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On intersections of compacta of complementary dimensions in Euclidean space

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Abstract

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A pair of maps $f: X\to\mathbb{R}^n$ and $g: Y\to\mathbb{R}^n$ of compacta X and Y into the Euclidean n-space is said to have a *stable intersection* if there exists $\varepsilon>0$ such that for any other pair of maps $f': X\to\mathbb{R}^n$ and $g': Y\to\mathbb{R}^n$, satisfying $\rho(f,f')<\varepsilon$ and $\rho(g,g')<\varepsilon$, it follows that $f'(X)\cap g'(Y)\neq\emptyset$. The main result of this paper is the following theorem: Let X and Y be compacta and let $n=\dim X+\dim Y$. Then there exists a pair of maps $f: X\to\mathbb{R}^n$ and $g: Y\to\mathbb{R}^n$ with stable intersection if and only if $\dim(X\times Y)=n$.

Keywords: Irrational compactum, stable intersection of maps, dimension of the product, Eilenberg-MacLane complex, essential maps.

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1. Introduction

Our work presented in this paper was inspired by the following question: *which compacta X have the property that every map $f: X\to\mathbb{R}^n$ of X into the Euclidean n-space can be approximated by an embedding*? If we denote by $C(X,\mathbb{R}^n)$ the space of all continuous maps of X into $\mathbb{R}^n$, equipped by the standard "sup-norm" metric $\rho(f,g)=\sup\{d(f(x),g(x))\mid x\in X\}$, and denote by $E(X,\mathbb{R}^n)$ the subspace of $C(X,\mathbb{R}^n)$, consisting of all embeddings of X into $\mathbb{R}^n$, then the question above can be restated as follows: *which compacta X have the property that $E(X,\mathbb{R}^n)$ is dense in $C(X,\mathbb{R}^n)$*?

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It follows by the classical Nöbeling-Pontrjagin embedding theorem that a *sufficient* condition for X is that $\dim X < \frac{1}{2}n$ [17]. In 1983, D. McCullough and L.R. Rubin published a theorem, asserting that the condition $\dim X < \frac{1}{2}n$ is also *necessary* in the case when n is *even* [15]. However, some years later, J. Krasinkiewicz and K. Lorentz found a gap in the proof of one of the crucial lemmas (but did not determine whether the main result of [15] was incorrect) [12]. Recently, McCullough and Rubin themselves found a counterexample to [15]: they constructed for each $n \geq 2$ an n-dimensional compactum X such that $E(X, \mathbb{R}^{2n})$ is dense in $C(X, \mathbb{R}^{2n})$ [16]. Their example turns out to possess the property that $\dim(X \times X) < 2n$. This led us to the following theorem:

Theorem 1.1. *Let X be a compactum and $n = \dim X$. Then $E(X, \mathbb{R}^n)$ is dense in $C(X, \mathbb{R}^n)$ if and only if* $\dim(X \times X) < n$.

Remark. If in Theorem 1.1, one omits the hypothesis that $n = 2 \dim X$, then the condition $\dim(X \times X) < n$ *still implies* that $E(X, \mathbb{R}^n)$ is dense in $C(X, \mathbb{R}^n)$. (Indeed, if $\dim(X \times X) < n$ then $n < 2 \dim X$ is impossible since by Bokštejn inequalities $\dim(X \times X)$ is either $2 \dim X$ or $2 \dim X - 1$. Consequently, $n > 2 \dim X$ and the assertion follows by the Pontrjagin-Nöbeling embedding theorem.) Note that our proof of the "if part" of Theorem 1.1 also does *not* use the hypothesis that $n = 2 \dim X$. (See also the remarks in Section 2, after Kiguradze's theorem.)

Theorem 1.1 will be proved using the following criterion for stability of intersections of maps of pairs of compacta of complementary dimensions into the Euclidean n-space:

Theorem 1.2. *Given compacta X and Y such that $n = \dim X + \dim Y$, there exists a pair of maps $f: X \to \mathbb{R}^n$ and $g: Y \to \mathbb{R}^n$ with stable intersections if and only if* $\dim(X \times Y) = n$.

A pair of maps $f: X \to S$ and $g: Y \to S$ of compacta X and Y into a metric space S is said to have a *stable intersection* if there exists $\varepsilon > 0$ such that for any other pair of maps $f': X \to S$ and $g': Y \to S$, satisfying the conditions that $\rho(f, f') < \varepsilon$ and $\rho(g, g') < \varepsilon$, it follows that $f'(X) \cap g'(Y) \neq \emptyset$.

The paper is organized as follows: in Section 2 we give a proof of the "if" part of Theorem 1.2 based on the theory of essential maps and irrational compacta. In Section 3 we prove some results about regular branched mappings which then play an important role in Sections 4 and 5 in which the proof of the "only if" part of Theorem 1.2 is presented, first for the higher dimensional case ($n > 4$), and then for the 4-dimensional case ($n = 4$). In Section 6 we prove Theorem 1.1.

The main results of this paper were announced in [5-8]. Theorems 1.1 and 1.2 also follow from the recent work of Krasinkiewicz [11] and Spież [18, 19]—they used techniques different from ours.

We wish to point out that besides the main results concerning the stability of intersections of compacta our paper contains several results of independent interest. In particular, the second section—on essential maps and irrational compacta, the third section—on regular branched maps, and the fifth section—on abelianizing the fundamental group via Casson's finger moves.

2. Essential maps and irrational compacta

A map $f: X \to B^n$ of a space X onto the closed n-ball B^n is said to be *essential* if there is no map $g: X \to \partial B^n$ with the property that $g|f^{-1}(\partial B^n) = f|f^{-1}(\partial B^n)$. Next, a point $x \in \operatorname{int} B^n$ is called a *stable value* of a surjective map $f: X \to B^n$ if there exists $\varepsilon > 0$ such that for every map $g: X \to B^n$ such that $\rho(f, g) < \varepsilon$ it follows that $x \in g(X)$. In other words, x cannot be avoided by $f(X)$ under small perturbations of f [1]. Finally, a map $f: X \to \mathbb{R}^n$ is said to have a *stable value at* $y \in \mathbb{R}^n$ if there is a closed n-ball $C \subset \mathbb{R}^n$, centered at y, such that the restriction $f|f^{-1}(C): f^{-1}(C) \to C$ has a stable value at y.

Clearly, if $f: X \to B^n$ is an essential map, then every point $x \in \operatorname{int} B^n$ is a stable value of f. Conversely, if some point $x \in \operatorname{int} B^n$ is a stable value of an onto map $f: X \to B^n$, then there exists a small n-ball $C^n \subset \operatorname{int} B^n$ such that $x \in \operatorname{int} C^n$ and $f|f^{-1}(C^n): f^{-1}(C^n) \to C^n$ is essential.

For every point $x \in \mathbb{R}^n$, let $r(x)$ be the number of rational coordinates of x. For every subset $X \subset \mathbb{R}^n$ let $r(X) = \max\{r(x) \mid x \in X\}$. Finally, for every $k \leq n$, let $R_k^n = \{x \in \mathbb{R}^n \mid r(x) \leq k\}$. It is well known that for every $k \leq n$, $\dim R_k^n = k$ and consequently that for every subset $X \subset \mathbb{R}^n$, $r(X) \geq \dim X$ [1, 9].

A subset $X \subset \mathbb{R}^n$ is said to be *irrational* if $r(X) = \dim X$. For example, if $X = \{p\}$ is irrational, where $p \in \mathbb{R}^n$, then $r(X) = \dim X = 0$, hence all coordinates of the point p are irrational. This shows our notion of irrationality is the correct generalization of the standard one for points. Using this new concept we rewrite the following two classical results of dimension theory [1, 20]:

Nöbeling–Hurewicz theorem. *Every bounded map $f: X \to \mathbb{R}^{2n+1}$ of a separable metric n-dimensional space X into $\mathbb{R}^{2n+1}$ can be approximated arbitrarily closely by a map $f': X \to \mathbb{R}^{2n+1}$ such that the closure of the image of f' is an irrational n-dimensional compactum.*

Štan'ko's theorem. *Every embedding of a compactum into $\mathbb{R}^n$ can be approximated arbitrarily closely by an embedding whose image is irrational.*

We shall need the following recent unpublished result of D.O. Kiguradze:

Theorem (Kiguradze). *Let X be a k-dimensional irrational compactum, lying in the Euclidean n-space $\mathbb{R}^n$, $k \leq n$. Then there exists a k-dimensional plane $L \subset \mathbb{R}^n$ and a* *closed, k-dimensional ball $C \subset L$ such that $p|(p^{-1}(C) \cap X): p^{-1}(C) \cap X \to C$ is an essential map, where $p: \mathbb{R}^n \to L$ is the orthogonal projection of $\mathbb{R}^n$ onto L.*

Remark. In the first version of our paper we used an earlier result of Kiguradze [10]; it was his theorem above *without* the irrationality hypothesis. Its proof was based on the classical Čogošvili's theorem [3] from the 1930s. With such more general result we could prove, using essentially the same argument as for the special case, the "if" part of Theorem 1.2 *without* the "$\dim X + \dim Y = n$" condition (and, consequently, Theorem 1.1 *without* the "$n = 2 \dim X$" condition). However, we have subsequently discovered that there is a serious gap in Čogošvili's proof [3] and so his theorem remains *unproved.*

With the author's kind permission, we have included here the complete proof of Kiguradze's theorem. We shall need two lemmas:

Lemma 2.1. *Given a compactum $X \subset \mathbb{R}^n$, let $p: \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$, be a surjective linear map such that $p|X: X \to \mathbb{R}^m$ has an unstable value $y \in p(X)$. Then for every $\varepsilon > 0$, there exists a map $g: X \to \mathbb{R}^n$ such that:*

(i) *$\rho(g, j) < \varepsilon$, where $j: X \to \mathbb{R}^n$ is the inclusion;*

(ii) *For every $x \in X$, $g(x) = x$, provided that $dist(x, p^{-1}(y)) \geq \frac{1}{2}\varepsilon$; and*

(iii) *$g(X) \cap p^{-1}(y) = \emptyset$.*

Proof. Without losing generality, we may assume that the point y is the origin $0 \in \mathbb{R}^n$ and that p is the projection of $\mathbb{R}^n$ onto the first m coordinates, i.e., $p(x_1, \ldots, x_m, \ldots, x_n) = (x_1, \ldots, x_m)$. Since y is by hypothesis an unstable value of $p|X: X \to \mathbb{R}^m$, the map $p|X$ inessentially covers the closed m-ball $C \subset \mathbb{R}^m$, centered at y and with radius $\frac{1}{2}\varepsilon$. Hence there exists a map $f: X \to \mathbb{R}^m$ such that $f(X \cap p^{-1}(C)) \subset \partial C$ and for every $x \in X \cap p^{-1}(\overline{\mathbb{R}^m \setminus C})$, $f(x) = p(x)$. Define now the desired map $g: X \to \mathbb{R}^n$ by $g(x) = (f_1(x), \ldots, f_m(x), x_{m+1}, \ldots, x_n)$ for every $x \in X$, where $x = (x_1, \ldots, x_n)$ and $f(x) = (f_1(x), \ldots, f_m(x), 0, \ldots, 0)$. It is easy now to verify that g satisfies properties (i)-(iii). □

For any k-dimensional plane $L \subset \mathbb{R}^n$, $k \leq n$, denote by $p_L: \mathbb{R}^n \to L^\perp$ the orthogonal projection of $\mathbb{R}^n$ to the $(n-k)$-dimensional plane $L^\perp$ orthogonal to L through the origin. Clearly, $p_L(L)$ is then just a point.

Lemma 2.2. *Suppose that a compactum $X \subset \mathbb{R}^n$ and a collection $L_1, \ldots, L_k \subset \mathbb{R}^n$ of planes satisfy the following conditions:*

(1) *For every $i \neq j$, $X \cap L_i \cap L_j = \emptyset$; and*

(2) *For every i, the projection $p_{L_i}|X: X \to L_i^\perp$ has an unstable point at $p_{L_i}(L_i)$.*

Then for every $\varepsilon > 0$, there exists a map $g: X \to \mathbb{R}^n$ such that

(i) *$\rho(g, j) < \varepsilon$, where $j: X \to \mathbb{R}^n$ is the inclusion; and*

(ii) *$g(X) \cap (\bigcup_{i=1}^k L_i) = \emptyset$.*

Proof. We may assume that $\varepsilon > 0$ is so small that for every $i \neq j$,

$$X \cap N_\varepsilon(L_i) \cap N_\varepsilon(L_j) = \emptyset, \tag{2.1}$$

where $N_\varepsilon(L_t) \subset \mathbb{R}^n$ is the open ε-neighborhood of L_t, $t \in \{1, \ldots, k\}$. Apply Lemma 2.1 to obtain for every $i \in \{1, \ldots, k\}$, a map $g_i : X \to \mathbb{R}^n$ such that

$$\rho(g_i, \text{incl}) < \varepsilon; \tag{2.2}$$

$$\text{for every } x \in X,\ g_i(x) = x \text{ if } \operatorname{dist}(x, L_i) \geq \varepsilon; \tag{2.3}$$

and

$$g_i(X) \cap L_i = \emptyset. \tag{2.4}$$

Define $g : X \to \mathbb{R}^n$ as follows: for every $x \in X$, let $g(x) = g_i(x)$ where L_i is the closest plane to x, i.e., $\operatorname{dist}(x, L_i) \leq \operatorname{dist}(x, L_j)$. Clearly, g is well defined. Indeed, if for some $i \neq j$, the planes L_i and L_j both have the minimal distance from x, then by (2.1) above, this distance must be at least ε. Consequently, by (2.3) above, $g_i(x) = x = g_j(x)$. Also, g is clearly continuous and it satisfies the required properties (i) and (ii), by (2.1)-(2.4). □

Proof of Kiguradze's theorem. Let $X \subset \mathbb{R}^n$ be a k-dimensional irrational compactum, $0 < k \leq n$. (The case $k = 0$ is trivial.) Then

(1) for every $x \in X$, $r(x) \leq k$

and there exists $\delta > 0$ such that

(2) X has no open δ-covering of order $\leq k$.

Choose a rational $\lambda > 0$ such that

(3) $\lambda < \frac{1}{2}\delta\sqrt{n}$

and define $C_\lambda = \{x \in \mathbb{R}^n \mid \text{for every } i, 0 \leq x_i \leq \lambda/n\}$. Then $d = \operatorname{diam} C_\lambda = \lambda/\sqrt{n}$ hence by (3),

(4) $d < \frac{1}{2}\delta$.

Consider a *Lebesgue lattice* $\Omega = \{\omega_i\}_{i \in \mathbb{N}}$ in $\mathbb{R}^n$, i.e., the covering of $\mathbb{R}^n$ by copies of the n-cube C_λ such that (i) for every i, $\omega_i = C_\lambda + r_i$, $r_i \in \mathbb{Q}^n$ (i.e., ω_i is obtained by a parallel translation of C_λ along some rational vector r_i); (ii) for every $i \neq j$, $\omega_i \cap \omega_j = \partial\omega_i \cap \partial\omega_j$; and (iii) the order of Ω is $n+1$.

For every $m \geq 1$, define

$$S_m = \{x \in \mathbb{R}^n \mid x \text{ belongs to at least } m \text{ different elements of } \Omega\}.$$

Then

(5) $S_m \subset \bigcup_{j=1}^{\infty} L_j^{n-m+1}$,

where $\{L_j^{n-m+1}\}_{j \in \mathbb{N}}$ is a discrete collection of $(n-m+1)$-dimensional planes in $\mathbb{R}^n$, each of them being the intersection of some $(m-1)$ hyperplanes $\{\Sigma_l^{n-1}\}_{l=1}^{m-1}$:

(6) $L_j^{n-m+1} = \bigcap_{l=1}^{m-1} \Sigma_l^{n-1}$

where for every $l \in \{1, \ldots, m-1\}$ and for some $t(l) \in \{1, \ldots, n\}$ and $q(l) \in \mathbb{Q}$

$$\Sigma_l^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{t(l)} = q_{t(l)}\}.$$

We now focus our attention on the case $m=k+1$. Note that for every $i\in\mathbb{N}$ and, every $y\in L_i^{n-k}$, $r(y)\geq k$, hence for every $i\neq j$ and every $z\in L_i^{n-k}\cap L_j^{n-k}$, $r(z)\geq k+1$, therefore it follows by (1) that for every $i\neq j$,

(7) $X\cap L_i^{n-k}\cap L_j^{n-k}=\emptyset$.

Let $\varepsilon_0=\frac{1}{2}(\delta-2d)$. It follows by (4) that $\varepsilon_0>0$.

Assertion. *For every map $g:X\to\mathbb{R}^n$ such that $\rho(g,\text{incl})<\varepsilon_0$, $g(X)\cap S_{k+1}\neq\emptyset$.*

Proof. Suppose, to the contrary, that the intersection of $g(X)$ and S_{k+1} were void. Then $g^{-1}(\Omega')$ would provide an open cover of X of order $\leq k$ and with mesh $\mu<2\varepsilon_0+2d=\delta$, where $\Omega'=\{\omega_i'\}_{i\in\mathbb{N}}$ is some family of open cubes $\omega_i'\supset\omega_i$ which would directly contradict (2). This proves the assertion.

The assertion implies that, in particular,

(8) $X\cap S_{k+1}\neq\emptyset$

and since X is compact, it intersects only finitely many $(n-k)$-dimensional planes $\{L_j^{n-k}\}_{j\in\mathbb{N}}$, say $L_{\sigma(1)}^{n-k},\ldots,L_{\sigma(t)}^{n-k}$. By discreteness, there is $\varepsilon_1\in(0,\varepsilon_0)$ such that $\text{dist}(X,L^{n-k})\geq\varepsilon_1$ for every $j\in\mathbb{N}-\{\sigma(1),\ldots,\sigma(t)\}$. Therefore for every $g:X\to\mathbb{R}^n$ such that $\rho(g,\text{incl})<\varepsilon_1$ it follows by the assertion that

(9) $g(X)\cap(\bigcup_{i=1}^{t}L_{\sigma(i)}^{n-k})\neq\emptyset$.

It now follows by (7), (9), and Lemma 2.2 that for some $i_0\in\{1,\ldots,t\}$, the projection $p_{L_{\sigma(i_0)}}|X:X\to L_{\sigma(i_0)}^{\perp}$ has a stable value at the point $p_{L_{\sigma(i_0)}}(L_{\sigma(i_0)})$. By letting $L=L_{\sigma(i_0)}^{\perp}$ and $p=p_{L_{\sigma(i_0)}}$ we thus complete the proof of Kiguradze's theorem. □

Theorem 2.3. *Let $i:X\to\mathbb{R}^n$ and $j:Y\to\mathbb{R}^m$ be embeddings of compacta X and Y and consider the corresponding product embedding $i\times j:X\times Y\to\mathbb{R}^n\times\mathbb{R}^m$. Suppose that $p:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^k$ is a linear map such that $p\circ(i\times j):X\times Y\to\mathbb{R}^k$ has a stable value. Then there exist maps $f:X\to\mathbb{R}^k$ and $g:Y\to\mathbb{R}^k$ with stable intersections.*

Proof. Let $h=i\times j$ and $Z=h(X\times Y)$. We may assume that $0\in Z$ and hence for every $(x,y)\in X\times Y$, $h(x,y)=(i(x),0)+(0,j(y))$. We may also assume that the stable value of p is the origin $0\in\mathbb{R}^k$. Therefore there exists a closed k-ball $C\subset\mathbb{R}^k$, centered at $0\in\mathbb{R}^k$ and with radius $\delta>0$, such that the restriction $p|p^{-1}(C)\cap Z:p^{-1}(C)\cap Z\to C$ has a stable value at 0. Consequently, there exists $\varepsilon>0$ such that for every map $p':p^{-1}(C)\cap Z\to C$ such that $\rho(p,p')<\varepsilon$ it follows that $0\in\text{Im }p'$, too.

Define maps $f:X\to\mathbb{R}^k$ and $g:Y\to\mathbb{R}^k$ as follows: $f(x)=(ph)(x,j^{-1}(0))$ and $g(y)=-(ph)(i^{-1}(0),y)$, for every $(x,y)\in X\times Y$. Note that then $(ph)(x,y)=f(x)-g(y)$, for every $(x,y)\in X\times Y$.

Assertion. *The maps f and g have a stable intersection.*

Proof. Let $f':X\to\mathbb{R}^k$ and $g':Y\to\mathbb{R}^k$ be any maps such that $\rho(f,f')<\frac{1}{2}\varepsilon$ and $\rho(g,g')<\frac{1}{2}\varepsilon$. Define $p':p^{-1}(C)\cap Z\to C$ to be the map $p'(h(x,y))=\varphi(f'(x)-g'(y))$, for every $h(x,y)\in p^{-1}(C)\cap Z$, where $\varphi:\mathbb{R}^k\to C$ is the map

$$\varphi(t)=\begin{cases}\delta\dfrac{t}{\|t\|} & \text{if } t\notin C,\\ t, & \text{if } t\in C.\end{cases}$$

Then p' is well defined and continuous, and for every $h(x, y) \in p^{-1}(C) \cap Z$, it follows that (since $\mathrm{Im}(p) \subset C$):

$$\begin{aligned} &\|(ph)(x, y) - (p'h)(x, y)\| \\ &\quad \leq \|(f(x) - g(y)) - (f'(x) - g'(y))\| \\ &\quad \leq \|f(x) - f'(x)\| + \|g(y) - g'(y)\| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

hence $\rho(p, p') < \varepsilon$. Therefore, $0 \in \mathrm{Im}\, p'$ hence for some $(x_0, y_0) \in X \times Y$, $\varphi(f'(x_0) - g'(y_0)) = p'(h(x_0, y_0)) = 0$ so $f'(x_0) = g'(y_0)$. It follows that $\mathrm{Im}\, f' \cap \mathrm{Im}\, g' \neq \emptyset$. This completes the proof of the assertion (and also of the theorem). □

Proof of the "if" part of Theorem 1.2. By the Nöbeling-Hurewicz theorem (stated above) there exist $m \in \mathbb{N}$ and embeddings $i: X \to \mathbb{R}^m$ and $j: Y \to \mathbb{R}^m$ such that

(1) $r(i(X)) = \dim X$ and $r(j(Y)) = \dim Y$.

Consider the product embedding $i \times j: X \times Y \to \mathbb{R}^{2m}$. Then for every $(x, y) \in X \times Y$, $r((i \times j)(x, y)) = r(i(x)) + r(j(y))$ thus

(2) $r((i \times j)(X \times Y)) = r(i(X)) + r(j(Y))$.

Since, by hypothesis, $\dim X + \dim Y = \dim(X \times Y)$, it follows by (1) and (2) that

(3) $r((i \times j)(X \times Y)) = \dim((i \times j)(X \times Y))$.

Apply Kiguradze's theorem to conclude that there exists an n-dimensional plane $L \subset \mathbb{R}^{2m}$ such that the restriction $p|(i \times j)(X \times Y): (i \times j)(X \times Y) \to L$ of the orthogonal projection $p: \mathbb{R}^{2m} \to L$ has a stable value. The proof is then completed by invoking Theorem 2.3. □

3. Regular branched maps

For every $k \geq 0$ and every map $f: X \to Z$, define the following subset $B_k(f) \subset Z$:

$$B_k(f) = \{z \in Z \mid \mathrm{card}\, f^{-1}(z) \geq k\}$$

and call the map f *regular branched* if for every $k \geq 0$,

$$\dim B_k(f) \leq k \cdot \dim X - (k-1) \cdot \dim Z.$$

For example, if $\dim X < \frac{1}{2} \dim Z$ and $f: X \to Z$ is regular branched, then f is an embedding, provided X is compact. Also note that regular branched maps do not raise dimension. Indeed, since $B_1(f) = f(X)$ it follows that $\dim f(X) \leq \dim X$. Denote by $R(X, Z)$ the subset of $C(X, Z)$, consisting of all regular branched maps from X to Z.

Theorem 3.1. *For every compactum X, the complement of $R(X, \mathbb{R}^n)$ in $C(X, \mathbb{R}^n)$ is a countable union of nowhere dense sets (i.e., $C(X, \mathbb{R}^n) - R(X, \mathbb{R}^n)$ has the first Baire category).*

Proof. Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable base of open sets in X. For every closed sets $X_1, \ldots, X_k \subset X$ and every closed set $D \subset \mathbb{R}^n$, the set

$$M(X_1, \ldots, X_k, D) = \left\{ f \in C(X, \mathbb{R}^n) \,\middle|\, \left(\bigcap_{i=1}^{k} f(X_i) \right) \cap D \neq \emptyset \right\}$$

is clearly closed in $C(X, \mathbb{R}^n)$.

Assertion 1. *If $X_1, \ldots, X_k$ are pairwise disjoint, then the set $M(X_1, \ldots, X_k, D)$ is nowhere dense in $C(X, \mathbb{R}^n)$, provided that* $\dim D < k(n - \dim X)$ *and D is a plane.*

Proof. First approximate $f(X_i)$'s by polyhedra of $\dim \leqslant \dim X$ and apply the standard general position to obtain, via small moves, that $\dim(\bigcap_{i=1}^{k} f(X_i)) \cap D = \dim D + k(\dim f(X_i) - n) < kn - k \cdot \dim X + k \cdot \dim X - kn = 0$. This clearly proves that $\operatorname{int} M(X_1, \ldots, X_k, D) = \emptyset$.

We shall call a $(k+1)$-tuple $(X_1, \ldots, X_k, D)$ *admissible*, if

(i) for every $i \neq j$, $X_i \cap X_j = \emptyset$;

(ii) for every i, X_i is the union of the closure of finitely many elements of the basis $\{U_i\}_{i\in\mathbb{N}}$; and

(iii) D is a plane of dimension $< k(n - \dim X)$, defined as the solution of a system of linear equations with rational coefficients. (We shall call such D a *rational plane.*)

Clearly, the set A of all admissible $(k+1)$-tuples $(X_1, \ldots, X_k, D)$ is countable. Therefore the set

$$S = \bigcup \{ M(X_1, \ldots, X_k, D) \,|\, (X_1, \ldots, X_k, D) \in A \}$$

is a countable union of subsets of $C(X, \mathbb{R}^n)$ which, by Assertion 1 above, are all nowhere dense in $C(X, \mathbb{R}^n)$, provided $\dim D < k(n - \dim X)$.

Choose now any $k \geqslant 0$, any $f \in C(X, \mathbb{R}^n) - S$, and any rational plane $D \subset \mathbb{R}^n$ of dimension $< k(n - \dim X)$.

Assertion 2. $B_k(f) \cap D = \emptyset$.

Proof. Suppose to the contrary, that there were some $t \in B_k(f) \cap D$. Then $\operatorname{card} f^{-1}(t) \geqslant k$. Take any k different points $x_1, \ldots, x_k \in f^{-1}(t)$ and find elements $U_1, \ldots, U_k \subset X$ of the base $\{U_i\}_{i\in\mathbb{N}}$, such that $(X_1, \ldots, X_k, D) \in A$, where $X_i = \operatorname{Cl} U_i$ and $x_i \in U_i$, for every $i \in \{1, \ldots, k\}$. It follows that $f \in M(X_1, \ldots, X_k, D)$ hence $f \in S$, a contradiction.

Consequently, $B_k(f) \subset \mathbb{R}^n - \bigcup\{D \,|\, D \subset \mathbb{R}^n$ rational plane of dimension $< k(n - \dim X)\}$. In particular, this implies that $r(B_k(f)) \leqslant n - k(n - \dim X)$ and so $\dim B_k(f) \leqslant n - k(n - \dim X) = k \dim X - (k-1)n$ and hence $f \in R(X, \mathbb{R}^n)$. This shows that $C(X, \mathbb{R}^n) - S \subset R(X, \mathbb{R}^n)$ so the complement of $R(X, \mathbb{R}^n)$ lies in S and thus is of the first Baire category. This completes the proof of the theorem. □

Corollary 3.2. *Suppose that X and Y are compacta such that* $\dim(X\times Y)<n$ *and* $2\dim X+\dim Y\leq 2n-2$. *Then the set*

$$N=\{f\in C(X,\mathbb{R}^n)\,|\,\dim(f(X)\times Y)<n \text{ and } \dim f(X)\leq \dim X\}$$

contains a dense G_δ-subset of $C(X,\mathbb{R}^n)$.

Proof. By Theorem 3.1, it suffices to prove that $R(X,\mathbb{R}^n)\subset N$. So choose any $f\in R(X,\mathbb{R}^n)$ and consider the map $f\times \mathrm{id}_Y: X\times Y\to f(X)\times Y$. Since f is regular branched, $\dim B_2(f\times \mathrm{id}_Y)=\dim(B_2(f)\times Y)\leq \dim B_2(f)+\dim Y\leq (2\dim X-n)+\dim Y\leq n-2$. Apply Freudenthal's theorem [9, 1] to conclude that $\dim(f(X)\times Y)\leq \max\{\dim B_2(f\times \mathrm{id}_Y)+1, \dim(X\times Y)\}<n$. (Note that the point-inverses of f are finite sets so, in particular, f is light.) □

4. The proof of the "only if" part of Theorem 1.2: the case $n>4$

We begin this section by a result which will play a key role in the proof of the case $n>4$ of the "only if" part of Theorem 1.2:

Theorem 4.1. *Let $n>4$ and suppose that $K\subset\mathbb{R}^n$ and X are any compacta satisfying the following conditions*:

(i) $\dim K+\dim X\leq n$;

(ii) $\dim(K\times X)<n$; *and*

(iii) $k=\mathrm{dem}\, K=\dim K\leq n-3$, *where* dem *is Štan'ko's embedding dimension* [20].

Then $C(X,\mathbb{R}^n-K)$ is dense in $C(X,\mathbb{R}^n)$.

Proof. We only need to consider the case when $\dim K+\dim X=n$. Choose an arbitrary $f\in C(X,\mathbb{R}^n)$ and any $\varepsilon>0$. We shall find $f'\in C(X,\mathbb{R}^n-K)$ such that $\rho(f,f')<\varepsilon$.

Assertion 1. *There exist a finite polyhedron $L\subset\mathbb{R}^n$ with a triangulation T and a map $g:X\to L$ such that*

(1) $l=\dim L=\dim X$;

(2) $T^{(l-1)}\cap K=\emptyset$, *where $T^{(l-1)}$ is the $(l-1)$-skeleton of T*;

(3) mesh $T<\frac{1}{4}\varepsilon$; *and*

(4) $\rho(f,g)<\frac{1}{2}\varepsilon$.

Proof. We may assume that $X\subset I^\infty=$ the Hilbert cube. Then $X=\varprojlim\{X_m, p_{m,m+1}\}$ where each $X_m\subset I^\infty$ is an l-dimensional finite polyhedron with a triangulation T_m of mesh $<2^{-m}$. By [14], there exist $m_0\in\mathbb{N}$ and a simplicial map $f_{m_0}: X_{m_0}\to\mathbb{R}^n$ such that $\rho(f,f_{m_0}p_{m_0})<\frac{1}{2}\varepsilon$, where $p_{m_0}: X\to X_{m_0}$ is the canonical projection.

Let $L=f_{m_0}(X_{m_0})$ and let T be a triangulation of L with respect to which $f_{m_0}: |T_{m_0}|\to |T|$ is simplicial. We may also assume that mesh $T<\frac{1}{4}\varepsilon$. And, since by hypotheses,

$k = \text{dem } K = \dim K = n - \dim X = n - l$ it follows by [20] that $T^{(l-1)}$ may be assumed to be missing the compactum K. This proves the assertion.

Let $\sigma \in T^{(l)}$ be any top dimensional simplex of T and choose any n-dimensional polyhedral cell $B \subset \mathbb{R}^n$ of diameter $< \frac{1}{2}\varepsilon$ such that $B \cap L$ is a regular neighborhood of σ in L (with respect to T) and $\sigma \subset \text{int } B$.

Assertion 2. $H_c^k(K \cap \text{int } B; \mathbb{Z}) \cong \Pi_{l-1}(\text{int } B - K)$.

Proof. Since for every $i < n$, $H_c^i(\text{int } B; \mathbb{Z}) \cong H^i(S^n; \mathbb{Z}) = 0$, it follows by the Alexander duality that for every $i < n$,

$$\check{H}^i(K \cap B, K \cap \partial B; \mathbb{Z}) \cong H_c^i(K \cap \text{int } B; \mathbb{Z}) \cong H_{n-i-1}^c(\text{int } B - K; \mathbb{Z}). \qquad (*)$$

Next, $\Pi_1(\text{int } B - K) = 0$ since by hypothesis, $\text{dem } K \leqslant n - 3$. Therefore by the Hurewicz theorem, the first nonzero homotopy group of int $B - K$ is isomorphic to the corresponding (integral) homology group. Now, it follows by (*) above that for every $i < l - 1$, $H_i^c(\text{int } B - K; \mathbb{Z}) = 0$ (since then $n - i - 1 > n - l = k$). Thus by Hurewicz, $\Pi_{l-1}(\text{int } B - K) \cong H_{l-1}(\text{int } B - K; \mathbb{Z}) \cong H_{l-1}^c(\text{int } B - K; \mathbb{Z})$ so the assertion follows by applying (*) for $i = n - l = k$.

Let $\Pi = \Pi_{l-1}(\text{int } B - K)$. We now proceed to prove the next fact about X:

Assertion 3. $c - \dim_\Pi X \leqslant l - 1$.

Proof. Take any closed subset $A \subset X$ of X and define $G = \check{H}^l(X, A; \check{H}^k(K \cap B, K \cap \partial B, \mathbb{Z}))$. Then by the Universal Coefficients theorem, $G \cong \check{H}^l(X, A; \mathbb{Z}) \otimes \check{H}^k(K \cap B, K \cap \partial B; \mathbb{Z})$ so G is a direct summand in the Künneth formula for the product $(X, A) \times (K \cap B, K \cap \partial B)$:

$$\begin{aligned} H &= \check{H}^n(X \times (K \cap B), (A \times (K \cap B)) \cup (X \times (K \cap \partial B)); \mathbb{Z}) \\ &\cong \check{H}^n((X, A) \times (K \cap B, K \cap \partial B); \mathbb{Z}) \\ &\cong \bigoplus_{i+j=n} \check{H}^i(X, A; \mathbb{Z}) \otimes \check{H}^j(K \cap B, K \cap \partial B; \mathbb{Z}). \end{aligned}$$

Now, $H \cong 0$ since by hypothesis, $\dim(X \times K) < n$. Therefore all the summands in the Künneth formula above must be zero, in particular, $G \cong 0$. Since $A \subset X$ was an arbitrary closed subset of X it follows by [4] and Assertion 2 that $c - \dim_\Pi X \leqslant l - 1$.

We shall now complete the proof of the theorem. Glue to int $B - K$ cells of dimension $\geqslant l + 1$ in order to build the Eilenberg-MacLane complex $\mathcal{K} = \mathcal{K}(\Pi, l-1)$ along with the canonical embedding $j : \text{int } B - K \to \mathcal{K}$. (Clearly, $j(\text{int } B - K)$ then contains the l-skeleton of $\mathcal{K}$. Consider the map

$$j \circ (g | g^{-1}(\partial\sigma)) : g^{-1}(\partial\sigma) \to \mathcal{K}.$$

By Assertion 3, it extends to a map $\bar{g}: g^{-1}(\sigma) \to \mathscr{K}^{(l)} \subset \text{int } B - K$. (Since $\dim g^{-1}(\sigma) \leq l$ and X is compact, we may assume that $\text{Im } \bar{g}$ lies in some finite subcomplex of $\mathscr{K}$ and by the cellular approximation theorem, it can be pushed into the l-skeleton of $\mathscr{K}$, i.e., into $\text{int } B - K$.)

By repeating the procedure described above for every l-simplex $\sigma \in T^{(l)}$, we obtain the desired ε-approximation $f' \in C(X, \mathbb{R}^n - K)$ of f. □

We are now in a position to give a proof of the higher dimensional case, i.e., $n > 4$, of the "only if" part of Theorem 1.2: We may assume, without losing generality, that $\dim Y \leq \dim X \leq n-2$.

Therefore, by Corollary 3.2, there exists a map $g' \in C(Y, \mathbb{R}^n)$ such that $\dim(X \times g'(Y)) < n$, $\rho(g, g') < \varepsilon$ and $\dim g'(Y) \leq \dim Y$.

Since $n > 4$, it follows that $\dim Y \leq n-3$ hence $\dim g'(Y) \leq n-3$. By Štan'ko's approximation theorem [20], we may assume that $\text{dem } g'(Y) = \dim g'(Y)$. Apply now Theorem 4.1 for $K = g'(Y)$ to obtain a map $f' \in C(X, \mathbb{R}^n - K)$ such that $\rho(f, f') < \varepsilon$. As a consequence, $f'(X) \cap g'(Y) = \emptyset$, as asserted.

5. The proof of the "only if" part of Theorem 1.2: the case $n = 4$

As in the preceding section we shall again begin by a result which will be of key importance in the proof of the 4-dimensional case of the "only if" part of Theorem 1.2. The trouble with this case lies in the fact that the fundamental group is the only homotopy group which can fail to be abelian. Therefore we need a result to the effect that under certain conditions Π_1 can be effectively abelianized.

Theorem 5.1. *Let $Z \subset M$ be a 2-dimensional compactum in a simply connected PL 4-manifold with boundary M. Then for every pair of compact polyhedra (P, Q) where* $\dim P \leq 2$ *and every PL immersion $f: (P, Q) \to (M - Z, \partial M - Z)$ there is a PL homotopy $G: P \times I \to M - Z$ such that:*

(i) $G_0 = f$;

(ii) *For every $t \in I$, $G_t: P \to M - Z$ is an immersion and $G_t | Q = f | Q$;*

(iii) *$G_1: P \to M - Z$ is a general position map (hence has only double singular points); and*

(iv) *$M - G_1(P)$ has an abelian fundamental group.*

Proof. Let T be a triangulation of the polyhedron $f(P)$. For every 2-simplex $\sigma \in T^{(2)}$ let $z_\sigma: S^1 \to \mathring{N}$ be a meridian of σ with a fixed orientation (i.e., z_σ is an embedding and $z_\sigma(S^1)$ bounds some PL embedded 2-disk D_σ in M such that σ and D_σ intersect transversely in an interior point), where $N = M - (f(P) \cup Z)$. We may assume that for every $\sigma \neq \sigma' \in T^{(2)}$, $z_\sigma(S^1) \cap z_{\sigma'}(S^1) = \emptyset$.

For every $\sigma \in T^{(2)}$, fix some point $x_\sigma \in z_\sigma(S^1)$. Let $\tilde{Z} = \bigcup \{z_\sigma(S^1) \mid \sigma \in T^{(2)}\}$. Choose a base point $y_0 \in \operatorname{int} N$ of $\Pi_1(M - f(P))$. For every $\sigma \in T^{(2)}$ and every path $u : I \to \operatorname{int} N$ from $u(0) = y_0$ to $u(1) = x_\sigma$ define a loop $z_\sigma^u : I \to \operatorname{int} N$, given by $z_\sigma^u = u^{-1} * z_\sigma * u$, where $*$ denotes the usual "join" product of paths.

Assertion 1. *The loops* $\{z_\sigma^u\}$ *generate* $\Pi_1(M - f(P))$.

Proof. Pick an arbitrary element $[\alpha] \in \Pi_1(M - f(P), y_0)$ and represent it by a PL embedding $\alpha : (S^1, 1) \to (M - f(P), y_0)$. By general position we may assume that $\alpha(S^1) \cap \tilde{Z} = \emptyset$. Since by hypothesis, M is 1-connected, α extends to a PL immersion $\bar{\alpha} : B^2 \to M$. Again, by general position, we may assume that $\bar{\alpha}(B^2) \cap T^{(1)} = \emptyset$ and that $\bar{\alpha}(B^2)$ intersects every 2-simplex $\sigma \in T^{(2)}$ transversely, at a finite set of interior points $A_\sigma = \{p_1^\sigma, \ldots, p_{n_\sigma}^\sigma\} \subset \operatorname{int} \sigma$. We may also assume that the singular set of $\bar{\alpha}$ consists of finitely many double singular points and that they all miss $f(P)$.

Consider the preimage of A_σ in B^2, i.e., $C_\sigma = \{q_1^\sigma, \ldots q_{n_\sigma}^\sigma\} \subset \operatorname{int} B^2$, where for every $i \in \{1, \ldots, n_\sigma\}$, $q_i^\sigma = \bar{\alpha}^{-1}(p_i^\sigma)$. There exists a collection $\{J_{\sigma,i} \mid \sigma \in T^{(2)}, i \in \{1, \ldots, n_\sigma\}\}$ of pairwise disjoint simple closed curves in int B^2 such that $[\bar{\alpha} \mid J_{\sigma,i}] = [z_\sigma^{\pm 1}] \in \Pi_1(M - f(P))$ and for every σ and i, q_i^σ lies in the interior of the disk $E_{\sigma,i}$ which is bounded by $J_{\sigma,i}$ in B^2 and $E_{\sigma,i} \cap E_{\sigma',j} = \emptyset$ whenever $\sigma \neq \sigma'$ or $i \neq j$.

For every σ, i, let $\gamma_{\sigma,i} : I \to B^2 - \bigcup_{\sigma,i} \operatorname{int} E_{\sigma,i}$ be an arc from $\gamma_{\sigma,i}(0) = 1$ to a point $t_{\sigma,i} \in J_{\sigma,i}$, such that for every $\sigma \neq \sigma'$ or $i \neq j$, $\operatorname{Im} \gamma_{\sigma,i} \cap \operatorname{Im} \gamma_{\sigma',j} = \{1\}$. The deformation retraction of B^2 onto $Y = \bigcup \{E_{\sigma,i} \cup \operatorname{Im} \gamma_{\sigma,i}\}$ followed by the map $\bar{\alpha} \mid Y : Y \to M$ is a homotopy (based at y_0) from $[\alpha]$ to a product of finitely many elements of the type z_σ^u or their inverses, namely, $(\bar{\alpha} \circ \gamma_{\sigma,i}^{-1}) * (\bar{\alpha} \mid J_{\sigma,i}) * (\bar{\alpha} \circ \gamma_{\sigma,i})$. This proves the assertion.

Now, $\Pi_1(M - f(P))$ is finitely generated and we can easily get a finite set $\{z_{\sigma_1}^{u_1}, \ldots, z_{\sigma_m}^{u_m}\}$ of generators. We may also assume that for every $i \neq j$, $u_i((0,1]) \cap u_j((0,1]) = \emptyset$. In order to kill the commutators of $\Pi_1(M - f(P))$ it clearly suffices to kill the commutators of the type $[z_{\sigma_i}^{u_i}, z_{\sigma_j}^{u_j}]$ for all $i, j \in \{1, \ldots, m\}$. This is achieved by pushing $f(P)$ along the paths u_i and u_j (the so-called Casson "finger" moves [2]), until the "fingers" intersect transversely at two points p and q, near the point y_0, bounding a Whitney 2-disk W. Denote this push (a regular homotopy) by $G_t : P \times I \to M - Z$, where $G_0 = f$.

Obviously, $\Pi_1(M - f(P)) = \Pi_1(M - (G_1(P) \cup W))$. Now do the "anti-Whitney" move, i.e., remove W. It is easy to verify that the following claim holds:

Assertion 2. $\Pi_1(M - (G_1(P) \cup W)) \to \Pi_1(M - G_1(P))$ *is an epimorphism.*

Proof. Indeed, let $[\varphi] \in \Pi_1(M - G_1(P))$. Then by general position, the map $\varphi : S^1 \to M - G_1(P)$ can be homotoped to a map $\varphi' : S^1 \to M - (G_1(P) \cup W)$, where the homotopy has a support inside $M - G_1(P)$.

As a consequence, every element of $\Pi_1(M - G_1(P))$ is also generated by the elements $\{z_{\sigma_1}^{u_1}, \ldots, z_{\sigma_m}^{u_m}\}$.

After removing W we detect the "characteristic" torus $S^1 \times S^1$ in the complement of the fingers, i.e., in $M - G_1(P)$. Now, $z_{\sigma_i}^{u_i}$ and $z_{\sigma_j}^{u_j}$ can be homotoped onto the generators of $H_1(S^1 \times S^1; \mathbb{Z})$ and hence they will commute in $M - G_1(P)$. Since the paths u_i and u_j were chosen to miss Z, so will the homotopy G_t. This completes the proof of the theorem. □

We can now begin the proof of the "only if" part of the 4-dimensional case of Theorem 1.2: Clearly, the only nontrivial case is when $\dim X = \dim Y = 2$. We may assume that $X, Y \subset I^\infty$. Then $X = \varprojlim\{X_m, p_{m,m+1}\}$ and $Y = \varprojlim\{Y_m, q_{m,m+1}\}$ where every X_m (respectively Y_m) is a 2-dimensional compact polyhedron in I^∞, equipped with a finite triangulation S_m (respectively T_m) of mesh $<2^{-m}$. By [14], there exist $m_0 \in \mathbb{N}$ and a pair of simplicial maps $f_{m_0}: X_{m_0} \to \mathbb{R}^4$ and $g_{m_0}: Y_{m_0} \to \mathbb{R}^4$ such that

(1) $\rho(\bar{f}, f) < \frac{1}{4}\varepsilon$, where $\bar{f} = f_{m_0} p_{m_0}^\infty$ and $p_{m_0}^\infty : X \to X_{m_0}$ is the canonical projection; and

(2) $\rho(\bar{g}, g) < \frac{1}{2}\varepsilon$, where $\bar{g} = g_{m_0} q_{m_0}^\infty$ and $q_{m_0}^\infty: Y \to Y_{m_0}$ is the canonical projection.
Let $K = f_{m_0}(X_{m_0})$ and $L = g_{m_0}(Y_{m_0})$ and choose some finite triangulation S (respectively T) of K (respectively L) with respect to which f_{m_0} (respectively g_{m_0}) is simplicial. We may also take the mesh of S and T to be $<\frac{1}{4}\varepsilon$. Now,

(3) $\dim K \leq \dim X$; and

(4) $\dim L \leq \dim Y$,

so by general position,

(5) $f_{m_0} \amalg g_{m_0}: |S_{m_0}| \amalg |T_{m_0}| \to |S| \cup |T|$ is a general position map (hence has only finitely many double singular points); and

(6) K and L intersect transversely, $K \cap L = \{t_1, \ldots, t_r\}$, where for every i, $f_{m_0}^{-1}(t_i) = \hat{\sigma}_i$ and $g_{m_0}^{-1}(t_i) = \hat{\tau}_i$, for some $(\sigma_i, \tau_i) \in S^{(2)} \times T^{(2)}$ and $\hat{\sigma}_i$ (respectively $\hat{\tau}_i$) is the barycenter of σ_i (respectively τ_i).

Therefore there exist pairwise disjoint PL 4-balls $B_1^4, \ldots, B_r^4 \subset \mathbb{R}^4$ such that for every $i \in \{1, \ldots, r\}$,

(7) $t_i \in \operatorname{int} B_i$;

(8) $f_{m_0}^{-1}(B_i) \subset \operatorname{int} \sigma_i$;

(9) $g_{m_0}^{-1}(B_i) \subset \operatorname{int} \tau_i$;

(10) $\operatorname{diam} B_i < \frac{1}{4}\varepsilon$; and

(11) f_{m_0} and g_{m_0} are one-to-one over ∂B_i.

For every $i \in \{1, \ldots, r\}$, let $C_i = \bar{f}^{-1}(B_i)$ and $D_i = \bar{g}^{-1}(B_i')$ where $B_i' \subset \operatorname{int} B_i$ is a closed PL 4-cell such that $t_i \in \operatorname{int} B_i'$. It follows by (11) and by general position, that

(12) $\dim((\bar{f}(C_i) \cap \partial B_i) \times D_i) < 4$;

hence there exists, by Corollary 3.2, for every i, a map $\tilde{f}_i: C_i \to B_i$ such that

(13) $\tilde{f}_i | \bar{f}^{-1}(\partial B_i) = \bar{f} | \bar{f}^{-1}(\partial B_i)$ and $\tilde{f}_i^{-1}(\partial B_i) = \bar{f}^{-1}(\partial B_i)$; and

(14) $\dim(\tilde{f}_i(C_i) \times D_i) < 4$ and $\tilde{f}_i(C_i) \cap Q_i = \emptyset$, where $Q_i = \bar{g}(Y) \cap \overline{(B_i - B_i')}$.

Let $\tilde{K}_i = \tilde{f}_i(C_i)$ and $\partial \tilde{K}_i = \tilde{K}_i \cap \partial B_i$. (Note that $\tilde{K}_i$ is not a polyhedron because of (14).) Define $W_i = \operatorname{int} B_i - \tilde{K}_i$ and $G_i = H_1(W_i; \mathbb{Z})$. Let $Y_i = \bar{g}^{-1}(\partial B_i')$. Then $\bar{g}(Y_i) \subset W_i$.

Assertion. $c-\dim_{G_i} D_i \leq 1$.

Proof. By the Alexander duality,

(15) $G_i \cong \check{H}^2(\tilde{K}_i, \partial \tilde{K}_i; \mathbb{Z})$.

Let $E_i \subset D_i$ be any closed subset of D_i. Apply the Künneth formula and the Universal Coefficients theorem to compute

$$(16) \quad H = \check{H}^4(D_i, E_i) \times (\tilde{K}_i, \partial \tilde{K}_i); \mathbb{Z})$$

$$\cong \bigoplus_{j=0}^{4} \check{H}^{4-j}(D_i, E_i; \mathbb{Z}) \otimes \check{H}^j(\tilde{K}_i, \partial \tilde{K}_i; \mathbb{Z})$$

$$\cong \bigoplus_{j=0}^{4} H_c^{4-j}(D_i, E_i; \check{H}^j(\tilde{K}_i, \partial \tilde{K}_i; \mathbb{Z})).$$

By (14) and [4] (see also [13]), it follows that $H \cong 0$ hence all direct summands in (16) are zero, in particular

(17) $H_c^2(D_i, E_i; \check{H}^2(\tilde{K}_i, \partial \tilde{K}_i; \mathbb{Z})) \cong 0$

therefore by (15),

(18) $H_c^2(D_i, E_i; G_i) \cong 0$.

Since E_i was arbitrary, the assertion now follows by Cohen's theorem [4, 13].

We now kill the commutators of $\Pi_1(W_i)$ by glueing 2-cells $\{A_t^i\}$ to W_i so that for every t,

(19) $\partial A_t^i \subset W_i^{(1)}$, where $W_i^{(1)}$ is the 1-skeleton of a fixed triangulation of W_i;

(20) $[\partial A_t^i] = [0] \in G_i$; and

(21) $\{\partial A_t^i\}$ generate $\mathrm{Ker}[\Pi_1(W_i) \to G_i]$.

Let $W_i^* = W_i \cup (\bigcup A_t^i)$. Then by (21),

(22) $\Pi_1(W_i^*) \cong H_1(W_i^*; \mathbb{Z}) = G_i$.

By attaching to W_i^* cells of dimension ≥ 3, we build the Eilenberg-MacLane complex $\mathcal{K}_i = \mathcal{K}(G_i, 1)$ together with the canonical embedding $\varphi_i : W_i^* \to \mathcal{K}_i$. For practical purposes we shall identify $\varphi_i(W_i^*)$ with W_i^*. Clearly, W_i^* contains the 2-skeleton of $\mathcal{K}_i$.

Consider the map $\psi_i = \varphi_i(\bar{g} | Y_i) : Y_i \to \mathcal{K}_i$. By the assertion above, ψ_i extends to a map $\bar{\psi}_i : D_i \to \mathcal{K}_i^{(2)}$. (Since D_i is compact $\bar{\psi}_i(D_i)$ lies in some finite subcomplex of $\mathcal{K}_i$. Furthermore, by the cellular approximation theorem, we can homotop $\bar{\psi}_i(D_i)$ into the 2-skeleton of $\mathcal{K}_i$.) Hence we may assume also that

(23) $\bar{\psi}_i(D_i) \subset W_i^*$.

By the compactness, $\bar{\psi}_i(D_i)$ intersects at most finitely many cells A_t^i of W_i^*, hence

(24) $\bar{\psi}_i(D_i) \subset W_i \cup A_{t_1}^i \cup \cdots \cup A_{t_{r(i)}}^i$.

By (14), (20) and (24), there is an open neighborhood $U_i \subset B_i$ of $\tilde{K}_i$ such that for every $s \in \{t_1, \ldots, t_{r(i)}\}$,

(25) ∂A_s^i is nullhomologous (over $\mathbb{Z}$) in $B_i - U_i$; and

(26) $U_i \cap \partial \bar{\psi}_i(D_i) = \emptyset$.

Since $\tilde{K}_i = \varprojlim \{P_m^i, \omega_{m,m+1}\}$ where every $P_m^i \subset B_i$ is a compact 2-dimensional polyhedron, there exists by [14], an integer $m_0 \in \mathbb{N}$ such that $P_{m_0}^i \subset U_i$. We may assume that $\omega^{\infty}_{m_0}|_{\tilde{K}_i \cap \partial B_i} = \text{incl}$. Apply now Theorem 5.1 and use (24)-(26) to obtain a regular homotopy $F^i_t : P^i_{m_0} \to B_i - (Q_i \cup \bar{\psi}_i(D_i))$ such that

(27) $F^i_0 =$ the inclusion of $P^i_{m_0}$ into U_i;

(28) $\Pi_1(B_i - F^i_1(P^i_{m_0}))$ is abelian; and

(29) for every $s \in \{t_1, \ldots, t_{r(i)}\}$, ∂A^i_s is nullhomologous (over $\mathbb{Z}$) in $B_i - F^i_1(P^i_{m_0})$.

Let $f^*_i : C_i \to B_i$ be given by $f^*_i = F^i_1 \omega^{\infty}_{m_0} \tilde{f}_i$, where $\omega^{\infty}_{m_0} : \tilde{K}_i \to P^i_{m_0}$ is the canonical projection. Then by (28) and (29),

(30) for every $s \in \{t_1, \ldots, t_{r(i)}\}$, ∂A^i_s is nullhomotopic in $\tilde{W}_i = B_i - f^*_i(C_i)$,

so there exists a retraction $h_i : \tilde{W}_i \cup A^i_{t_1} \cup \cdots \cup A^i_{t_{r(i)}} \to \tilde{W}_i$. Let $g^*_i : D_i \to B_i$ be given by $g^*_i = h_i \bar{\psi}_i$. Then

(31) $g^*_i \mid Y_i = \bar{g} \mid Y_i$; and

(32) $f^*_i(C_i) \cap (Q_i \cup g^*_i(D_i)) = \emptyset$.

Finally, define the maps $f' : X \to \mathbb{R}^4$ and $g' : Y \to \mathbb{R}^4$ by

$$f'(x) = \begin{cases} f^*_i(x), & \text{if } x \in C_i \text{ for some } i \in \{1, \ldots, r\}, \\ \bar{f}(x), & \text{otherwise,} \end{cases}$$

and

$$g'(y) = \begin{cases} g^*_i(y), & \text{if } y \in D_i \text{ for some } i \in \{1, \ldots, r\}, \\ \bar{g}(y), & \text{otherwise.} \end{cases}$$

It follows by (1), (2), (6), (10), (31), and (32) that $\rho(f, f') < \varepsilon$, $\rho(g, g') < \varepsilon$, and $f'(X) \cap g'(Y) = \emptyset$. This completes the proof of the "only if" part of the 4-dimensional case of Theorem 1.2.

6. The proof of Theorem 1.1

Suppose first, that $\dim(X \times X) < n$. Cover X by a countable family $\{(\bar{U}_i, \bar{V}_i)\}_{i \in \mathbb{N}}$ of pairs of the closures of open sets $U_i, V_i \subset X$ such that for every i, $\bar{U}_i \cap \bar{V}_i = \emptyset$ and for every $x \neq y$, $(x, y) \in X \times X$, there exists a pair $(\bar{U}_k, \bar{V}_k)$ such that $(x, y) \in (U_k, V_k)$. Define for every i, $C_i = \{h \in C(X, \mathbb{R}^n) \mid h(\bar{U}_i) \cap h(\bar{V}_i) = \emptyset\}$. Clearly, C_i is open in $C(X, \mathbb{R}^n)$. That C_i is also dense in $C(X, \mathbb{R}^n)$ follows by the "only if" part of Theorem 1.2 (applied to $X = \bar{U}_i$, $Y = \bar{V}_i$, $f = h \mid \bar{U}_i$, and $g = h \mid \bar{V}_i$). Therefore, by the Baire category theorem, $E(X, \mathbb{R}^n) = \bigcap_{i=1}^{\infty} C_i$ is dense in $C(X, \mathbb{R}^n)$.

Conversely, suppose now that $\dim(X \times X) \geq n$, hence $\dim(X \times X) = n$ since $\dim(X \times X) \leq 2 \dim X = n$, by hypothesis. We shall first verify the following claim:

Assertion. *There exists a pair of disjoint compacta* $X_1, X_2 \subset X$ *such that* $\dim(X_1 \times X_2) = n$.

Proof. Let $B = \{U_i\}_{i \in \mathbb{N}}$ be a countable basis of open sets in X. Thus for every $x_1 \neq x_2 \in X$, there exist $U_{i_1}, U_{i_2} \in B$ such that $x_k \in U_{i_k}$, $k = 1, 2$ and $\bar{U}_{i_1} \cap \bar{U}_{i_2} = \emptyset$. Then we have the following equality:

$$(X \times X) - \Delta = \bigcup \{\bar{U}_i \times \bar{U}_j \mid U_i, U_j \in B, \bar{U}_i \cap \bar{U}_j = \emptyset\}, \qquad (*)$$

where $\Delta = \{(x, x) | x \in X\}$ is the *diagonal* of X. Indeed, for every $(x_1, x_2) \in (X \times X) - \Delta$, there exist disjoint open sets $U_{i_1}, U_{i_2} \subset X$ such that $\bar{U}_{i_1} \cap \bar{U}_{i_2} = \emptyset$ and $x_k \in U_{i_k}$ for $k = 1, 2$. The other inclusion in (*) is obvious since $\bar{U}_i \cap \bar{U}_j = \emptyset$.

Therefore, by [9], $\dim(X \times X) = \max\{\dim \Delta, \dim(\bar{U}_i \times \bar{U}_j) | U_i, U_j \in B\}$. Hence there exist elements $U_{i_1}, U_{i_2} \in B$ such that $\bar{U}_{i_1} \cap \bar{U}_{i_2} = \emptyset$ and $\dim(X \times X) = \dim(\bar{U}_{i_1} \times \bar{U}_{i_2})$ since $\dim \Delta = \dim X < \dim(X \times X)$. Let $X_k = \bar{U}_{i_k}$ for $k = 1, 2$. This proves the assertion above.

Apply now the "if part" of Theorem 1.2 for $X = X_1$ and $Y = X_2$, to obtain a pair of maps $f_1 : X_1 \to \mathbb{R}^n$ and $f_2 : X_2 \to \mathbb{R}^n$ with a stable intersection. This completes the proof of Theorem 1.1 since we can extend the map $f_1 \amalg f_2 : X_1 \cup X_2 \to \mathbb{R}^n$ over all of X (recall $\mathbb{R}^n$ is an AE) thus obtaining a map $f \in C(X, \mathbb{R}^n)$ which cannot be approximated by an embedding. □

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