Доклады Болгарской академии наук
Comptes rendus de l'Académie bulgare des Sciences
Tome 41, No 2, 1988

MATHEMATIQUES
Topologie

AMBIENT ISOTOPIES IN TOPOLOGICAL MANIFOLDS WHICH RESPECT OPEN COVERINGS

D. Repovš

(Submitted by Corresponding Member P. Kenderov on July 27, 1987)

Let M be closed (i. e. compact, connected and having no boundary) topological n-manifold ($n \in \mathbb{N}$) and let $\mathcal{U}$ be a (finite) open covering of M. It was shown by Edwards a. Kirby [1] (1.3) that every isotopy $H: M \times I \to M$ of M (i. e. H is a continuous mapping and for every $t \in I=[0, 1]$, the restriction $H|(M\times\{t\}):(M\times\{t\})\to M$ is a homeomorphism) can be decomposed into a finite collection of isotopies $H=H_m \dots H_2H_1$, where each $H_k: M\times I\to M$ is supported by some element of the covering $\mathcal{U}$ (i. e. $H_k|((M-U_k)\times\{t\})$ is the identity map for some $U_k \in \mathcal{U}$). That is — given any two isotopic homeomorphisms $f, g: M\to M$, f and g are isotopic via arbitrarily small moves.

In 1979 R. C. Lacher asked the following question: Suppose that, in addition, elements of $\mathcal{U}$ are topological n-cells and that f and g are $\mathcal{U}$-close, i. e. that for every $x \in M$, there is an element of the covering $U_x \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U_x$. Is it then possible to isotope f to g via an isotopy $H: M\times I\to M$ whose moves are not only small but are also $\mathcal{U}$-bounded (i. e. H has the following properties: (i) $H|(M\times\{0\})=f$; (ii) $H|(M\times\{1\})=g$; and (iii) for every $x \in M$, there is an element of the covering $U_x \in \mathcal{U}$ such that $H(\{x\}\times I)\subset U_x$)?

In this paper we answer this question in dimensions 1 and 2. We construct an example in each dimension of a homeomorphism $h: M\to M$ which is isotopic to the identity and is $\mathcal{U}$-close to the identity, but there is no isotopy $H: M\times I\to M$ from h to the identity which would be $\mathcal{U}$-bounded.

The question of R. C. Lacher remains unanswered in higher dimensions. It is quite possible that our construction of the two-dimensional example on $M=T^2=S^1\times S^1$ can be generalized to the n-dimensional torus $T^n=S^1\times \dots \times S^1$ ($n\geq 3$), by appropriate modifications of $\mathcal{U}$ and h. This would confirm our expectations that the answer should be negative in all dimensions. Finally, an analogous question can be asked for topological manifolds with boundary, where one should assume, in addition, that the maps (f, g, H) are fixed on the boundary of M.

The 1-Dimensional Example. Let $M=S^1=\{e^{it}\,|\,0\leq t<2\pi\}$ and define an open covering $\mathcal{U}$ of M by 1-cells

$$U_1=S^1-\{p_1\} \text{ and } U_2=S^1-\{p_2\},$$

where $p_1=1$ and $p_2=i$. Let $h: M\to M$ be the homeomorphism given by

$$h(e^{it})=e^{i\left(t+\frac{\pi}{2}\right)}, \text{ for every } t\in[0, 2\pi).$$

Clearly, h is $\mathcal{U}$-close to the identity and $H_0 : M\times I\rightarrow M$, given by

$$H_0(e^{it}, s)=e^{i\left(t+\frac{\pi . s}{2}\right)}, \quad \text{for every } (t, s)\in[0, 2\pi)\times I$$

is an isotopy from the identity to h.

We now demonstrate that there can be no isotopy H of M from the identity to h, which would also be $\mathcal{U}$-bounded. Suppose on the contrary that such an H did exist. Let $A_k=H(\{p_k\}\times I)$, $k=1, 2$, be the paths of points p_k under such an isotopy H. Consider the A_k's in $M\times I$=the image of the homeomorphism $G: M\times I\rightarrow M\times I$, given by

$$G(x, t)=(H(x, t), t), \text{ for every } (x, t)\in M\times I.$$

Let $B_k=h(p_k)\times I\subset G(M\times I)$, $k=1, 2$. Assume that we are in the PL category and that all orientations are induced by the fixed one on M. Let $n(k)=\#(A_k, B_{3-k})$, $k=1, 2$, be the integral intersection number [2]; p. 68 of the arcs A_k, B_{3-k}. Clearly,

$$n(1) . n(2)=-1$$

since each A_k lies in $U_{3-k}\times I$. However, the restriction $G|(D\times I): D\times I\rightarrow M\times I$, where

$$D=\left\{e^{it}\,|\,0\leq t\leq\frac{\pi}{2}\right\}$$

yields a homotopy in $(M\times I, M\times\partial I)$ from $(A_1, \partial A_1)$ to $(A_2, \partial A_2)$, implying that $n(1)=n(2)$. This contradiction shows that H cannot exist.

The 2-Dimensional Example. Let M be the 2-torus

$$M=\{\vec{r}(u, v)\in\mathbf{R}^3\,|\,0\leq u<2\pi, \quad 0\leq v<2\pi\},$$

where

$$\vec{r}(u, v)=((b-a\cos u)\cos v, (b-a\cos u)\sin v, a\sin u)$$

and $0<a<b$. Let

$$V_1=\left\{\vec{r}(u, v)\,|\,(u, v)\in\left(-\frac{1}{\pi}, \frac{\pi}{2}+\frac{1}{\pi}\right)\times\left(-\frac{1}{\pi}, \frac{\pi}{2}+\frac{1}{\pi}\right)\right\}$$

$$V_2=\left\{\vec{r}(u, v)\,|\,(u, v)\in\left(-\frac{1}{\pi}+\frac{\pi}{2}, \pi+\frac{1}{\pi}\right)\times\left(-\frac{1}{\pi}, \frac{\pi}{2}+\frac{1}{\pi}\right)\right\}$$

$$V_3=\left\{\vec{r}(u, v)\,|\,(u, v)\in\left(-\frac{\pi}{2}-\frac{1}{\pi}, \frac{1}{\pi}\right)\times\left(-\frac{1}{\pi}, \frac{\pi}{2}+\frac{1}{\pi}\right)\right\}$$

$$V_4=\left\{\vec{r}(u, v)\,|\,(u, v)\in\left(\pi-\frac{1}{\pi}, \frac{3\pi}{2}+\frac{1}{\pi}\right)\times\left(-\frac{1}{\pi}, \frac{\pi}{2}+\frac{1}{\pi}\right)\right\}$$

$$V_5=\left\{\vec{r}(u, v)\,|\,(u, v)\in\left(-\frac{1}{\pi}, \pi+\frac{1}{\pi}\right)\times\left(\frac{\pi}{2}-\frac{1}{\pi}, 2\pi+\frac{1}{\pi}\right)\right\}$$

$$V_6=\left\{\vec{r}(u, v)\,|\,(u, v)\in\left(\pi-\frac{1}{\pi}, 2\pi+\frac{1}{\pi}\right)\times\left(\frac{\pi}{2}-\frac{1}{\pi}, 2\pi+\frac{1}{\pi}\right)\right\}$$

and define the open cover $\mathcal{U}=\{U_k\,|\,1\leq k\leq 10\}$ by

$$U_k=\begin{cases} V_k\cup h(V_k); & 1\leq k\leq 4\\ V_{k-4}\cup h^{-1}(V_{k-4}); & 5\leq k\leq 8\\ V_{k-4}; & 9\leq k\leq 10\end{cases}$$

Finally, let $h\ M\to M$ be given by

$$h(\vec{r}(u,\ v))=\vec{r}(u,\ u+v),\ \text{for every } (u,\ v)\in[0,\ 2\pi)\times[0,\ 2\pi).$$

Clearly, the elements of $\mathcal{U}$ are open 2-cells and $M=\bigcup_{k=1}^{10}U_k$.

Assertion 1. For every $x\in M$ there is an index $k\in\{1,\ 2,\dots,\ 10\}$ such that $\{x,\ h(x)\}\subset U_k$.

Proof. Let $x\in M$. Then $x\in V_k$ for some $k\in\{1,\ 2,\dots,\ 6\}$. If $1\leq k\leq 4$ then $\{x,\ h(x)\}\subset U_k$. So assume that $k=5,6$. Then either

$$h(x)\in V_k \text{ hence } \{x,\ h(x)\}\subset U_{k+4}$$

or

$$h(x)\in V_m \text{ for some } m\in\{2k-8,\ 2k-9\} \text{ hence } \{x,\ h(x)\}\subset U_{m+4}.$$

Assertion 2. h is isotopic to the identity map.

Proof. Let $H\colon M\times I\to M$ be given by

$$H(\vec{r}(u,\ v),\ t)=\vec{r}(u,\ tu+v),\ \text{for every } \vec{r}(u,\ v)\in M \text{ and } t\in I.$$

Then H is an ambient isotopy from $H|M\times\{0\}=id_M$ to $H|M\times\{1\}=h$.

Assertion 3. There is no isotopy of M from id_M to h which would be $\mathcal{U}$-bounded.

Proof. Suppose on the contrary that there was such an isotopy $G\colon M\times I\to M$. Consider the arc

$$C=\{\vec{r}(u,\ v)\,|\,u\in[0,\ 2\pi],\ v=0\}.$$

Then $\partial C=\{p_1,\ p_2\}$, where $p_1=\vec{r}(0,\ 0)$ and $p_2=\vec{r}(\pi,\ 0)$. By hypothesis, each arc $A_j=G(\{p_j\}\times I)$, $j=1,\ 2$, lies in some 2-cell $U_{i(j)}$, $i(j)\in\{1,\ 2,\dots,\ 10\}$. Since $D=G(C\times\{1\})$ is isotopic to C, it follows that D is homotopic to C rel$\{p_1,\ p_2\}$. Therefore, the loop $D.C^{-1}$ is homotopically trivial in M. But $D.C^{-1}$ is one of the two generators of $\pi_1(M)=\langle\alpha,\ \beta\,|\,\alpha\beta\alpha^{-1}\beta^{-1}\rangle$ [3]; (III. 8. 12.) hence nontrivial. This contradiction shows that such G cannot exist.

Institute of Mathematics, Physics and Mechanics
University of Ljubljana
Jadranska cesta 19
61000 Ljubljana, Yugoslavia

REFERENCES

[1] Edwards, R. D., R. C. Kirby. Ann. Math. **93**, 1971, 63-88. [2] Rourke, C. P., B. J. Sanderson. Introduction to Piecewise-Linear Topology, Springer-Verlag, 1972. [3] Spanier, E. H. Algebraic Topology, McGraw-Hill, 1966.