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On Alexandroff theorem for general Abelian groups

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Abstract

We present a technique for construction of infinite-dimensional compacta with given extensional dimension. We then apply this technique to construct some examples of compact metric spaces for which the equivalence $X\tau M(G,n) \Leftrightarrow X\tau K(G,n)$ fails to be true for some torsion Abelian groups G and $n \geqslant 1$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We shall work in the category of locally finite countable CW complexes and continuous maps. Recall that the *Kuratowski* notation $X\tau Y$ denotes that every extension problem on X has a *solution*, i.e., that for every closed subset $A \subset X$ and every map $f : A \to Y$ there exists an extension $\overline{f} : X \to Y$ of f over X [22, §VII.53.I]. This notation allows one to define very quickly the notion of the *covering* dimension [19] (respectively *cohomological* dimension [5,13], with respect to any Abelian group G) as follows: For every integer $n \geqslant 0$ and every compactum X, $\dim X \leqslant n \Leftrightarrow X\tau S^n$ (respectively $\dim_G X \leqslant n \Leftrightarrow X\tau K(G,n)$), where S^n is the standard n-sphere (respectively $K(G,n)$ is the Eilenberg–MacLane complex [30, §V.7]).

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In the 1930s Alexandroff [1,2] proved a fundamental result on homological dimension, which in the modern language reads as follows:

Theorem 1.1 (P.S. Alexandroff). *For every finite-dimensional compactum X, the following equality holds*:

$$\dim X = \dim_{\mathbb{Z}} X.$$

In the above notation this can be formulated as the equivalence $X\tau S^n \Leftrightarrow X\tau K(\mathbb{Z},n)$. Since the n-sphere S^n is a Moore space [30, §VII.7] of the type $M(\mathbb{Z},n)$, the equivalence $X\tau M(G,n) \Leftrightarrow X\tau K(G,n)$ would be a perfect extension of Theorem 1.1 to arbitrary Abelian groups. We shall assume throughout this paper that a Moore space $M(G,n)$ is simply connected if $n>1$, and that it has an Abelian fundamental group if $n=1$.

In the early 1990s the first author [8] proved this equivalence under the following restrictions:

Theorem 1.2 (A.N. Dranishnikov). *For every integer $n>1$ and every finite-dimensional compactum X, the following equivalence holds*:

$$X\tau M(G,n) \Leftrightarrow X\tau K(G,n).$$

In the present paper we investigate whether these restrictions can be omitted. First we note that the finite-dimensionality condition cannot be dropped for $G=\mathbb{Z}$ and $n>1$ (see [5]). Miyata [26] observed that this also holds for all finite groups. For $n=1$ the equality $M(\mathbb{Z},1)=K(\mathbb{Z},1)=S^1$ holds. This equality also holds for all torsion free Abelian groups G. However, for torsion groups this is false [26]. As it was proved in [8], the implication $X\tau M(G,n) \Rightarrow X\tau K(G,n)$ always holds.

Below we state our results. Details and necessary preliminaries will be given later on in the paper. Our main result (proved in Section 3) is a theorem which allows one to construct compacta with different extension properties—it is an extension of Theorem 2.4 from our earlier paper [14] to truncated cohomologies.

Theorem 1.3. *Let P and K be simplicial complexes and assume that K is countable. Let T^* be a truncated continuous cohomology theory such that $T^n(P)\neq 0$, for some $n<-1$ and $T^k(K)=0$, for all $k<n$. Then there exist a compactum X such that $e-\dim X \leqslant K$, and a T^n—essential map $f: X\to P$.*

In Section 4 we apply our main result to show that $M(\mathbb{Z}_p,1)$ and $K(\mathbb{Z}_p,1)$ are not extensionally equivalent in the class of all compacta, including the infinite-dimensional ones (see also [26] and [24] for $p=2$):

Theorem 1.4.

(1) *For every prime p, there exists an infinite-dimensional compactum X such that $\dim_{\mathbb{Z}_p} X=1$ and $e-\dim X > M(\mathbb{Z}_p,1)$.*

(2) *There exists a compactum Y such that $e - \dim Y \leqslant \mathbb{R}P^\infty$ and $e - \dim Y > \mathbb{R}P^m$, for all integers $m > 0$.*

2. On constructions of compacta having different cohomological and extensional dimensions

In this section it will be convenient to use the notation $e - \dim X \leqslant K$ for $X \tau K$ (see [9] or [12]) which reads *extensional dimension of X does not exceed K*.

Construction of compacta with different cohomological and extensional dimensions is presently a very active area of research. Here we outline three different approaches to the construction of such compacta. For convenience we give them the following names: *Combinatorial approach*, *Game with infinity* and *Splitting the space*. All three are important in the sense that there are problems where one approach is more suitable than the others.

Combinatorial approach. This approach was first used in the construction [27] of *Pontryagin surfaces*, i.e., 2-dimensional compacta with rational dimension one and 1-dimensional with respect to $\mathbb{Z}_p$ for all but one prime p. The idea of the construction is to start by a certain (finite) polyhedron, replace all of its simplices (in certain dimensions) by some building blocks, and then iterate this procedure infinitely many times. The resulting inverse limit space will usually have some exotic properties, depending on the properties of the building blocks. In the simplest Pontryagin's example one starts by the 2-sphere and the building block the Möbius band. Since the boundary of the 2-simplex is homeomorphic to the boundary of the Möbius band, it is easy to make replacements.

In the case of higher-dimensional simplicial complexes, finding proper building blocks is not so easy. Some interesting blocks were found by Boltyanskij [3], Kodama [20, 21] and Kuzminov [23]. Eventually, the first author [5] found the family of blocks which provides the solution to the Bockstein–Boltyanskij realization problem in cohomological dimension theory. All the blocks in [5] have in common certain features which first appeared in Walsh's proof [29] of the Edwards resolution theorem [18]. Having this in mind, Dydak and the first author extracted the axioms for the building blocks and named them the *Edwards–Walsh modification* (resolution) of a simplex (cf. [11,13,17]).

Game with infinity. This approach has a strong flavor of general topology. An exotic compactum is here also constructed as the limit space of an inverse sequence $\{X_i, q_i^{i+1}\}$. However, the spaces X_i are not necessarily as nice as above. On any compact metric space there exists a countable basis of extension problems to a given countable complex K. We may also assume that every one of these problems factors through some extension problem on X_i. If we can construct an inverse sequence in such a way that all extension problems on X_i, for all i are killed by passing to the limit, then the limit will be a compactum with desired properties. It is reasonable to require here that the projection q_i^{i+1} kills one given extension problem on X_i, i.e., for a given $f: A \to K$, where A is a closed subset of X_i, the map $f \circ q_i^{i+1}$ is extendable over X_{i+1}. In this way we produce infinitely many new extension problems on X_{i+1}. It seems that killing one extension problem and making infinitely many new ones will not make any progress in the task of getting rid of the unsolvable extension problems. But this is the standard game with infinity—like in the classical story about the hotel with infinitely many rooms [28]. So one can succeed—the correct strategy is to properly enumerate the extension problems. This approach first appeared in [11] (see also [14]).

Splitting the space. Here the idea is to produce an exotic space by splitting a nice space like $\mathbb{R}^n$ into exotic nuclear pieces. This approach appeared during the first author's work [10] on the mapping intersection problem (MIP). The MIP was reduced in [15] to a problem of imbedding a given cohomological dimension type in the n-dimensional Euclidean space. The clue to this problem was found in a generalization of the Urysohn Splitting theorem, which says that every n-dimensional compactum can be presented as the union of $n+1$ zero-dimensional spaces. The generalization of this, given in [10] says that if a join product $K(G_1, n_1) * \cdots * K(G_k, n_k)$ is $(n-1)$-connected then any n-dimensional compactum can be presented as the union $\bigcup X_i$, where $\dim_{G_i} X_i \leqslant n_i$. We note that the Urysohn Splitting theorem follows from the fact that the join product of $n+1$ zero-dimensional spheres is $(n-1)$-connected.

All approaches above give compacta with $e - \dim X \leqslant K$, for some countable K, which does not mean much unless we additionally require that $e - \dim X > L$, for some complex L. This property can be achieved by means of homology or cohomology. In [5] classical cohomology and K-theory were used. A breakthrough was made by Dydak and Walsh—they introduced truncated cohomology for this purpose [16] and used it in the combinatorial approach. As it was noted in [11], truncated cohomology can also be used in the game with infinity approach. Below we formulate a corresponding result (Theorem 1.3) which will be proved in Section 3 (for the most recent development see [24]).

We recall that a *truncated spectrum* is a sequence of pointed spaces $\mathbb{E} = \{E_i\}$, $i \leqslant 0$, such that $E_{i-1} = \Omega E_i$. Thus, any truncated spectrum is generated by the space E_0. The lower half of every Ω-spectrum is an example of a truncated spectrum. The *truncated cohomology* of a given space X with coefficients in a given truncated spectrum $T^i(X; \mathbb{E})$ is the set of pointed homotopy classes of mappings of X to E_i. Note that $T^i(X)$ is a group, for $i < 0$ and it is an Abelian group, for $i < -1$. Truncated cohomologies possess many features of a generalized cohomology. For every map $f: X \to Y$ there is the induced homomorphism $(i > 0)$ $f^*: T^i(Y) \to T^i(X)$. Homotopic maps induce the same homomorphism and a null-homotopic map induces zero homomorphism. There is the natural Mayer–Vietoris exact sequence:

$$\cdots \to T^r(A \cup B) \to T^r(A) \times T^r(B) \to T^r(A \cap B) \to T^{r+1}(A \cup B) \to \cdots$$

of groups, for $r \leqslant -1$ and Abelian groups, for $r \leqslant -2$. We call a truncated homology T^* *continuous* if for every direct limit of finite CW-complexes $L = \varinjlim\{L_i; \lambda_{i+1}^i\}$ the following formula holds $T^k(L) = \lim\limits_{\leftarrow} T^k(L_i)$, for $k < 0$. We note that the Milnor theorem holds for truncated cohomologies:

$$0 \to \lim{}^1\{T^{k-1}(L_i)\} \to T^k(L) \to \lim_{\leftarrow}\{T^k(L_i)\} \to 0.$$

Hence, if $T^k(M)$ is a finite group for every finite complex M and every $k < -1$, then by the Mittag-Leffler condition, T^* must be continuous. We can now restate our first main result:

Theorem 2.1. *Let P and K be simplicial complexes and assume that K is countable. Let T^* be a truncated continuous cohomology theory such that $T^n(P) \neq 0$, for some $n < -1$ and $T^k(K) = 0$, for all $k < n$. Then there exist a compactum X such that $e - \dim X \leqslant K$, and a T^n-essential map $f : X \to P$.*

3. Proof of Theorem 1.3

Definition 3.1. An extension problem (A, α) on a topological space X is a map $\alpha : A \to K$ defined on a closed subset $A \subset X$ with the range a CW-complex (or ANE). A *solution* of an extension problem (A, α) is a continuous extension $\bar{\alpha} : X \to K$ of a map α. A *resolution* of an extension problem (A, α) is a map $f : Y \to X$ such that the induced extension problem $f^{-1}(A, \alpha) = (f^{-1}(A), \alpha \circ f|\ldots)$ on Y has a solution.

Because of the Homotopy extension theorem, the solvability of extension problem (A, α) is an invariant of the homotopy class of α. We call two extension problems (A, α) and (A, β) *equivalent* if α is homotopic to β. A family of extension problems $\{(A_i, \alpha_i)\}_{i \in J}$ forms a *basis* if for every extension problem (B, β), there is $i \in J$ such that $B \subset A_i$ and the restriction $\alpha_i|_B$ is homotopic to β.

In view of the Homotopy extension theorem the following proposition is obvious:

Proposition 3.2. *Suppose that a map $f : Y \to X$ resolves all extension problems on X from a given basis $\{(A_i, \alpha_i)\}_{i \in J}$. Then f resolves all extension problems on X.*

Proposition 3.3. *Let K be any CW-complex and X the limit space of the inverse sequence of compacta $\{X_k, q_k^{k+1}\}$. Let $\{(A_i^k, \alpha_i^k)\}_{i \in J_k}$ be a basis of extension problems, for every k. Then*

$$\left\{(q_k^\infty)^{-1}(A_i^k, \alpha_i^k) \mid k \in \mathbb{N},\ i \in J_k\right\}$$

is a basis of extension problems on X, where $q_k^\infty : X \to X_k$ denotes the infinite projection in the inverse sequence.

Proof. Since $K \in$ ANE, there exist for every extension problem (A, α) on X, a number k and a map $\beta : q_k^\infty(A) \to K$ such that $\beta \circ q_k^\infty|_A$ is homotopic to α. Take a problem (A_i^k, α_i^k) serving for $(q_k^\infty(A), \beta)$ as a majoration: $\alpha_i^k|_{q_k^\infty(A)} \simeq \beta$. Then the extension problem $(q_k^\infty)(A_i^k, \alpha_i^k)$ is a majoration for (A, α). $\square$

The following lemma was proved in [14, Lemma 2.2]:

Lemma 3.4. *For every extension problem $(A, \alpha : A \to K)$ on X there is a resolution $g : Y \to X$ such that every preimage $g^{-1}(x)$ is either a point or it is homeomorphic to K. If additionally, X and K are simplicial complexes, A is a subcomplex and α is a simplicial map, then the resolving map g can be chosen to be simplicial.*

Proposition 3.5. *Let X be the limit space of an inverse sequence $\{X_k; q_k^{k+1}\}$ and let $\{(A_i^k, \alpha_i^k)\}_{i \in J_k}$ be a basis of extension problems for each k. Assume that q_k^∞ resolves all problems (A_i^k, α_i^k) for all k. Then $e - \dim X \leqslant K$.*

Proof. According to Proposition 3.3, X has a basis of solvable extension problems. Then by Proposition 3.2 applied to the identity map, all extension problems on X have solutions. This means that $e - \dim X \leqslant K$. □

Remark. If a map $f : Y \to X$ resolves some extension problem (A, α) on X, then for any map $g : Z \to Y$, the composition $f \circ g$ resolves (A, α).

Lemma 3.6. *Let $g : L \to M$ be a simplicial map onto a finite-dimensional complex M and let T^* be a truncated cohomology theory such that $T^k(g^{-1}(x)) = 0$, for all $k < n$. Then g induces an isomorphism $g^* : T^k(M) \to T^k(L)$, for $k < n$ and a monomorphism, for $k = n$.*

Proof. We proceed by induction on $m = \dim M$. If $\dim M = 0$, then lemma holds. Let $\dim M = m > 0$. We denote by A a regular neighborhood in M of the $(m-1)$-dimensional skeleton $M^{(m-1)}$. Since the map $g : L \to M$ is simplicial, $g^{-1}(A)$ admits a deformation retraction onto $g^{-1}(M^{(m-1)})$. By the inductive assumption, lemma holds for $g|\ldots : g^{-1}(M^{(m-1)}) \to M^{(m-1)}$. Hence, the conclusion of the lemma holds for $g|\ldots : g^{-1}(A) \to A$.

We define $B = M \setminus \operatorname{Int} A$, i.e., B is the union of disjoint m-dimensional PL-cells, $B = \bigcup B_i$. Since g is simplicial, $g^{-1}(B_i) \simeq g^{-1}(c_i) \times B_i$, where $c_i \in B_i$. Therefore the conclusion of lemma holds for $g|\ldots : g^{-1}(B) \to B$. Note that $\dim(A \cap B) = m - 1$ and hence lemma holds for $g|\ldots : g^{-1}(A \cap B) \to A \cap B$.

The Mayer–Vietoris sequence for the triad (A, B, M) produces the following diagram:

$$\begin{tikzcd}
T^k(A' \cap B') & T^k(A') \oplus T^k(B') \arrow[l] & T^k(L) \arrow[l] & T^{k-1}(A \cap B') \arrow[l] & \arrow[l] \\
T^k(A \cap B) \arrow[u] & T^k(A) \oplus T^k(B) \arrow[l] \arrow[u] & T^k(M) \arrow[l] \arrow[u, "g^*"] & T^{k-1}(A \cap B) \arrow[l] \arrow[u] & \arrow[l]
\end{tikzcd}$$

Here $A' = g^{-1}(A)$ and $B' = g^{-1}(B)$. The Five lemma implies that g^* is an isomorphism for $k < n$. The mono-version of the Five lemma implies that g^* is a monomorphism for $k = n$. □

Proof of Theorem 1.3. Since $T^n(P) \neq 0$, there exists a finite subcomplex $P_1 \subset P$ such that the inclusion is T^n-essential. This follows by continuity of T^*. We construct X as the limit space of an inverse sequence of polyhedra $\{P_k; q_k^{k+1}\}$, where $f : X \to P$ will be the composition of q_1^∞ and the inclusion $P_1 \subset P$. We construct this sequence by induction on k such that:

(1) For every k, there is a fixed countable basis of extension problems $\mathcal{A}^k = \{(A_i^k, \alpha_i^k)\}$ on P_k.
(2) For every k, some nonzero element $a_k \in T^n(P_k)$ is fixed such that $(q_k^{k+1})^*(a_k) = a_{k+1}$, for all k.
(3) For every problem $(A_i^k, \alpha_i^k) \in \mathcal{A}^k$, there is $j > k$ such that q_k^j is the corresponding resolution.

If we manage to construct such a sequence, then by Proposition 3.5, $e - \dim X \leqslant K$. Property (2) will then imply that f is T^n-essential. Thus, Theorem 1.3 will be proved.

Enumerate all prime numbers $2 = p_1 < p_2 < p_3 < \cdots < p_k < \cdots$. We fix some nonzero element $a_1 \in T^*(P_1)$ which comes from an element $a \in T^n(P)$. Denote by τ_1 a triangulation on P_1 and by $\beta^k\tau$ the kth barycentric subdivision of τ. There are only countably many subpolyhedra in P_1 with respect to all subdivisions $\beta^k\tau$. Since the set of homotopy classes $[L, K]$ is countable, we have only countably many inequivalent extension problems (A, α) defined on these subpolyhedra, for every compact L. Denote the set of all these extension problems (L, α) on P_1 with simplicial maps α by $\mathcal{A}^1$. Since $K \in$ ANE, it easy to show that $\mathcal{A}^1$ forms a basis of extension problems on P_1. We enumerate elements of $\mathcal{A}^1$ by all powers of 2. Let $N : \mathcal{A}^1 \to \mathbb{N}$ be the enumeration function.

Consider an extension problem from $\mathcal{A}^1$ having number one in our list and resolve it by a simplicial map $g : L \to P_1$ by means of Lemma 3.4. By Lemma 3.6, $g^* : T^n(P_1) \to T^n(L)$ is a monomorphism. Let $g^*(a_1) = a_2'$. Since a truncated cohomology T^* is continuous, there is a finite subcomplex $P_2 \subset L$ and a nonzero element $a_2 \in T^n(P_2)$ which comes from a_2' under the inclusion homomorphism. We define the bonding map $q_1^2 : P_2 \to P_1$ as the restriction $f|_{P_2}$ of f onto P_2. Then the condition (2) holds: $(q_1^2)^*(a_1) = a_2$.

Define a countable basis $\mathcal{A}^2 = \{(A_i^2, \alpha_i^2)\}$ of extension problems such that every A_i^2 is a subcomplex of P_2 with respect to iterated barycentric subdivision of the triangulation on P_2. Enumerate elements of $\mathcal{A}^2$ by all numbers of the form $2^k 3^l$ with $k \geqslant 0$ and $l > 0$. Lift all the problems from the list $\mathcal{A}^1$ to a space P_2, i.e., consider $(q_1^2)^{-1}(\mathcal{A}^1)$. Thus the family $(q_1^2)^{-1}(\mathcal{A}^1) \cup \mathcal{A}^2$ is enumerated by all numbers of the form $2^k 3^l$. Let

$$N : (q_1^2)^{-1}(\mathcal{A}^1) \cup \mathcal{A}^2 \to \mathbb{N}$$

be the enumeration function. Now consider the extension problem having number 2 in the updated list and apply the entire procedure described above to obtain P_3. Etc.

Thus, all problems in $\mathcal{A}^k$ will be enumerated by numbers of the form $p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}$ with $l_k > 0$. Since $k \leqslant p_k$, we have $k \in N((q_1^k)^{-1}(\mathcal{A}^1) \cup (q_2^k)^{-1}(\mathcal{A}^2) \cup \cdots \cup \mathcal{A}^k)$. Hence we can keep going, for any k. As the result of this construction we have that if a problem (A_i^l, α_i^l) has number k, then $l \leqslant k$ and the problem is resolved by q_l^{k+1}. Thus, the conditions (1)–(3) hold. □

4. Proof of Theorem 1.4

We consider the truncated cohomology T_p^* generated by the mapping space $E_0 = (S^n)^{M_p}$, where $M_p = M(\mathbb{Z}_p, 1)$ is a Moore space of the type $(\mathbb{Z}_p, 1)$ and S^n is the n-dimensional sphere.

Lemma 4.1. *The truncated cohomology theory T_p^* is continuous.*

For the proof we need the following proposition:

Proposition 4.2. *Let $\nu_p : S^1 \to S^1$ be a map of degree p. Then the map $f = \nu_p \wedge id : S^1 \wedge M_p \to S^1 \wedge M_p$ is null-homotopic.*

Proof. The space $S^1 \wedge M_p$ is the suspension ΣM_p of the space M_p and it can be defined as the quotient space of a map $h : B^3 \to \Sigma M_p$. Temporarily we denote a fixed map of degree p between 2-spheres by p, and we denote the identity map on the 2-sphere by 1. Let C_q denote the mapping cone of a map $q : X \to Y$, i.e., $C_q = \mathrm{Cone}(X) \cup_q Y$. Consider the following commutative diagram:

$$\begin{tikzcd}
S^2 \arrow[r, "1"'] \arrow[d, "1"] & S^2 \arrow[r] \arrow[d, "p"] & C_1 \arrow[d, "h"] \\
S^2 \arrow[r, "p"'] & S^2 \arrow[r] & C_p \\
S^2 \arrow[r, "p"'] \arrow[u, "p"] & S^2 \arrow[r] \arrow[u, "p"] & C_p \arrow[u, "g"]
\end{tikzcd}$$

Here, the mapping cone C_1 is homeomorphic to the 3-ball B^3 and C_p is homeomorphic to ΣM_p. First we note that the map g is homotopic to the map $\nu_p \wedge id$. Then we show that g has a lift $g' : \Sigma M_p \to B^3$ with respect to h. In fact, g' is defined by the following diagram:

$$\begin{tikzcd}
S^2 \arrow[r, "1"'] & S^2 \arrow[r] & C_1 \\
S^2 \arrow[r, "p"'] \arrow[u, "p"] & S^2 \arrow[r] \arrow[u, "1"] & C_p \arrow[u, "g'"]
\end{tikzcd}$$

Since B^3 is contractible, g' is null-homotopic and hence g is null-homotopic. □

Proof of Lemma 4.1. We show that every element of the group $T_p^k(L)$ has order p for $k < 0$. Indeed, $T_p^k(L) = [L, \Omega^{-k}(S^n)^{M_p}] = [\Sigma M_p, (S^n)^{\Sigma^{-k-1}L}]$. For any space N and any element $a \in [\Sigma M_p, N]$, represented by a map $f : \Sigma M_p \to N$, the element pa is represented by a map $f \circ (\nu_p \wedge id)$ and it is homotopic to zero, by virtue of Proposition 4.2. Note that $T_p^k(L) = [S^k \wedge L \wedge M_p, S^n]$. When the complex L is finite, this group is finitely generated. Hence in the case of $k < -1$, the group $T_p^k(L)$ of any finite complex L is finite. As we have already observed, this suffices for the continuity. □

Proposition 4.3. *For every integer $k < 0$, the following equality holds*:

$$T_p^k\big(K\big(\mathbb{Z}\big[\tfrac{1}{p}\big], 1\big)\big) = 0.$$

Proof. We can represent $K(\mathbb{Z}[\frac{1}{p}], 1)$ as the direct limit of complexes L_i, where each L_i is homotopy equivalent to the circle S^1 and every bonding map $\xi_i : L_i \to L_{i+1}$ is homotopy equivalent to a map of degree p of S^1 to itself. Then

$$\begin{aligned} T_p^k\big(K\big(\mathbb{Z}\big[\tfrac{1}{p}\big], 1\big)\big) &= \big[\varinjlim\{L_i, \xi_i\}, \Omega^k(S^n)^{M_p}\big] \\ &= \big[(\varinjlim\{L_i, \xi_i\}) \wedge M_p, \Omega^k S^n\big] = \big[\varinjlim\{L_i \wedge M_p, \xi_i \wedge id\}, \Omega^k S^n\big]. \end{aligned}$$

Consider a bonding map $\xi_i \wedge id : L_i \wedge M_p \to L_{i+1} \wedge M_p$. This map is homotopy equivalent to the map $\nu_p \wedge id$ and hence it is homotopically trivial, by Proposition 4.2. Therefore the space $\varinjlim\{L_i \wedge M_p, \xi_i \wedge id\}$ is homotopically trivial. Hence $T_p^k(K(\mathbb{Z}[\frac{1}{p}], 1)) = 0$. □

We also need the following result of Miller [25] (Sullivan conjecture):

Theorem 4.4 (H. Miller). *Let K be a finite-dimensional CW-complex and π a finite group. Then the mapping space $K^{K(\pi,1)}$ is weakly homotopy equivalent to the point.*

Proposition 4.5. *For every integer k, the following equality holds*:

$$T_p^k\big(K(\mathbb{Z}_p, 1)\big) = 0.$$

Proof. We note that by Theorem 4.4,

$$T_p^k\big(K(\mathbb{Z}_p, 1)\big) = \big[K(\mathbb{Z}_p, 1), (S^n)^{\Sigma^k M_p}\big] = \big[\Sigma^k M_p, (S^n)^{K(\mathbb{Z}_p,1)}\big] = 0$$

so the assertion follows. □

Proof of Theorem 1.4. (1) We take $T^* = T_p^*$ for $n = 7$, and define $P = \Sigma M_p$ and $K = K(\mathbb{Z}_p, 1) \vee K(\mathbb{Z}[\frac{1}{p}], 1)$. Note that

$$\begin{aligned} T^{-2}(\Sigma M_p) &= \big[\Sigma M_p, \Omega^2(S^7)^{M_p}\big] = \big[\Sigma M_p \wedge S^2 \wedge M_p, S^7\big] \\ &= \big[\Sigma^3 M_p \wedge M_p, S^7\big] = H^7\big(\Sigma^3 M_p \wedge M_p\big) \neq 0. \end{aligned}$$

By Propositions 4.2 and 4.4 we have $T^k(K) = 0$, for all k. We now apply Theorem 1.3 to obtain a compactum X such that

$$\dim_{\mathbb{Z}_p} X \leqslant 1 \quad \text{and} \quad \dim_{\mathbb{Z}[1/p]} X \leqslant 1$$

and to get an essential map $f : X \to \Sigma M_p$.

Let $A = f^{-1}(M_p)$ where M_p is embedded in P as the equator. Then the map $f|_A : A \to M_p$ does not have any extension (otherwise an extension g would be null-homotopic as a map to P and homotopic to f). Hence $e - \dim X > M(\mathbb{Z}_p, 1)$. Since for 2-dimensional compacta the inequality $\dim_{\mathbb{Z}_p} X \leqslant 1$ implies $e - \dim X \leqslant M(\mathbb{Z}_p, 1)$, we have that $\dim X > 2$. The short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}\big[\tfrac{1}{p}\big] \to \mathbb{Z}_{p^\infty} \to 0$$

and Bockstein's inequality $\dim_{\mathbb{Z}_{p^\infty}} X \leqslant \dim_{\mathbb{Z}_p} X$ imply that $\dim_{\mathbb{Z}} X \leqslant 2$. Finally, Theorem 1.1 implies that X is infinite-dimensional.

(2) For a given m, we take $T^* = T_2^*$ for $n = m + 5$ and $P = \Sigma\mathbb{R}P^m$ and $K = \mathbb{R}P^\infty$. Then

$$T^{-2}(P) = [\Sigma^3\mathbb{R}P^m \wedge \mathbb{R}P^2, S^{m+5}] = H^{m+5}(\Sigma^3\mathbb{R}P^m \wedge \mathbb{R}P^2) \neq 0.$$

By Theorem 1.3 we obtain a compactum X_m such that $e - \dim X_m \leqslant \mathbb{R}P^\infty$ and $e - \dim X_m > \mathbb{R}P^m$. Finally, we define Y to be the one-point compactification of the disjoint union of all X_m. □

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