Topology and its Applications 54 (1993) 3–11
North-Holland

On unstable intersections of 2-dimensional compacta in Euclidean 4-space *

A.N. Dranišnikov and D. Repovš **
Steklov Mathematical Institute, Vavilova St. 42, Moscow 117966, Russia

This paper is dedicated to the memory of Professor M.F. Bokštein who died in May 1990.

Received 18 July 1991

Abstract

Dranišnikov, A.N. and D. Repovš, On unstable intersections of 2-dimensional compacta in Euclidean 4-space, Topology and its Applications 54 (1993) 3–11.

We give an alternative proof, based on Bokštein's theory, of the following result which was originally proved by the authors, jointly with E.V. Ščepin (and independently, by S. Spież): Let X and Y be 2-dimensional compact metric spaces such that $\dim(X \times Y) = 3$. Then for every $\varepsilon > 0$ and every pair of maps $f: X \to \mathbb{R}^4$ and $g: Y \to \mathbb{R}^4$ there exist maps $f': X \to \mathbb{R}^4$ and $g': Y \to \mathbb{R}^4$ such that $d(f, f') < \varepsilon$, $d(g, g') < \varepsilon$ and $f'(X) \cap g'(Y) = \emptyset$.

Keywords: Cohomological dimension; Compactum; Eilenberg–MacLane complex; Unstable intersection; Telescope; Membrane; Grafting.

AMS (MOS) Subj. Class.: Primary 55M10, 57Q55; secondary 57Q65, 54C25, 54F45.

1. Introduction

Compacta (i.e., compact metric spaces) exhibit many properties quite unlike those of polyhedra. One of them is that they fail to satisfy the logarithmic law for dimension, i.e., $\dim(X \times Y)$ can be strictly less than $\dim X + \dim Y$ if we choose appropriate compacta X and Y (see [1,2,11,18–20]).

Correspondence to: Professor A.N. Dranišnikov, Steklov Mathematical Institute, Vavilova St. 42, Moscow 117 966, Russia.

* This paper is in its final form and no version of it will appear elsewhere.

** Supported in part by a grant from the Ministry of Science and Technology of the Republic of Slovenia. Permanent address: Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P.O. Box 64, Ljubljana 61111, Slovenia.

Another feature which distinguishes compacta from polyhedra was discovered more recently by Ščepin and the authors (see [8–10]) and, independently—via a different approach, by Spież (see [23–25]). They proved that maps of compacta of complementary dimensions (i.e., dim X + dim $Y = n$) into Euclidean n-space have *unstable* intersections, i.e., their images can be made disjoint by arbitrarily small perturbations if $\dim(X \times Y) < n$. Their proofs, although quite different, nevertheless both naturally split into two separate cases: when $n = 4$ and when $n \geqslant 5$.

The major reason for treating the 4-dimensional situation separately is the problem with the fundamental group—the only possibly non-Abelian homotopy group. Both sets of authors use here an elaboration of the celebrated Casson *finger moves* (see e.g. [5,12]) in order to abelianize Π_1.

It is the purpose of the present paper to give an alternative argument, avoiding the use of the Casson trick, and trade *geometry* for *algebra*—by invoking instead the classical Bokštein theory of the cohomological dimension theory for compacta [3,7,15].

More precisely, we propose to give an alternative proof of the following general position theorem for maps of compacta into $\mathbb{R}^4$:

Theorem 1.1. *Let X and Y be 2-dimensional compacta such that* $\dim(X \times Y) = 3$. *Then for every $\varepsilon > 0$ and every pair of maps $f : X \to \mathbb{R}^4$ and $g : Y \to \mathbb{R}^4$ there exist maps $f' : X \to \mathbb{R}^4$ and $g' : Y \to \mathbb{R}^4$ such that $d(f, f') < \varepsilon$, $d(g, g') < \varepsilon$ and $f'(X) \cap g'(Y) = \emptyset$.*

Note that there are plenty of 2-dimensional compacta X and Y such that $\dim(X \times Y) = 3$ (see e.g. [7,11,18]). The first example for $X \neq Y$ was discovered in 1930 by Pontrjagin [21] and for $X = Y$ in 1949 by Boltjanskii [4].

We also wish to point out that compacta X such that $\dim(X \times X) = 3$ have recently played an important role in geometric topology—in the attacks on the 4-dimensional *cell-like mapping problem* which asks whether cell-like maps on 4-manifolds can raise dimension (see e.g. [6,16,17])—all other dimensions have already been solved, for $n \leqslant 3$ in the negative and for $n \geqslant 5$ in the affirmative—see the survey [16].

The preliminary version of this paper was written during the second author's visit to the Sterlov Mathematical Institute in 1989 (and announced in [8]), on the basis of the long term agreement between the Slovenian Academy of Arts and Sciences and the Russian Academy of Sciences (1986–1995). The final version was then prepared during the visit by both authors to the University of Tsukuba in 1990. We wish to thank Professor Yukihiro Kodama for his kind hospitality.

2. Preliminaries

We shall work in the category of separable metrizable spaces and continuous maps throughout this paper. A *compactum* is a compact metric space. A space X

is said to have *cohomological dimension at most* n, $n \in \mathbb{N} \cup \{0\}$, *with respect to a group of coefficients* R, $c\text{-}\dim_R X \leqslant n$ if for every closed subset $A \subset X$ and every map $f: A \to K(R, n)$ of A into the Eilenberg–MacLane complex $K(R, n)$ (see [13] for the definition and properties of Eilenberg–MacLane complexes), there is an extension of f over all of X. (For several equivalent versions of the definition of $c\text{-}\dim_R X$ see [7,15] where also its properties are studied in details.)

For any $m \in \mathbb{Z}$, define an *m-telescope* $T(m)$ (respectively *m-membrane* $M(m)$) to be the mapping cylinder (respectively mapping cone) of a degree m map $\phi: S^1 \to S^1$. The obvious embedding of S^1 into $T(m)$ (respectively $M(m)$) as the preimage of ϕ in $T(m)$ (respectively as the image of ϕ in $M(m)$) will be called the *canonical embedding of* S^1 *into* $T(m)$ (respectively $M(m)$) and denoted by $\Gamma(T(m))$ (respectively $\Gamma(M(m))$. Note that $M(m)$ is the mod m 2-dimensional Moore space [22].

For any finite sequence of integers $\mathscr{K} = (n_1, \ldots, n_t) \subset \mathbb{Z}$, define a *finite* $\mathscr{K}$*-telescope* $T(\mathscr{K})$ to be the finite tower of n_i-telescopes $T(n_i)$, glued together in the obvious way—via identifying homeomorphisms $\alpha_i: (\partial T(n_i) - \Gamma(T(n_i))) \to \Gamma(T(n_{i+1}))$, i.e., identification of the bottom circle of $T(n_i)$ to the top circle of $T(n_{i+1})$:

$$T(\mathscr{K}) = T(n_1) \cup_{\alpha_1} T(n_2) \cup_{\alpha_2} \cdots \cup_{\alpha_{t-1}} T(n_t).$$

For any infinite sequence of integers $\mathscr{K} = (n_i)_{i \in \mathbb{N}}$, define an *infinite* $\mathscr{K}$*-telescope* $T(\mathscr{K})$ to be the direct limit of the direct system of finite $(n_1, \ldots, n_t)$-telescopes, i.e.,

$$T(\mathscr{K}) = \underset{t \in \mathbb{N}}{\overrightarrow{\lim}} \{T(n_1, \ldots, n_t), \text{incl.}\}.$$

The canonical embedding of S^1 into the base $T(n_1)$ of the infinite $\mathscr{K}$-telescope $T(\mathscr{K})$ will be called the *canonical embedding of* S^1 and denoted by $\Gamma(T(\mathscr{K}))$ (compare [26]).

Given a subset $\mathscr{L} \subset \mathscr{P}$ of the set of all primes $\mathscr{P}$, a sequence $(p_i)_{i \in \mathbb{N}}$ of primes is said to be $\mathscr{L}$*-complete* if all p_i are elements of $\mathscr{L}$ and every element of $\mathscr{L}$ appears in $(p_i)_{i \in \mathbb{N}}$ infinitely many times.

Let $\mathscr{A} \subset \mathscr{P} \cup \{0\}$ be any subset. Recall that the *localization of* $\mathbb{Z}$ *at* $\mathscr{A}$ is the subset of the rationals

$$\mathbb{Z}_{(\mathscr{A})} = \left\{ \frac{m}{n} \in \mathbb{Q} \,|\, \text{for every } a \in \mathscr{A},\ n \text{ is not divisible by } a \right\}.$$

For example, $\mathbb{Z}_{(0)} = \mathbb{Q}$. Next, let $\mathbb{Z}_{\mathscr{A}^\infty}$ denote the quotient $\mathbb{Q}/\mathbb{Z}_{\mathscr{A}}$. Finally, for any $m \in \mathbb{Z}$ and $\mathscr{A} \subset \mathbb{Z}$, we say that m and $\mathscr{A}$ are *relatively prime*, $(m, \mathscr{A}) = 1$, if for every $a \in \mathscr{A}$, m and a are relatively prime, i.e., $(m, a) = 1$.

Example 2.1. For every $\mathscr{L} \subset \mathscr{P}$ and every $\mathscr{L}$-complete sequence $\mathscr{K}$ of primes, the infinite $\mathscr{K}$-telescope $T(\mathscr{K})$ is the Eilenberg–MacLane complex $K(\mathbb{Z}_{(\bar{\mathscr{L}})}, 1)$, where $\bar{\mathscr{L}} = \mathscr{P} - \mathscr{L}$. In particular, $T(\mathscr{P}) = K(\mathbb{Q}, 1)$ and for every prime $p \in \mathscr{P}$, $T(p, p, \ldots) = K(\mathbb{Z}[1/p], 1)$ (see [7]).

Let $m \in \mathbb{Z}$ and $\mathscr{L} \subset \mathscr{P}$ be arbitrary and let $\mathscr{K}$ be any $\mathscr{L}$-complete sequence. Suppose that Z is either the finite m-telescope $T(m)$, or the finite m-membrane $M(m)$, or the infinite $\mathscr{K}$-telescope $T(\mathscr{K})$. Let L be a 2-dimensional polyhedron with some triangulation τ. For every 2-simplex $\sigma \in \tau$, let σ^* be the 2-simplex which is obtained by slightly pushing σ into int σ along some collar on $\partial\sigma$. Define a *Z-modification of L* to be a CW complex $L(Z)$ which is obtained from L by replacing every σ^*, $\sigma \in \tau$, by a copy Z_σ of Z, via an identification of $\partial\sigma^*$ with the canonical embedding $\Gamma(Z)$ of S^1 into Z:

$$L(Z) = \left(L - \left(\bigcup \{\text{int } \sigma^* \mid \sigma \in \tau\}\right)\right) \bigcup_{\{\partial\sigma^* = \Gamma(Z_\sigma) \mid \sigma \in \tau\}} \left(\bigcup \{Z_\sigma \mid \sigma \in \tau\}\right).$$

The following result can be deduced from [7] (see Example 2.1):

Proposition 2.2. *Suppose $\mathscr{L} \subset \mathscr{P}$ and denote $\bar{\mathscr{L}} = \mathscr{P} - \mathscr{L}$. Let L be a 2-dimensional polyhedron with triangulation τ and let X be a compactum with* c-dim$_{\mathbb{Z}(\bar{\mathscr{L}})} X \leqslant 1$. *Then for every $\mathscr{L}$-complete sequence $\mathscr{K}$ and for every map $f: X \to L$ there exists a map $\bar{f}: X \to L(T(\mathscr{K}))$ such that:*

(i) $\bar{f} \mid f^{-1}(\tau^{(1)}) = f \mid f^{-1}(\tau^{(1)})$, *where $\tau^{(1)}$ is the 1-skeleton of τ; and*

(ii) *for every 2-simplex $\sigma \in \tau$, $\bar{f}(f^{-1}(\sigma^*)) \subset T_\sigma(\mathscr{K})$.*

Let L be a 2-dimensional polyhedron with some triangulation τ and let $D \subset L$ be a 2-cell subpolyhedron (with the induced triangulation $\tau \mid D$). Let $D^* \subset \text{int } D$ be the 2-cell which is obtained by slightly pushing D into int D along some collar on ∂D. For any $m \in \mathbb{Z}$, define an *m-grafting of L* along D to be a CW complex $\tilde{L}(D, m)$ which is obtained from L by replacing D^* by a copy $M_D(m)$ of the m-membrane $M(m)$, via an identification of ∂D^* with the canonical embedding $\Gamma(M(m))$ of S^1 into $M(m)$:

$$\tilde{L}(D, m) = (L - \text{int } D^*) \bigcup_{\{\partial D^* = \Gamma(M_D(m))\}} M_D(m).$$

The membrane $M_D(m)$ is called the *grafted part of* $\tilde{L}(D, m)$.

The following result can also be deduced from [7] (recall that for every prime $p \in \mathscr{P}$, the finite p-membrane $M(p)$ is the 2-skeleton of the Eilenberg–MacLane complex $K(\mathbb{Z}_p, 1)$).

Proposition 2.3. *Let L be a 2-dimensional polyhedron and let a 2-cell $D \subset L$ be a subpolyhedron of L. Suppose that X is a compactum such that* dim $X \leqslant 2$ *and* c-dim$_{\mathbb{Z}_p} X \leqslant 1$, *for some $p \in \mathscr{P}$. Then for every map $f: X \to L$ there exists a map $\bar{f}: X \to \tilde{L}(D, p)$ such that:*

(i) $\bar{f} \mid f^{-1}(L - \text{int } D^*) = f \mid f^{-1}(L - \text{int } D^*)$; *and*

(ii) $\bar{f}(f^{-1}(D^*)) \subset M_D(p)$.

The last result in this section follows easily from [14] using duality.

Proposition 2.4. *Given $m \in \mathbb{N}$, there exist PL embeddings $f:(T(m), \Gamma(T(m))) \to (B^4, \partial B^4)$ and $g:(M(m), \Gamma(M(m))) \to (B^4, \partial B^4)$ of $T(m)$ and $M(m)$ into the 4-ball B^4 such that:*

(i) *The circles $f(\Gamma(T(m)))$ and $g(\Gamma(M(m)))$ are unknotted and form the Hopf link;*

(ii) $\Pi_1(B^4 - f(T(m))) \cong \mathbb{Z}_m$;

(iii) $\Pi_1(B^4 - g(M(m))) \cong \mathbb{Z}$ *and is generated by a circle $\gamma \subset \partial B^4$ such that* $\mathrm{lk}(g(\Gamma(M(m))), \gamma; \mathbb{Z}) = 1$; *and*

(iv) $f(T(m)) \cap g(M(m)) = \emptyset$.

3. Proof of Theorem 1.1

A sequence $\{X_i\}_{i\in\mathbb{N}}$ of spaces X_i is said to be *decreasing and properly nested* if for every $i \in \mathbb{N}$, $X_{i+1} \subset \mathrm{int}\, X_i$. We shall need the following lemma:

Lemma 3.1. *For every $m \in \mathbb{N}$, every subset $\mathcal{K} \subset \mathcal{P}$ and every $\mathcal{K}$-complete sequence $\{p_i\}_{i\in\mathbb{N}}$, there exist a decreasing and properly nested sequence $\{L_i\}_{i\geqslant 0}$ of compact 4-manifolds with boundary $L_i \subset B^4$, with intersection $Z = \bigcap_{i=0}^{\infty} L_i$ and a collection $\{R_i\}_{i\geqslant 0}$ of 2-dimensional polyhedra $R_i \subset B^4$, such that:*

(i) *$R_0 = T(m)$ and for every $i \geqslant 1$, R_i is a p_i-grafting of R_{i-1} along some PL subdisk of R_{i-1};*

(ii) *for every $i \in \mathbb{N}$, L_i is the regular neighborhood of R_i in B^4;*

(iii) *$Z \cap \partial B^4$ is an unknotted circle in ∂B^4; and*

(iv) *$\Pi_1(B^4 - Z) \cong \mathbb{Z}_{\mathcal{K}^\infty} \oplus \mathbb{Z}_r$ for some $r \in \mathbb{N}$ such that $(r, \mathcal{K}) = 1$.*

Proof. Let $\{\varepsilon_i\}_{i\geqslant 0}$ be a strictly monotone decreasing sequence of positive numbers such that $\lim_{i\to\infty}\varepsilon_i = 0$. Let $R_0 \subset B^4$ be the canonical embedding of $T(m)$ into B^4 (see [14]) and define $L_0 \subset B^4$ to be a regular ε_0-neighborhood of R_0 in B^4 with canonical retraction $\pi_0: L_0 \to R_0$ of L_0 onto R_0. Choose a 2-simplex σ_0 in R_0. Define $R_1 \subset B^4$ to be a p_1-grafting of R_0 along σ_0, the grafted part of which is realized inside $\pi_0^{-1}(\sigma_0) \cong \sigma_0 \times D_0^2 \cong B^4$ (see Proposition 2.4).

Let $L_1 \subset B^4$ be a regular ε_1-neighborhood of R_1 in L_0. Inductively, define for every $i \geqslant 2$, $R_i \subset B^4$ to be a p_i-grafting of R_{i-1} along σ_{i-1}, for some appropriate 2-simplex σ_{i-1} in R_{i-1}, $\pi_{i-1}^{-1}(\sigma_{i-1})$, where $\pi_{i-1}: L_{i-1} \to R_{i-1}$ is the canonical projection of the regular neighborhood on its spine. Then define $L_i \subset L_{i-1}$ as a regular ε_i-neighborhood of R_i in L_{i-1}.

Clearly, the properties (i)–(iii) hold. It therefore remains to verify (iv). We first observe that for every $i \geqslant 1$,

$$\Pi_1(B^4 - L_i) \cong \Pi_1(B^4 - R_i) \cong \mathbb{Z}_{r_i},$$

generated by a circle in B^4 linking R_i, where $r_i = mp_1 \cdots p_i$. The inclusion $B^4 - L_i \subset B^4 - L_{i+1}$ induces a homomorphism $\mathbb{Z}_{r_i} \to \mathbb{Z}_{r_{i+1}}$ which sends the generator of $\mathbb{Z}_{r_i}$ to the p_{i+1}-multiple of the generator of $\mathbb{Z}_{r_{i+1}}$. Indeed, $B^4 - L_{i+1}$ is homotopy equivalent to $(B^4 - L_i) \cup ((\sigma_i \times D_i) - M(p_{i+1}))$ and attaching $((\sigma_i \times D_i) - M(p_{i+1}))$ to $(B^4 - L_i)$ divides the generator of $\mathbb{Z}_{r_i}$ by p_{i+1} (see Proposition 2.4(ii)). It therefore remains to verify the following algebraic fact:

Lemma 3.2. *Let $\{p_i\}_{i \in \mathbb{N}}$ be a $\mathscr{K}$-complete sequence of prime numbers, for some $\mathscr{K}$ and let*

$$\mathbb{Z}_{m_0} \xrightarrow{\delta_0} \mathbb{Z}_{m_1} \xrightarrow{\delta_1} \mathbb{Z}_{m_2} \xrightarrow{\delta_2} \mathbb{Z}_{m_3} \xrightarrow{\delta_3} \cdots$$

be a direct sequence of groups such that the bonding homomorphisms $\mathbb{Z}_{m_i} \xrightarrow{\delta_i} \mathbb{Z}_{m_{i+1}}$ send the generator e_i of $\mathbb{Z}_{m_i}$ to $p_{i+1}e_{i+1}$, where $m_0 = m$ and for every $i \geqslant 1$, $m_i = mp_1p_2 \cdots p_i$. Then

$$\overrightarrow{\lim}_{i \in \mathbb{N}} \{\mathbb{Z}_{m_i}, \delta_i\} \cong \left(\oplus \mathbb{Z}_{\{p_i\}_{i \in \mathbb{N}}} \right) \oplus \mathbb{Z}_r,$$

where $r \in \mathbb{Z}$ is such that $(r, \{p_i\}_{i \in \mathbb{N}}) = 1$.

Proof. For every $i \geqslant 1$, $\mathbb{Z}_{m_i} \cong (\oplus_{q \in \mathscr{K}} \mathbb{Z}_{q^{\alpha(q,i)}}) \oplus \mathbb{Z}_r$ for some $r \in \mathbb{N}$ such that $(r, \mathscr{K}) = 1$. Clearly, then the restriction of the homomorphism

$$\delta_i = \phi_i + \psi_i : \mathbb{Z}_{m_i} \to \mathbb{Z}_{m_{i+1}}$$

to

$$\delta_i \,|\, \mathbb{Z}_{p_{i+1}^{\alpha(p_{i+1},i)}} : \mathbb{Z}_{p_{i+1}^{\alpha(q,i)}} \to \mathbb{Z}_{p_{i+1}^{\alpha(p_{i+1},i)+1}}$$

is the inclusion homomorphism. Consequently,

$$\begin{aligned}
\overrightarrow{\lim}_{i \in \mathbb{N}} \left(\bigoplus_{q \in \mathscr{K}} \mathbb{Z}_{q^{\alpha(q,i)}} \right) \oplus \mathbb{Z}_r &\cong \left(\bigoplus_{q \in \mathscr{K}} \left(\overrightarrow{\lim}_{i \in \mathbb{N}} \mathbb{Z}_{q^{\alpha(q,i)}} \right) \right) \oplus \mathbb{Z}_r \\
&\cong \left(\bigoplus_{q \in \mathscr{K}} \mathbb{Z}_{\{q\}^\infty} \right) \oplus \mathbb{Z}_r \\
&\cong \left(\bigoplus_{q \in \mathscr{K}} \mathbb{Q}/\mathbb{Z}_{(q)} \right) \oplus \mathbb{Z}_r \\
&\cong (\mathbb{Q}/\mathbb{Z}_{(\mathscr{K})}) \oplus \mathbb{Z}_r \\
&\cong \mathbb{Z}_{\mathscr{K}^\infty} \oplus \mathbb{Z}_r. \qquad \square
\end{aligned}$$

Proof of Theorem 1.1. Without losing generality (compare [9]) we may assume that $M = f(X)$ and $N = g(Y)$ are 2-dimensional subpolyhedra in $\mathbb{R}^4$ and that M intersects N transversely, i.e., for some triangulations τ_M and τ_N of M and N, respectively, we have that

(1) $\tau_M^{(1)} \cap \tau_N = \tau_M \cap \tau_N^{(1)} = \emptyset$,

(2) $\tau_M \cap \tau_N$ consists of finitely many double points $\{t_1, \ldots, t_s\}$ where for every i, $t_i \in \sigma_i \cap \omega_i = \hat{\sigma}_i = \hat{\omega}_i$, $\sigma_i \in \tau_M$, $\omega_i \in \tau_N$, and $t_i \neq t_j$ for all $i \neq j$.

Suppose that $c\text{-dim}_{\mathbb{Q}}\, X = 1$ (otherwise $c\text{-dim}_{\mathbb{Q}}\, Y = 1$).

Step 1. Let $\mathscr{L} = \{p \in \mathscr{P} \mid c\text{-dim}_{\mathbb{Z}_{(p)}} X = 1\}$, let $\bar{\mathscr{L}} = \mathscr{P} - \mathscr{L}$ and choose an $\bar{\mathscr{L}}$-complete sequence $\mathscr{K}$. Let $\tilde{M}$ be a $T(\mathscr{K})$-modification of M. Apply Proposition 2.2 to find a map $f' : X \to \tilde{M}$ such that

(3) $f' \mid f^{-1}(\tau_M^{(1)}) = f \mid f^{-1}(\tau_M^{(1)})$; and

(4) for every 2-simplex $\sigma \in \tau_M$, $f'(f^{-1}(\sigma^*)) \subset T_\sigma(\bar{\mathscr{P}})$.

Step 2. Since $X_\sigma = f^{-1}(\sigma^*)$ is compact, so is $f'(X_\sigma)$ and thus it lies in some compact part of $\tilde{M}$ which is a finite telescope. Since every finite telescope is homotopy equivalent to some m-telescope, $f'(X_\sigma)$ lies in a compact polyhedron $\overline{M_\sigma} \subset \mathbb{R}^4$, homotopy equivalent to a $T(m)$-telescope, for some m. This means that there exists an extension $f': X_\sigma \to \overline{M_\sigma}$ of the restriction $f \mid f^{-1}(\partial\sigma)$.

Step 3. For every $j \in \{1, \ldots, s\}$, there exists a closed PL 4-dimensional ball $B_j^4 \subset \mathbb{R}^4$ of radius $\leqslant \frac{1}{2}\text{mesh}\{\tau_M, \tau_N\}$ such that

(5) for every $j \neq l$, $B_j \cap B_l = \emptyset$;

(6) $\partial\sigma_j \cup \partial\omega_j \subset \partial B_j$;

(7) $B_j \cap M = \sigma_j$; and

(8) $B_j \cap N = \omega_j$.

Let $\mathscr{K} = \{p \in \bar{\mathscr{P}} \mid c\text{-dim}_{\mathbb{Z}_p} X = 1\}$. For every j, apply Lemma 3.1 with m, $\mathscr{K}$, and B_j as above, to get a decreasing, properly nested sequence $\{L_{j,i}\}_{i \geqslant 0}$ of compact 4-manifolds with boundary $L_{j,i} \subset B_j^4$ with intersection $Z_j = \bigcap_{i \geqslant 0} L_{j,i}$ such that

(9) $\Pi_1(B_j - Z_j) \cong \mathbb{Z}_{\mathscr{K}^\infty} \oplus \mathbb{Z}_r$, for some $r_j \in \mathbb{N}$ such that $(r_j, \mathscr{K}) = 1$ and $Z_j \cap \partial B_j^4 = \partial\sigma_j$.

Since there is an extension $f': f^{-1}(\sigma_j) \to \overline{M_{\sigma_j}}$ of $f \mid: f^{-1}(\partial\sigma_j) \to \partial\sigma_j$ there exists an extension $f_{j_0}: f^{-1}(\sigma_j) \to R_{j_0} \subset L_{j_0}$ of $f \mid f^{-1}(\partial\sigma_j)$. Proposition 2.3 implies that there exists an extension $f_{j,i}: f^{-1}(\sigma_j) \to R_{j,i} \subset L_{j,i}$ for every i.

Step 4. We shall first prove two assertions:

Assertion 1. $c\text{-dim}_{\mathbb{Z}_r} Y \leqslant 1$.

Proof. For every $p \in \bar{\mathscr{P}} - \mathscr{K}$, we have that

$$\begin{aligned} 3 &\geqslant c\text{-dim}_{\mathbb{Z}_p}(X \times Y) \\ &= c\text{-dim}_{\mathbb{Z}_p} X + c\text{-dim}_{\mathbb{Z}_p} Y \\ &\geqslant 2 + c\text{-dim}_{\mathbb{Z}_p} Y. \end{aligned}$$

Assertion 2. $c\text{-dim}_{\mathbb{Z}_{\mathscr{K}^\infty}} Y \leqslant 1$.

Proof. For every $p \in \mathscr{K}$, $c\text{-dim}_{\mathbb{Z}_{p^\infty}} Y = 1$ for otherwise the Bokštein product theorem and one of his inequalities [3,7,15] would imply that

$$\begin{aligned} c\text{-dim}_{\mathbb{Z}_{(p)}}(X \times Y) &= c\text{-dim}_{\mathbb{Z}_{(p)}} X + c\text{-dim}_{\mathbb{Z}_{(p)}} Y \\ &\geqslant 2 + c\text{-dim}_{\mathbb{Z}_{p^\infty}} Y \geqslant 4 \end{aligned}$$

which is a contradiction.

Step 5. Attach to $B_j - Z_j$ cells of dimension greater than 2 to obtain the Eilenberg–MacLane complex $K(G, 1)$, where G is the group $G = \mathbb{Z}_{\mathscr{K}^\infty} \oplus \mathbb{Z}_{r_j}$. It follows by Assertions 1 and 2 that $c\text{-dim}_G\ Y = 1$. Therefore, there exists an extension of the map

$$g|g^{-1}(\partial\omega_j) : g^{-1}(\partial\omega_j) \to B_j - Z_j \subset K(G, 1)$$

to the map $g_j : g^{-1}(\omega_j) \to K(G, 1)$. Since $\dim\ g^{-1}(\omega_j) \leqslant 2$, we can push Im g_j into the 2-skeleton $B_j - Z_j$ of $K(G, 1)$.

Step 6. Define $\bar{g} : Y \to B^4$ to be the map g_j on every $g^{-1}(\omega_j)$, $1 \leqslant j \leqslant s$, and g elsewhere. Clearly, for every $j \in \{1, \ldots, s\}$, $\bar{g}(Y) \cap B_j \subset B_j - Z_j$. Since Y is compact, there exists for every $j \in \{1, \ldots, s\}$, an integer n_j such that $L_{j,n_j} \cap \bar{g}(Y) = \emptyset$. Consider the map $f_{j,n_j} : f^{-1}(\sigma_j) \to L_{j,n_j}$.

Step 7. Define $\bar{f} : X \to \mathbb{R}^4$ to be the map f_{j,n_j} on every $f^{-1}(\sigma_j)$ and f elsewhere. Then the maps $\bar{f}$ and $\bar{g}$ have disjoint images and they are ε-close to f and g, respectively, where ε is the mesh of τ_M and τ_N. Since the choice of τ_M and τ_N was arbitrary, this completes the proof of the theorem. □

References

[1] P.S. Aleksandrov, Introduction to Homological Dimension Theory (Nauka, Moskow, 1975) (in Russian).

[2] P.S. Aleksandrov and B.A. Pasynkov, Introduction to Dimension Theory (Nauka, Moscow, 1973) (in Russian).

[3] M.F. Bokštein, On homology invariants of topological spaces I, Trudy Moskov. Mat. Obshch. 5 (1956) 3–80 (in Russian); also: Amer. Math. Soc. Transl. 11 (2) (1959) 173–385 (in English).

[4] V.G. Boltjanskii, An example of a two dimensional compactum whose topological square has dimension equal to three, Dokl. Akad. Nauk SSSR 67 (4) (1949) 597–599 (in Russian); also: Amer. Math. Soc. Transl. 48 (1951) 3–6 (in English).

[5] A. Casson, Three lectures on new infinite construction in 4-dimensional manifolds, in: L. Guillou and A. Marin, eds., A la Recherche de la Topologie Perdue (Birkhäuser, Boston, 1986) 201–244.

[6] R.J. Daverman, Hereditarily aspherical compacta and cell-like maps, Topology Appl. 41 (1991) 247–254.

[7] A.N. Dranišnikov, Homological dimension theory, Uspekhi Mat. Nauk 43 (4) (1988) 11–55 (in Russian); also: Russian Math. Surveys 43 (4) (1988) 11–63 (in English).

[8] A.N. Dranišnikov and D. Repovš, On unstable intersections of 2-dimensional compacta in Euclidean 4-space, Abstracts Amer. Math. Soc. 10 (1989) 411.

[9] A.N. Dranišnikov, D. Repovš and E.V. Ščepin, On intersection of compacta of complementary dimension in Euclidean space, Topology Appl. 38 (1991) 237–253.

[10] A.N. Dranišnikov, D. Repovš and E.V. Ščepin, On intersections of compacta in Euclidean space: the metastable case, Tsukuba J. Math., to appear.

[11] R. Engelking, Dimension Theory (North-Holland, Amsterdam, 1978).

[12] M.H. Freedman and R.C. Kirby, Topology of 4-Manifolds (Princeton University Press, Princeton, NJ, 1989).

[13] D.B. Fuks, Eilenberg–MacLane complexes, Uspekhi Mat. Nauk 21 (5) (1966) 213–215 (in Russian); also: Russian Math. Surveys 21 (5) (1966) 205–207 (in English).

[14] J. Krasinkiewicz and K. Lorentz, Disjoint membranes in cubes, Bull. Polish Acad. Sci. Math. 36 (1988) 397–402.

[15] V.I. Kuz'minov, Homological dimension theory, Uspekhi Mat. Nauk 23 (5) (1968) 3–49 (in Russian); also: Russian Math. Surveys 23 (5) (1968) 1–45 (in English).

[16] W.J.R. Mitchell and D. Repovš, The topology of cell-like mappings, in: Proceedings Conference on Differential Geometry and Topology, Cala Gonone, 1988, Suppl. Rend. Sem. Fac. Sci. Univ. Cagliari 58 (1988) 265–300.

[17] W.J.R. Mitchell, D. Repovš and E.V. Ščepin, On 1-cycles and the finite dimensionality of homology 4-manifolds, Topology 31 (1992) 605–623.
[18] K. Nagami, Dimension Theory (Academic Press, New York, 1970).
[19] J.-I. Nagata, Modern Dimension Theory (North-Holland, Amsterdam, 1965).
[20] A.R. Pears, Dimension Theory of General Spaces (Cambridge University Press, Cambridge, 1975).
[21] L.S. Pontrjagin, Sur une hypothèse fondamentale de la théorie de la dimension, C.R. Acad. Sci. Paris 190 (1930) 1105–1107.
[22] E.H. Spanier, Algebraic Topology (McGraw-Hill, New York, 1966).
[23] S. Spież, Imbeddings in $\mathbb{R}^{2m}$ of m-dimensional compacta with $\dim(X \times Y) < 2m$, Fund. Math. 134 (1990) 105–115.
[24] S. Spież, The structure of compacta satisfying $\dim(X \times X) < 2 \dim X$, Fund. Math. 135 (1990) 127–145.
[25] S. Spież, On pairs of compacta with $\dim(X \times Y) < \dim X + \dim Y$, Fund. Math. 135 (1990) 213–222.
[26] D. Sullivan, Geometric Topology, Part I: Localization, Periodicity and Galois Symmetry (MIT, Cambridge, MA, 1970).