \leqslant n-3$ *and* $\underline{\dim}\, X \geqslant 2$ *and every Bockstein $(n-1)$-complement X^* of X, the following statements are equivalent*:

(1) DIM X *embeds in* $\mathbb{R}^n$;

(2) DIM X^* *embeds in* $\mathbb{R}^n$; *and*

(3) $X \| X^*$ *in* $\mathbb{R}^n$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows by the Realization of the Complements Theorem 1.19 because the condition $\underline{\dim}\, X \geqslant 2$ implies that $\dim X^* = (n-1) - \underline{\dim}\, X \leqslant n-3$. Next, the implication (1) $\Rightarrow$ (3) follows by the Subset Disjoining Theorem 1.11 and the Dimension Type Determination Theorem 1.3. It remains to check the implication (3) $\Rightarrow$ (1). If $X \| X^*$ in $\mathbb{R}^n$ then by Theorem 3.1, $X \| Y$ in $\mathbb{R}^n$ for every compactum Y such that $\dim(X \times Y) < n$ since the latter implies that DIM $Y \leqslant$ DIM X^* by the maximality property of the complement. Now invoke Theorem 1.3. □

Note that Corollary 4.2 implies Theorem 1.20.

Theorem 4.3. *For every prime p, the following equalities hold*:

(1) $[n-D](\mathbb{Z}_{(p)}) = n - \min\{D(\mathbb{Q}), D(\mathbb{Z}_{p^\infty})\}$;

(2) $[n-D](\mathbb{Z}_{p^\infty}) = n - \min\{D(\mathbb{Z}_{(p)}), D(\mathbb{Z}_{p^\infty}) + 1\}$;

(3) $[n-D](\mathbb{Z}_p)=n-D(\mathbb{Z}_p)$; *and*
(4) $[n-D](\mathbb{Q})=n-D(\mathbb{Q})$.

Proof. By the Splitting Theorem 2.11, one obtains that $[n-D]=\bigvee\{\Phi(G,k) \mid \|\Phi(G,k)\boxplus D\|\leqslant n\}=\bigvee\{\Phi(G,k) \mid D(G)+k\leqslant n\}=\bigvee\{\Phi(G,n-D(G)) \mid G\in\{\mathbb{Z}_p,\mathbb{Z}_{(p)},\mathbb{Z}_{p^\infty} \mid p \text{ prime}\}\cup\{\mathbb{Q}\}\}$. We now check the formulae (1) and (2) (and leave (3) and (4) as an exercise).

(1)

$$\Phi(\mathbb{Z}_{(p)},n-D(\mathbb{Z}_{(p)}))(Z_{(p)})=n-D(\mathbb{Z}_{(p)}),$$

$$\Phi(\mathbb{Z}_p,n-D(\mathbb{Z}_p))(\mathbb{Z}_{(p)})=n-D(\mathbb{Z}_p),$$

$$\Phi(\mathbb{Z}_{p^\infty},n-D(\mathbb{Z}_{p^\infty}))(\mathbb{Z}_{(p)})=n-D(\mathbb{Z}_{p^\infty}),$$

$$\Phi(\mathbb{Q},n-D(\mathbb{Q}))(\mathbb{Z}_{(p)})=n-D(\mathbb{Q}),$$

$$\Phi(\mathbb{Z}_{(q)},n-D(\mathbb{Z}_{(q)}))(\mathbb{Z}_{(p)})=n-D(\mathbb{Z}_{(q)}),$$

$$\Phi(\mathbb{Z}_q,n-D(\mathbb{Z}_q))(\mathbb{Z}_{(p)})=1,$$

$$\Phi(\mathbb{Z}_{q^\infty},n-D(\mathbb{Z}_{q^\infty}))(\mathbb{Z}_{(p)})=1.$$

By the Bockstein inequalities, $D(\mathbb{Z}_{(q)})\geqslant D(\mathbb{Q})$, $D(\mathbb{Z}_{(p)})\geqslant D(\mathbb{Z}_p)\geqslant D(\mathbb{Z}_{p^\infty})$. Therefore the maximum is $\max\{n-D(\mathbb{Q}),\ n-D(\mathbb{Z}_{p^\infty})\}=n-\min\{D(\mathbb{Q}),\ D(\mathbb{Z}_{p^\infty})\}$.

(2)

$$\Phi(\mathbb{Z}_{(p)},n-D(\mathbb{Z}_{(p)}))(\mathbb{Z}_{p^\infty})=n-D(\mathbb{Z}_{(p)}),$$

$$\Phi(\mathbb{Z}_p,n-D(\mathbb{Z}_p))(\mathbb{Z}_{p^\infty})=n-D(\mathbb{Z}_p)-1,$$

$$\Phi(\mathbb{Z}_{p^\infty},n-D(\mathbb{Z}_{p^\infty}))(\mathbb{Z}_{p^\infty})=n-D(\mathbb{Z}_{p^\infty})-1,$$

$$\Phi(\mathbb{Q}_,n-D(\mathbb{Q}))(\mathbb{Z}_{p^\infty})=1,$$

$$\Phi(\mathbb{Z}_{(q)},n-D(\mathbb{Z}_{(q)}))(\mathbb{Z}_{p^\infty})=1,$$

$$\Phi(\mathbb{Z}_q,n-D(\mathbb{Z}_q))(\mathbb{Z}_{p^\infty})=1,$$

$$\Phi(\mathbb{Z}_{q^\infty},n-D(\mathbb{Z}_{q^\infty}))(\mathbb{Z}_{p^\infty})=1.$$

Table 3

	$\mathbb{Z}_{(p)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^\infty}$	$\mathbb{Q}$	$\mathbb{Z}_{(q)}$	$\mathbb{Z}_q$	$\mathbb{Z}_{q^\infty}$
$\Phi(\mathbb{Q},n)^*$	n	n	$n-1$	1	n	n	$n-1$
$\Phi(\mathbb{Z}_{(p)},n)^*$	1	1	1	1	n	n	$n-1$
$\Phi(\mathbb{Z}_p,n)^*$	n	1	1	n	n	n	n
$\Phi(\mathbb{Z}_{p^\infty},n)^*$	n	2	1	n	n	n	n

By the Bockstein inequalities, $D(\mathbb{Z}_p) \geqslant D(\mathbb{Z}_{p^\infty})$ so $\max\{n - D(\mathbb{Z}_{(p)}),\ n - D(\mathbb{Z}_{p^\infty}) - 1\} = n - \min\{D(\mathbb{Z}_{(p)}),\ D(\mathbb{Z}_{p^\infty}) + 1\}$. $\square$

Corollary 4.4. *For the singular case,* $[n - D](\mathbb{Z}_{p^\infty}) = n - D(\mathbb{Z}_{p^\infty}) - 1$.

Corollary 4.5. *For every prime p, the following formulae hold*:

(1) $(D \boxplus [n - D])(\mathbb{Z}_p) = n$;
(2) $(D \boxplus [n - D])(\mathbb{Z}_{(p)}) = n$;
(3) $(D \boxplus [n - D])(\mathbb{Z}_{p^\infty}) = n$ *if p is regular, and* $n - 1$ *if p is singular*;
(4) $(D \boxplus [n - D])(\mathbb{Q}) = n$.

In Table 3 we present the calculations for the fundamental compacta. For every fundamental function $\Phi(G, n)$, let $\Phi(G, n)^*$ denote its Bockstein $(n+1)$-complementary function.

We summarize the calculations above as follows:

Theorem 4.6.

(1) $\Phi(\mathbb{Q}, n)^* = \bigvee\{\Phi(\mathbb{Z}_p, n) \mid p \text{ prime}\}$;
(2) $\Phi(\mathbb{Z}_{(p)}, n)^* = \bigvee\{\Phi(Z_q, n) \mid p \neq q \text{ prime}\}$;
(3) $\Phi(Z_p, n)^* = \Phi(\mathbb{Q}, n) \vee \{\Phi(\mathbb{Z}_{(q)}, n) \mid p \neq q \text{ prime}\}$;
(4) $\Phi(\mathbb{Z}_{p^\infty}, n)^* = \Phi(\mathbb{Q}, n) \vee \Phi(\mathbb{Z}_p, 2) \vee \{\Phi(\mathbb{Z}_{(q)}, n) \mid p \neq q \text{ prime}\}$.

Theorem 4.7. *Suppose that* $\mathbb{R}^n$ *contains all compacta of the type* $F(\mathbb{Z}_p, n-3) \sqcup I^2$, *for all primes p. Then for every* $(n-3)$*-dimensional compactum X such that* $\underline{\dim}\, X \geqslant 2$, DIM *X can be realized in* $\mathbb{R}^n$.

Proof. Consider $\Omega = \coprod\{F(\mathbb{Z}_p, n-3) \sqcup I^2 \mid p \text{ prime}\}$. Note that Ω is embeddable in $\mathbb{R}^n$, by hypothesis. By the previous theorem and by Theorem 4.1(3), the Bockstein $(n-1)$-complement of Ω is $F(\mathbb{Q}, n-3) \sqcup I^2$. By Theorem 1.19, $F(\mathbb{Q}, n-3) \sqcup I^2$ embeds in $\mathbb{R}^n$. Similarly, we can prove that $F(\mathbb{Z}_{(p)}, n-3) \sqcup I^2$ embeds in $\mathbb{R}^n$ and Theorem 4.6(4) implies that $F(\mathbb{Z}_{p^\infty}, n-3) \sqcup I^2$ embeds in $\mathbb{R}^n$. The Splitting Theorem 1.14 then implies that every $(n-3)$-dimensional type with $\underline{\dim}\, X \geqslant 2$ is embeddable in $\mathbb{R}^n$. $\square$

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