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THE URYSOHN-MENGER SUM FORMULA: AN EXTENSION OF THE DYDAK-WALSH THEOREM TO DIMENSION ONE

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Abstract

Let X be a finite-dimensional separable metric space, presented as a disjoint union of subsets, $X = A \cup B$. We prove the following theorem: For every prime p, $\text{c-dim}_{\mathbb{Z}_p} X \leq \text{c-dim}_{\mathbb{Z}_p} A + \text{c-dim}_{\mathbb{Z}_p} B + 1$. This improves upon some of the earlier work by Dydak and Walsh.

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1. Introduction

Cohomological dimension theory (of separable metrizable spaces) is in many respects parallel to the classical (Lebesgue covering) dimension theory (see, for example, the survey by Kuz'minov [13]). This is particulary true for the cohomological dimension $\text{c-dim}_{\mathbb{Z}}$ over the ring of integers $\mathbb{Z}$. The basic reason for that is the equivalence of $\text{c-dim}_{\mathbb{Z}} X$ and $\dim X$ for finite dimensional spaces X, a fact which was established already in the 1930's by Aleksandrov, the founder of (co)homological dimension theory. However, in general, $\text{c-dim}_{\mathbb{Z}} X$ and $\dim X$ need not be the same – there exist infinite dimensional spaces X of finite cohomological dimension over $\mathbb{Z}$, the first such example having been found by Dranišnikov in 1987 [1, 2]. This result has had other important implications, since it provided dimension raising cell-like maps, thus solving another outstanding problem for many years in geometric (Bing) topology (see, for example, the survey by Mitchell and Repovš [14]).

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One of the classical results of the Lebesgue covering dimension theory is the Urysohn-Menger sum formula [11]: it asserts that for all subsets A and B of X such that $X = A \cup B$,

$$\dim X \leq \dim A + \dim B + 1. \tag{1}$$

It was only very recently that (1) was verified for the cohomological dimension over $\mathbb{Z}$: in 1992 Rubin [15] proved that

$$\text{c-dim}_{\mathbb{Z}} X \leq \text{c-dim}_{\mathbb{Z}} A + \text{c-dim}_{\mathbb{Z}} B + 1. \tag{2}$$

On the other hand, it was shown in 1992 by Dranišnikov, Repovš and Ščepin [6] that the Urysohn-Menger sum formula (1) fails for cohomological dimension over arbitrary abelian groups: they have constructed subsets A and B of $\mathbb{R}^4$ such that

$$\text{c-dim}_{Q/\mathbb{Z}}(A \cup B) > \text{c-dim}_{Q/\mathbb{Z}} A + \text{c-dim}_{Q/\mathbb{Z}} B + 1. \tag{3}$$

(Subsequently, Dydak [9] presented a different approach to this construction.)

Rubin's argument [15] for (2) is based on the resolution method, which can be traced back to the 1970's pioneering work of Edwards (see, for example, the survey by Walsh [16]). Roughly speaking, a *resolution* of a polyhedron L for some integer n, is a replacement of all $(n + 1)$ and higher dimensional simplices of L by the Eilenberg-MacLane spaces $K(\oplus\mathbb{Z}, n)$. In the 1980's Dranišnikov [1] adapted this method for the Bockstein rings $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}_{(p)}$, that is the localization of $\mathbb{Z}$ at the prime p (see also the surveys [2] and [14]). Subsequently, Dydak and Walsh [10] used this method to prove the Urysohn-Menger sum formula (1) for cohomological dimension c-dim_R over all Bockstein rings $R \in \{\mathbb{Z}_{(\ell)}, \mathbb{Z}_p\}_{\ell \subset \mathcal{P},\ p \in \mathcal{P}}$ (where $\mathcal{P}$ is the set of all primes and $\mathbb{Z}_{(\ell)}$ is the localization of the integers at ℓ), however they had to impose the restriction that both $\text{c-dim}_R A \geq 2$ and $\text{c-dim}_R B \geq 2$. (Note that such a restriction was unnecessary in Rubin's paper [15] since $\text{c-dim}_{\mathbb{Z}} X = 1$ obviously implies $\dim X = 1$.)

Let $\text{AE}(X)$ denote the class of all absolute extensors for X. Recall that the standard definition of the cohomological dimension of X over an abelian group G is

$$\text{c-dim}_G X \leq n \quad \text{if and only if} \quad K(G, n) \in \text{AE}(X). \tag{4}$$

It follows from the work by Dranišnikov [3, 4], that the Eilenberg-MacLane complexes $K(G, n)$ in the definition (4) can be replaced by the Moore spaces $M(G, n)$:

$$\text{c-dim}_G X \leq n \quad \text{if and only if} \quad M(G, n) \in \text{AE}(X). \tag{5}$$

Recall that $M(G, n)$ is a polyhedron such that

$$\widetilde{H}_i(M(G, n)) \cong \begin{cases} G & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

The equivalence (5) is true for $n \geq 2$ and the conclusion $\text{c-dim}_G X \leq n$ also holds for $n = 1$. This result allows instead for a different type of resolution of a polyhedron L, based on Moore spaces as building blocks.

The purpose of this paper is to remove the dimensional restrictions from the Dydak-Walsh theorem [10]. We were able to achieve this by using the new type of resolutions described above, that is, using the Moore spaces $M(G, n)$. We shall only give the proof for the case of the finite groups $\mathbb{Z}_p$ since the proof for $\mathbb{Z}_{(\ell)}$ is similar. This result was obtained in August 1992. Subsequently, Dydak [8] announced a generalization of (6) to all rings R with unity.

THEOREM 1.1. *For all subsets $A, B \subset X$ of a finite-dimensional separable metric space X and for every prime p, the following holds:*

$$\text{(6)} \qquad \text{c-dim}_{\mathbb{Z}_p}(A \cup B) \leq \text{c-dim}_{\mathbb{Z}_p} A + \text{c-dim}_{\mathbb{Z}_p} B + 1.$$

2. Preliminaries

We shall require the following result from [3]:

THEOREM 2.1. *Suppose $\dim X < \infty$ and that for some $n \geq 2$ and some abelian group G, $\text{c-dim}_G X \leq n$. Then the corresponding Moore space $M(G, n)$ is an absolute extensor for X, $M(G, n) \in AE(X)$.*

We shall also need the following version of the Blakers-Massey theorem from [12, Proposition 16.30]:

PROPOSITION 2.2. *Let $(X, A) \in$ AHEP and suppose (X, A) is $(n-1)$-connected and A is $(s-1)$-connected. Then the homomorphism $\pi_r(X, A, *) \to \pi_r(X/A, *)$ is an $(n+s-1)$-isomorphism, for every $r > 0$.*

PROPOSITION 2.3. *Let p be any prime, $n \geq k$ and let K be any $(k-1)$-connected polyhedron such that $\pi_i(K) \cong \bigoplus \mathbb{Z}_p$ for all $k \leq i \leq n$. Then there exists an inclusion $K \hookrightarrow \tilde{K}$ such that:*

(i) *$\pi_i(\tilde{K}) \cong \bigoplus \mathbb{Z}_p$ for $i \leq n+1$; and*
(ii) *the inclusion-induced homomorphism $H^*(\tilde{K}; \mathbb{Z}_p) \to H^*(K; \mathbb{Z}_p)$ is bijective for $* \leq k$ and surjective for $* > k$.*

PROOF. Let $K^1 = K \cup_{\varphi_i} B_i^{n+2}$ where $\{\varphi_i : \partial B_i^{n+2} \to K\}_{i \in \mathbb{N}}$ are the generators of $\pi_{n+1}(K)$. Let $\tilde{K} = (K^1, K)_{\mathbb{Z}_p}$, that is, $\tilde{K}$ is obtained from K^1 by replacing all $(n+2)$-dimensional simplices of $K^1 \setminus K$ by the Moore spaces $M(\mathbb{Z}_p, n+1)$. (See [2] for more on this construction.) Join all copies of $M(\mathbb{Z}_p, n+1)$ in $\tilde{K}$ by arcs in $K^1 \backslash K$ to obtain the wedge $\vee M(\mathbb{Z}_p, n+1)$.

Consider a couple $(\tilde{K}, \vee M(\mathbb{Z}_p, n+1))$ and its exact sequence

$$\cdots \bigoplus \mathbb{Z}_p \cong \pi_{n+1}(\vee M(\mathbb{Z}_p, n+1)) \xrightarrow{\alpha} \pi_{n+1}(\tilde{K}) \to \pi_{n+1}(\tilde{K}, \vee M(\mathbb{Z}_p, n+1)) \to \cdots$$

By Proposition 2.2, we have that

$$\pi_r(\tilde{K}, \vee M) \to \pi_r(\tilde{K}/\vee M) \cong \pi_r(K^1/\text{wedge of arcs}) \cong \pi_r(K^1)$$

is an isomorphism for $r = n+1$, since $\vee(M(\mathbb{Z}_p, n+1))$ is n-connected and $(\tilde{K}, \vee M)$ is 0-connected. By construction, $\pi_{n+1}(K^1) = 0$; thus α is an epimorphism, and hence the image is $\bigoplus \mathbb{Z}_p$.

PROPOSITION 2.4. *Let K be such that $H_k(K) = \bigoplus \mathbb{Z}_p$. Then there exists an inclusion $K \hookrightarrow \tilde{K}$ such that*

(i) *$\pi_i(\tilde{K}) = \bigoplus \mathbb{Z}_p$ for all i; and*
(ii) *$H^i(\tilde{K}; \mathbb{Z}_p) \to H^i(K; \mathbb{Z}_p)$ is an isomorphism for $i \leq k$ and is an epimorphism for $i > k$.*

PROOF. Let $\bar{K}$ be the abelianization of K, that is, $\bar{K}$ is obtained from K by attaching 2-cells along the commutators of all the generators of the fundamental group. Then $\pi_i(\bar{K}) = \bigoplus \mathbb{Z}_p$. The map $H^1(\bar{K}; \mathbb{Z}_p) \to H^1(K; \mathbb{Z}_p)$ is an isomorphism because 2-disks are attached by homology trivial maps (commutators). Apply Proposition 2.3, starting from $n = 1$ and $\bar{K}$ to get a sequence

$$K \hookrightarrow K_1 \overset{j_1}{\hookrightarrow} K_2 \overset{j_2}{\hookrightarrow} K_3 \hookrightarrow \cdots,$$

where $K_1 = \bar{K}$, $\pi_i(K_n) = \bigoplus \mathbb{Z}_p$ for $i \leq n$, and j_i^* is an isomorphism in dimension one, and j_i^* is an epimorphism for $* > 1$. Finally, define $\tilde{K} = \varinjlim K_i$.

PROPOSITION 2.5. *If* $\dim X < \infty$ *and* $\text{c-dim}_{\mathbb{Z}_p} X = 1$ *then* $K \in \text{AE}(X)$ *if* $\pi_i(K) \cong \oplus \mathbb{Z}_p$ *for all* i.

PROOF. See [3, 7].

3. (cd_R, n)-resolutions

DEFINITION. Let cd_R be an abbreviation for c-dim_R. Suppose that we have a polyhedron K with some triangulation τ. Then a map $\psi : \hat{K} \to K$ is called a (cd_R, n)-*resolution* if for every simplex $\sigma \in \tau$, $\psi^{-1}(\sigma) \in \text{AE}(\text{cd}_R \leq n, \dim < \infty)$ and $\psi^{-1}|_{K^{(n)}} : K^{(n)} \to \psi^{-1}(K^{(n)})$ is a homeomorphism.

Property $(*)$ of such a resolution means: For every simplex $\sigma \in \tau$ and for every integer $n \leq i$, the map $H^i(\psi^{-1}(\sigma); R) \to H^i(\psi^{-1}(\partial\sigma); R)$ is an epimorphism and $H^n(\psi^{-1}(\sigma); R) \to H^n(\psi^{-1}(\partial\sigma); R)$ is an isomorphism (for $i < n$ an isomorphism of trivial groups).

THEOREM 3.1. *For all p, n, K, and τ there exists a $(\mathrm{cd}_{\mathbb{Z}_p}, n)$-resolution $\psi : \hat{K} \to K$ with the property $(*)$.*

PROOF. The proof is by induction on $\dim K$. For $\dim K = n + 1$, replace $(n+1)$-simplexes by $K(\mathbb{Z}_p, n)$'s, by identifying the boundary of the simplex with the n-skeleton of $K(\mathbb{Z}_p, n)$. For the inductive step m to $m+1$: Suppose that $\dim K = m + 1$. Consider $\psi_i : \hat{K}^{i(m)} \to K^{(m)}$. Fix an $(m+1)$-simplex σ. Let $L = \psi_i^{-1}(\partial\sigma)$. Then by the construction $H_n(L) \cong \bigoplus \mathbb{Z}_p$. Apply Proposition 2.4 to obtain the embedding $L \hookrightarrow \hat{L}$. Then by Proposition 2.5, $\hat{L} \in \mathrm{AE}(\mathrm{cd}_{\mathbb{Z}_p} \leq n, \dim < \infty)$ and so the property $(*)$ holds.

REMARK. Such a resolution with a weaker form of the property $(*)$ was constructed in [2], whereas in [10] a $(\mathrm{cd}_{\mathbb{Z}_p}, n)$-resolution with the property $(*)$ was constructed just for $n \geq 2$. The argument in [10] is quite different from ours and it does not allow for an extension to the case $n = 1$ (hence the restrictions $\text{c-dim}_R A \geq 2$ and $\text{c-dim}_R B \geq 2$ in their proof of the special case of Theorem 1.1). The rest of our proof of Theorem 1.1 is to some extent similar to the argument in [10]; however, it is in many respects more elementary.

NOTATION. Let (K, τ) be a polyhedron with a triangulation τ, let $f : X \to K$ be a map. Then the notation $\text{c-dim}_R(f, \tau) \leq n$ means that the following extension problem has a solution for each subcomplex $A \subset K$ of K:

$$\begin{array}{ccccc} X & \dashrightarrow & f^{-1}(A) & & \\ f\downarrow & & \downarrow & \searrow & \\ K & \overset{\tau}{\hookleftarrow} & A & \overset{\varphi}{\longrightarrow} & K(R,n) \end{array}$$

(cf. the survey of Dranišnikov [2]).

PROPOSITION 3.2. *Suppose that $f : X \to K$ is a map and that for every simplex $\sigma \in \tau$ the map $H^n(f^{-1}(\sigma); R) \to H^n(f^{-1}(\partial\sigma); R)$ is an epimorphism. Then $\text{c-dim}_R(f, \tau) \leq n$.*

The proof of Proposition 3.2 is trivial. The following two theorems are taken from [2] with only minor changes:

THEOREM 3.3. *Suppose that for every open covering ω of the metric space Y there exists an ω-map $g : Y \to K$, with τ and there exists a τ-lifting $g' : Y \to X$ for some $f : X \to K$ with* $\text{c-dim}_R(f, \tau) \leq n$. *Then* $\text{c-dim}_R Y \leq n$.

Here *τ-lifting* means that if $g(x) \in \sigma \in \tau$ then $f \circ g'(x) \in \sigma$ and the ω-map has the property that $g^{-1}(\tau) \prec \omega$, where $\prec$ means refinement.

THEOREM 3.4. *Suppose* $\text{c-dim}_R Y \leq n$. *Then for every map $g : Y \to K$ with a triangulation τ, and for every (cd_R, n)-resolution $\psi : \hat{K} \to K$, there is a τ-lifting $g' : Y \to \hat{K}$.*

We shall also need the following assertion from [10]:

LEMMA 3.5. *If X is $(n-1)$-connected and Y is $(m-1)$-connected then $H^{n+m+1}(X * Y; R) \cong H^n(X; H^m(Y; R))$.*

PROPOSITION 3.6. *Suppose that we have resolutions: $\hat{\psi} : \hat{K} \to K$ $((\text{cd}_R, n)$-resolution) and $\hat{\psi}_1 : \hat{L} \to L$ $((\text{cd}_R, m)$-resolution) and both have the property $(*)$. Let $\hat{\sigma} = \hat{\psi}^{-1}(\sigma)$, $\hat{\delta} = \hat{\psi}_1^{-1}(\delta)$. Then the following are isomorphisms:*

$$H^{n+m+1}(\hat{\sigma} * \hat{\delta}) \to H^{n+m+1}(\hat{\sigma} * \partial\hat{\delta}),$$
$$H^{n+m+1}(\hat{\sigma} * \hat{\delta}) \to H^{n+m+1}(\widehat{\partial\sigma} * \hat{\delta}) \qquad \textit{and}$$
$$H^{n+m+1}(\hat{\sigma} * \hat{\delta}) \to H^{n+m+1}(\widehat{\partial\sigma} * \widehat{\partial\delta})$$

over the ring R, where $\widehat{\partial\delta} = \hat{\psi}_1^{-1}(\partial\delta)$ and $\widehat{\partial\sigma} = \hat{\psi}^{-1}(\partial\sigma)$.

PROOF. Notice that α is an isomorphism:

$$\begin{array}{ccc} H^{n+m+1}(\hat{\sigma} * \hat{\delta}) & \to & H^{n+m+1}(\hat{\sigma} * \partial\hat{\delta}) \\ \| & & \| \\ H^n(\hat{\sigma}; H^m(\hat{\delta})) & \xrightarrow{\alpha} & H^n(\hat{\sigma}; H^m(\partial\hat{\delta})). \end{array}$$

Similarly the second one is isomorphism. Now the third one:

$$\begin{array}{ccc} H^{n+m+1}(\hat{\sigma} * \hat{\delta}) & \longrightarrow & H^{n+m+1}(\widehat{\partial\sigma} * \widehat{\partial\delta}) \\ \| & & \| \\ H^n(\hat{\sigma}; H^m(\hat{\sigma})) & & H^n(\widehat{\partial\sigma}; H^m(\widehat{\partial\delta})) \\ \| & & \| \\ H^n(\hat{\sigma}; \bigoplus R) & \xrightarrow{\beta} & H^n(\widehat{\partial\sigma}; \bigoplus R). \end{array}$$

Now β is an isomorphism, since it is for just one R; so it follows for finite sums $\bigoplus R$.

4. The proof of Theorem 1.1

LEMMA 4.1. *The map* $H^{n+m+1}(\hat{\sigma}*\hat{\delta};\, R) \xrightarrow{\gamma} H^{n+m+1}(\partial(\hat{\sigma}*\hat{\delta});\, R)$ *is an epimorphism, where* $\partial(\hat{\sigma} * \hat{\delta}) = (\hat{\sigma} * \partial\hat{\delta}) \cup (\partial\sigma * \hat{\delta}) \hookrightarrow \hat{\sigma} * \hat{\delta}$.

PROOF. Consider the following Mayer-Vietoris cohomology sequences over R (where $s = n + m + 1$):

$$\begin{array}{ccccccccccc}
\to & H^{n+m}(\partial\hat{\sigma} * \partial\hat{\delta}) & \to & H^s(\partial(\hat{\sigma} * \hat{\delta})) & \to & H^s(\hat{\sigma} * \partial\hat{\delta}) & \oplus & H^s(\partial\hat{\sigma} * \hat{\delta}) & \to & H^s(\partial\hat{\delta} * \partial\hat{\sigma}) & \to \cdots \\
 & & & \cong\uparrow\gamma & & & \uparrow\alpha & & & \uparrow\beta & \\
\to & 0 & \to & H^s(\hat{\sigma} * \hat{\delta}) & \to & H^s(\hat{\sigma} * \hat{\delta}) & \oplus & H^s(\hat{\sigma} * \hat{\delta}) & \to & H^s(\hat{\sigma} * \hat{\delta}) & \to \cdots
\end{array}$$

It suffices to show that γ is an isomorphism. Suppose that $\dim\sigma \leq n$ and $\dim\delta \leq m$. Then all cohomology groups above vanish. If $\dim\sigma > n$ and $\dim\delta > m$, we have that $\partial\hat{\sigma} * \partial\hat{\delta}$ is $(n + m)$-connected, hence $H^{n+m}(\partial\hat{\sigma} * \partial\hat{\delta}) = 0$ and the Five Lemma yields the assertion.

It remains to consider the case when $\dim\sigma \leq n$ and $\dim\delta > m$ or vice versa. Then $\hat{\sigma} \simeq$ point: hence $\hat{\sigma} * \partial\hat{\delta} \simeq$ point, and $\hat{\sigma} * \hat{\delta} \simeq$ point. We want to show that $H^{n+m+1}(\partial(\hat{\sigma} * \hat{\delta})) = 0$. It sufices to prove that $H^{n+m}(\partial\hat{\sigma} * \partial\hat{\delta})$ maps onto, since we know it maps into. To this end, consider the following diagram:

$$\begin{array}{ccc}
H^{n+m}(\partial\hat{\sigma} * \hat{\delta}) & \longrightarrow & H^{n+n+1}(\partial(\hat{\sigma} * \hat{\delta})) \\
\| & & \| \\
H^{n+m}(S^k * \hat{\delta}) & \longrightarrow & H^{n+k+1}(S^k * \partial\hat{\delta}) \\
\| & & \| \\
H^{n+m-k-1}(\hat{\delta}) & \longrightarrow & H^{n+m-k-1}(\partial\hat{\delta}),
\end{array}$$

and recall that $\partial\hat{\sigma} = S^k$, so by the property $(*)$ there exists an epimorphism. Hence we get a zero where we need it.

PROPOSITION 4.2. *Suppose that* $\psi : \hat{K} \to K$ *is a* (cd_R, n)*-resolution and* $\varphi : \hat{L} \to L$ *is a* (cd_R, m)*-resolution, both with the property* $(*)$*. Then* $\text{c-dim}_R\,(\psi * \varphi, \tau * \tau') \leq n + m + 1$.

PROOF. Follows by Lemma 4.1 and Proposition 3.2.

DEFINITION. Suppose that $A, B \subset X$ are disjoint subsets of X, $X = A \cup B$ and suppose that $f : A \to K$ and $g : B \to L$ are any maps. Then define the map $f \overset{*}{\cup} g : A \cup B \to K * L$ as follows: Let $\bar{f} : U \to K$ be an extension of a map f', which is close to, and hence homotopic to the map f, over an open neighbourhood $U \subset X$ of A in X and let $\bar{g} : V \to L$ be an extension of a map g', which is close to, and hence homotopic to, the map g, over an open neighbourhood $V \subset X$ of B in

X (cf. [10, Lemma 4.2]). Let $d_A(x) = \rho(x, X\backslash U)$ and $d_B(x) = \rho(x, X\backslash V)$ be the distance functions. Now, define for every $x \in X$,

$$(f \overset{*}{\cup} g)(x) = \big(f(x), g(x), d_B(x)/(d_A(x) + d_B(x))\big).$$

Let $\pi : K * L \to [0, 1]$ be the natural projection of the join $K * L$ onto the interval $[0, 1]$. (Collapse K and L to a point, respectively.)

LEMMA 4.3. *Suppose that $X = A \cup B$. Then for every cover w of X there exist maps $\varphi_A : A \to K$ with a triangulation τ and $\varphi_B : B \to L$ with a triangulation τ' such that $\varphi_A \overset{*}{\cup} \varphi_B$ is an ω-map onto $K * L$ with respect to the triangulation $\tau * \tau'$.*

PROOF. Choose a cover ω_A (respectively ω_B) which is a star-refinement of ω and consider the projection onto nerves, φ_A (respectively φ_B).

PROPOSITION 4.4. *Suppose that $X = A \cup B$ and that there are maps $\psi : \hat{K} \to K$, $\varphi : \hat{L} \to L$, $f : A \to K$ with a τ-lifting $f' : A \to \hat{K}$ and $g : B \to L$ with a τ'-lifting $g' : B \to \hat{L}$. Then the map $f \overset{*}{\cup} g : A \cup B \to K * L$ has a $(\tau * \tau')$-lifting $q : A \cup B \to \hat{K} * \hat{L}$.*

PROOF. Define the lifting as follows: $q(x) = (f'(x), g'(x), (\pi(f \overset{*}{\cup} g))(x))$.

PROOF OF THEOREM 1.1. It suffices to prove Theorem 1.1 for the case when the subsets $A, B \subset X$ are disjoint; $A \cap B = \emptyset$. Indeed, if $A \cap B \neq \emptyset$ we define $B' = B\backslash A$ and it follows that

$$\begin{aligned} \text{c-dim}_R(A \cup B) &= \text{c-dim}_R(A \cup B') \le \text{c-dim}_R A + \text{c-dim}_R B' + 1 \\ &\le \text{c-dim}_R A + \text{c-dim}_R B + 1. \end{aligned}$$

So suppose now that $A \cap B = \emptyset$, $A \cup B = X$, $\text{c-dim}_R A \le n$ and $\text{c-dim}_R B \le m$. We shall prove that $\text{c-dim}_R(A \cup B) \le n + m + 1$.

To this end, consider an arbitrary cover ω of X and apply Lemma 4.3 to get maps $\varphi_A : A \to K$ and $\varphi_B : B \to L$. Next, apply Theorem 3.1 to obtain the corresponding resolutions of K and L, that is, a (cd_R, n)-resolution $\psi : \hat{K} \to K$ with the property $(*)$ and a (cd_R, m)-resolution $\varphi : \hat{L} \to L$ with the property $(*)$.

By Proposition 4.2, $\text{c-dim}_R(\psi * \varphi, \tau * \tau') \le n + m + 1$, and by Theorem 3.4 and Proposition 4.4, there exists a lifting $q : A \cup B \to \hat{K} * \hat{L}$ of $\varphi_A \overset{*}{\cup} \varphi_B$ which is a $(\tau * \tau')$-lifting. Since ω as an arbitrary covering, it follows by Theorem 3.3 that $\text{c-dim}_R(A \cup B) \le n + m + 1$ as asserted.

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