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Note

An infinite sequence of non-realizable weavings☆

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Abstract

A *weaving* is a number of lines drawn *in the plane* so that no three lines intersect at a point, and the intersections are drawn so as to show which of the two lines is above the other. For each integer $n \geqslant 4$ we construct a weaving of n lines, which is not realizable as a projection of a number of lines in 3-space, all of whose subfigures are realizable as such projections.

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A *weaving* is a collection of lines drawn in the plane so that no three lines intersect at a point, and the intersections are drawn so as to show which of the two lines is "above"

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the other (see the figures below). A weaving is called *realizable* if it is a projection onto a plane of a collection of pairwise skew lines in the 3-space. This problem has appeared in algebraic knot theory and discrete geometry ([3,9,7], see also [1,2]).

A *subweaving* of weaving A is a weaving obtained from A by deleting some lines. The purpose of this note is to solve a problem from [3, p. 262]:

Theorem. *For each integer $n \geqslant 4$ there exists a nonrealizable weaving, consisting of n lines, all of whose subweavings are realizable* (*see Figs.* 1 *and* 2 *for* $n = 4$ *and* $n = 5$).

Proof. For $k = 1, \dots, n$ take the line k passing through the points $(n - k, 0)$ and $(0, k - 1)$ of the Cartesian plane. Note that these lines cross in the plane in the following order: for $k = 2, \dots, n - 1$ the line k crosses the lines $1, 2, \dots, k - 1, k + 1, \dots, n$ as one moves up the page (see Figs. 1–3).

For $k = 2, \dots, n - 1$ set the line k to be "above" the lines $1, \dots, k - 2, k + 1$ and "below" the other lines.

Set line 1 to be "above" the lines 2, n and "below" the other lines.

Set line n to be "above" the lines $2, 3, \dots, n - 2$ and "below" the other lines.

Let us prove that all subweavings are realizable. A line of a weaving is called *monotone*, if it contains a point such that the line is "above" all other lines on one side of the point, and "below"—on the other side (in particular, a line which is above all or below all other lines is also monotone). Clearly, the realizability of a weaving is not changed under a deletion of a monotone line. After deleting the line k, the line $k - 1$ becomes monotone; the line n becomes monotone after deleting the line 1. So after deleting any of the lines we can delete monotone lines of the remaining weaving one after another, hence the subweaving is realizable.

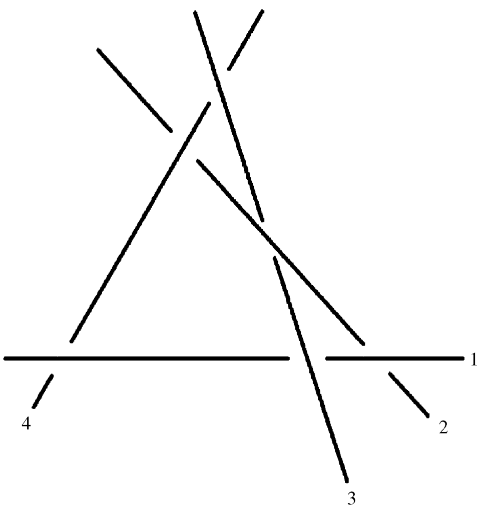

Fig. 1.

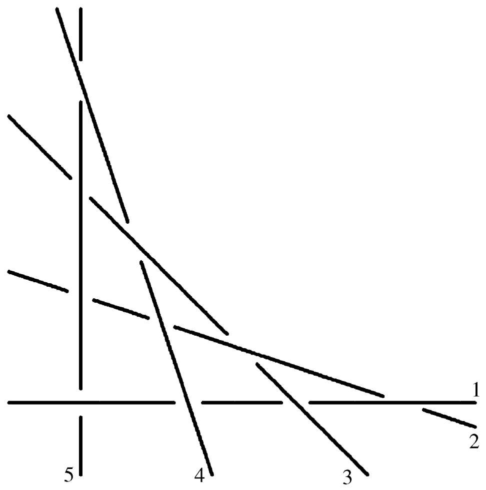

Fig. 2.

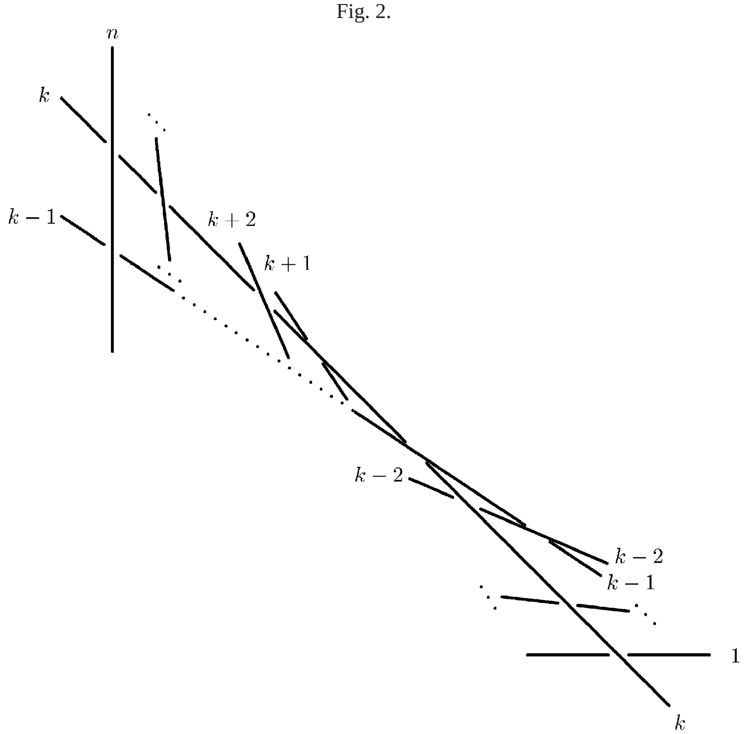

Fig. 3.

Let us prove that the resulting weaving is not realizable (we acknowledge one of the referees for shortening this proof). Take a collection of lines $1, 2, \ldots, n$ in the 3-space projecting onto our weaving. Move line 1 (in the vertical plane, preserving the projection) so that it would meet *n* (but no other lines). Denote by 1*n* the plane going through the (new) lines 1 and *n*.

Now we prove by induction on $k = 2, \ldots, n-2$ that

all points of line k 'between lines $k-1$ *and n' lie below* $1n$.

Line 2 is below both 1 and *n*, so the base $k = 2$ of the induction follows. Line *k* is below line $k-1$ (by the inductive hypothesis, at a point where $k-1$ is below $1n$) and below line *n*, so the inductive step follows.

Finally, line $n-1$ is above lines 1 and *n*, so all points of line $n-1$ 'between lines 1 and *n*' lie above $1n$. So line $n-1$ is below line $n-2$ at a point where $n-2$ is below $1n$ and $n-1$ is above $1n$, which is a contradiction. □

Another motivation for our above theorem was the following. Clearly, a weaving containing a nonrealizable subweaving is nonrealizable. There was a conjecture that nonrealizable weavings can be described in the spirit of the Kuratowski description of planar graphs. Namely, there exists a finite collection $L_1, \ldots, L_n$ of nonrealizable weavings such that an arbitrary weaving is realizable if and only if it does not contain any of $L_1, \ldots, L_n$ as a subweaving.

Our result above shows that this conjecture is false, because there exists an infinite sequence of nonrealizable weavings (consisting of different number of lines) all of whose subweavings are realizable. A construction of another such sequence (for which the proof is more complicated) with any odd $n \geqslant 5$ number of lines is sketched in Figs. 4 and 5 for $n = 5, 7$ (this sequence was also exhibited in [4, Theorem 3.7, p. 171] and, without a proof, in [6], [5, Fig. 6, p. 307]). This conjecture can alternatively be disproved using estimates as in [3, Section 4].

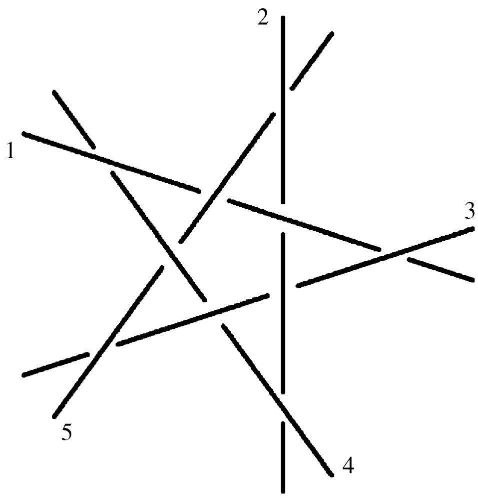

Fig. 4.

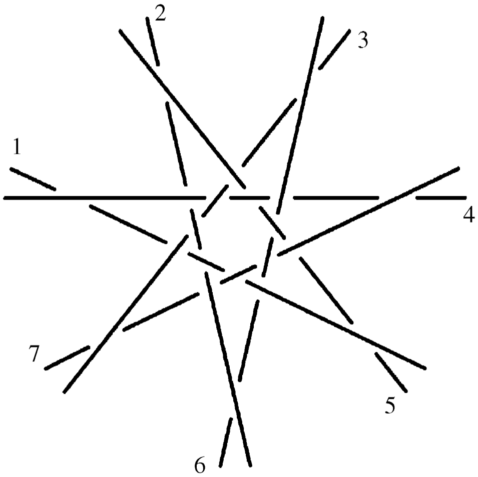

Fig. 5.

Note that the realizability of weavings is projectively invariant, provided that when the projective map cuts between two crossings, the crossings on ‘one side’ of the line going to infinity are reversed, and those on the other side are not reversed. This is related to the fact that there is a projective duality, though our proof does not require the duality.

Note also that there is a basic duality between weavings (and partial weavings) and plane tensegrity frameworks with inequality constraints (cables and struts in place of over and under) [8,9]. This gives an informal “physical” interpretation of this problem [cf. 9]. Suppose we have long and thin enough wooden rods (or strips). By slightly bending the rods, let us construct a model in the 3-space of the given weaving. A weaving is realizable if and only if such a construction does not contain any *rigid* subconstructions.

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