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On Homogeneity of Compacta in Manifolds (**).

Abstract. - *We introduce the concept of* CAT *homogeneity* (CAT = TOP, *PL, DIFF) for an n-dimensional compactum* $N \subsetneq M$ *in an m-dimensional* CAT *manifold M,* $n \leq m$: *N is said to be* CAT *homogeneous in M if for every two points* $x, y \in N$ *there is a* CAT *isomorphism* $h: M \to M$ *such that* $h(N) \subsetneq N$ *and* $h(x) = y$. Results: 1) *The image of a continuous, locally one-to-one map* $f: N \to M$ of a TOP *n-manifold N in a* TOP *m-manifold M,* $n < m$, *which is* TOP *homogeneous, is a* TOP *n-submanifold of M and, moreover,* $f: N \to f(N)$ *is a covering map.* 2) *If N is a* CAT *n-submanifold of a* CAT *m-manifold M,* $n < m$, *then N is* CAT *homogeneous (if* CAT = TOP *or* PL *one must, assume, in addition, that N is locally flat in M).* 3) *If N is a* TOP *n-submanifold of an orientable* DIFF $(n + 1)$*-manifold M and if N is* DIFF *homogeneous in M, then N has double tangent balls in M. We note that* 3) *gives a new criterion for the existence of double tangent balls, which were studied earlier by L. D. Loveland and D. G. Wright. For an n-dimensional manifold N in an* $(n + 1)$*-dimensinal manifold M we study the relations among the following properties of N: smoothly embedded, tamely embedded,* DIFF *homogeneous, and having double tangent balls.*

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THEOREM 2. *Let N be a compact manifold, and $f: N \to M$ a continuous locally one-to-one map. Then, there exists a point $w \in f(N)$ which has an open m-ball neighborhood W in M such that $W \cap f(N)$ has an n-ball neighborhood of w in $f(N)$.*

PROOF. Choose a point $x \in f(N)$ and set $A = f^{-1}(x)$. Suppose A is infinite. Then, due to compactness of N, there is an infinite sequence (y_s) in A which converges to some point $z \in A$. Therefore, for every neighborhood U of z in $N, f|_U : U \to f(U)$ fails to be one-to-one. So, we have proved the following fact:

Fact 1. For every $x \in f(N), f^{-1}(x)$ is a finite set. ■

For every $x \in f(N)$, define $p(x) = \text{card}(f^{-1}(x))$. Choose $x \in f(N)$, and denote $f^{-1}(x) = \{y_1, \ldots, y_{p(x)}\}$. Find open, pairwise disjoint n-ball neighborhoods U_i of y_i in N_i, such that $f|_{U_i} : U_i \to f(U_i)$ is one-to-one. Let $V_i = f(U_i)$, $U = \bigcup_{i=1}^{p(x)} U_i$, $V = \bigcup_{i=1}^{p(x)} V_i$. Note, that it is possible to have $V_r = V_s$ for some $r \neq s$.

Fact 2. For every $x \in f(N)$, $\{U_i\}$, $\{V_i\}$, U, V, as above there is $\delta(x) > 0$, such that $(f(N) \setminus V) \cap B(x, \delta(x)) = \emptyset$, where $B(x, \delta(x)) = \{u \in M \mid d(x, u) < \delta(x)\}$ = open m-ball in M with center at x and radius $\delta(x)$ (where M is equiped with a metric d).

PROOF. Suppose not. Then for every $r \in \mathbb{N}$, there is a point $z_r \in (f(N) \setminus V) \cap B(x, 1/r)$. Clearly, the sequence (z_r) converges to x. For every $z_r \in f(N)$, let $v_r \in f^{-1}(z_r) \subseteq N$ be an arbitrary point. By compactness of N, there is a convergent subsequence (v_{r_k}) of the sequence (v_r). Let (v_{r_k}) converge to v. Then due to the continuity of f, $(f(v_{r_k}))$ converges to $f(v)$, and since $f(v_{r_k}) = z_{r_k}$, and (z_{r_k}) converges to x, it follows that $f(v) = x$, i.e. $v \in f^{-1}(x)$. Therefore, $v = y_t$ for some $t \in \{1, \ldots, p(x)\}$. Since $U_t \subseteq N$ is an open neighborhood of y_t, there exists $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, $v_{r_k} \in U_t$, and consequently, $f(v_{r_k}) = z_{r_k}$ must lie in $f(U_t) = V_t \subseteq V$. However, this directly contradicts our choice of z_r's. ■

Now, let $x \in f(N)$, $\{U_i\}$, $\{V_i\}$, U, V, and $\delta(x)$ be as in Fact 2. Note, that if for some $\varepsilon > 0$, $B(x, \varepsilon) \cap V_i = B(x, \varepsilon) \cap V_j$, for some $i \neq j$, then for every $\varepsilon' < \varepsilon$, the same fact holds.

Let $q(x)$ be the number of distinct V_i's with respect to $B(x, \delta(x))$, i.e. their intersection with $B(x, \delta(x))$ are not the same sets. Clearly, $p(x) \geqslant$

$\geqslant q(x)$. Since $q(x) \leqslant p(x) < \infty$ (Fact 1), there is $0 < \varepsilon(x) < \delta(x)$ such that for each $\varepsilon < \varepsilon(x)$, the number of distinct V_i's with respect to $B(x, \varepsilon)$ is the same (i.e. is constant). Denote this number by $r(x)$. Reindex the V_i's to get $V_1', \ldots, V_{r(x)}'$, all distinct with respect to $B(x, \varepsilon(x))$.

Suppose that $r(x) = 1$. Since $f|_{U_1'} : U_1' \to V_1' = f(U_1')$ is a homeomorphism, $B(x, \varepsilon(x))$ is an m-ball neighborhood of x in M, such that $B(x, \varepsilon(x)) \cap f(N) = V'$ has an n-ball neighborhood of x in $f(N)$.

Next, suppose that $r(x) > 1$. Since, by our choice, $V_1' \neq V_j'$ with respect to $B(x, \varepsilon(x))$ for $i \neq j$, there is a point $x_1 \in V_1' \setminus V_2'$. Let $\rho_1 = \operatorname{dist}(x_1, \overline{V}_2') > 0$, $V_i'' = V_1' \cap B(x, \delta_1)$ for $i \in \{1, 3, 4, \ldots r(x)\}$, where $0 < \delta_1 < \rho_1$, and $B(x_1, \delta_1) \subseteq B(x, \varepsilon(x))$. Let $r(x_1)$ be the number of distinct V_i''''s, $i \in \{1, 3, 4, \ldots, r(x)\}$, with respect to $B(x_1, \varepsilon)$ for every $0 < \varepsilon \leqslant \varepsilon(x_1)$, for some $0 < \varepsilon(x_1) < \delta_1$. Clearly, $r(x_1) \leqslant r(x) - 1$. We now continue this process inductively. Eventually, we find a point $w \in B(x, \varepsilon(x))$ such that for some $\varepsilon' > 0$, $B(w, \varepsilon') \cap f(N) = B(w, \varepsilon') \cap V_1$, an n-ball neighborhood of w in $f(N)$. ■

PROOF OF THEOREM 1. Theorem 2 implies that there is a point $w \in f(N)$ which has an m-ball neighborhood W in M such that $W \cap f(N)$ has an n-ball neighborhood of w in $f(N)$. The assertion that $f(N)$ is an n-dimensional TOP submanifold of M follows by TOP homogenity.

Next, we prove that $f: N \to f(N)$ is a covering map. Using the notation from the proof of Theorem 2, for every $x \in f(N)$, there are pairwise disjoint open neighborhoods U_i of $y_i \in p^{-1}(x) = \{y_1, \ldots, y_{p(x)}\}$, such that $f|_{U_i} : U_i \to f(U_i)$ is a homeomorphism, and as it was shown in the proof of Theorem 2, all $f(U_i)$ agree setwise on some sufficiently small neighborhood V of x, i.e. for some $\delta > 0$, $V = f(U_1) \cap B(x, \delta) = f(U_i) \cap B(x, \delta)$ for each i. Let $U_i' = U_i \cap f^{-1}(V)$, for every $i \in \{1, \ldots, p(x)\}$. Then $f|_{U_i'} : U_i' \to V$ is a homeomorphism and $f^{-1}(V)$ is equal to the disjoint union of $U_1', U_2', \ldots, U_{p(x)}'$. ■

EXAMPLE. If $f: S^2 \to S^3$ is a continuous, locally one-to-one map, and if $f(S^2)$ is TOP homogeneous in S^3, then $f(S^2)$ is homeomorphic to S^2. Indeed, by Theorem 1, $f(S^2)$ is a closed 2-dimensional manifold and $f: S^2 \to f(S^3)$ is a covering map. But S^2 covers only itself or $\mathbf{R}P^2$ (the projective plane), and there is no embedding of $\mathbf{R}P^2$ in S^3. ■

PROPOSITION 3. Let N be a closed n-dimensional CAT submanifold of a CAT m-dimensional manifold, $n < m$, CAT = TOP, PL, or DIFF. If CAT = TOP or PL assume also that N is locally flat in M. Then N is CAT homogeneous in M.

PROOF. For CAT = PL or DIFF, this is a relative version (for pairs) of homogeneity of manifolds (see for example [11] and [3]). For CAT = TOP, let $x, y \in N$ be arbitrary two points. Then, there is an arc $A \subseteq N$, joining x and y. For each $z \in A$, there is a neighborhood triple $(U_z, V_z, W_z) \subseteq (M, N, A)$, homeomorphic to $(B^{m-n} \times B^{n-1} \times B^1, \{0\} \times B^{n-1} \times B^1, \{0\} \times \{0\} \times B^1)$. Since A is compact, there is a finite subcover $\{W_{z_1}, \ldots, W_{z_t}\}$ of $\{W_z\}$, such that $z_1 = x$, $z_t = y$, and $W_{z_i} \cap W_{z_j} \neq \emptyset$ if and only if $|i - j| \leqslant 1$. For each $i \in \{1, \ldots, t-1\}$, pick $u_i \in W_{z_i} \cap W_{z_{i+1}}$, and let $u_0 = x$, $u_t = y$. Using the homogeneity of $(B^{m-n} \times B^n, B^n)$, we can find for each $i \in \{1, \ldots, t\}$, a homeomorphism $h_i : M \to M$, such that $h_i(N) \subseteq N$ and $h_i(u_{i-1}) = u_i$. Composing these, we obtain the homeomorphism $h = h_t \circ h_{t-1} \circ \ldots \circ h_1 : M \to M$. Clearly, $h(N) \subseteq N$ and $h(x) = y$. ∎

REMARK. Theorem 1 is a converse to Proposition 3 for CAT = TOP. We are left with the following:

CONJECTURE. Let an n-dimensional closed CAT manifold N be TOP embedded and CAT homogeneous in an m-dimensional CAT manifold M, CAT = PL, DIFF. Then N is a CAT submanifold of M.

Next, let N be a closed n-dimensional DIFF manifold TOP embedded in an $(n+1)$-dimensional DIFF manifold M. Suppose also that N separates M. We consider the following properties of the embedding of N into M:

(DHO) N is DIFF homogeneous in M;

(TEM) N is tamely embedded in M;

(SEM) N is smoothly embedded in M; and

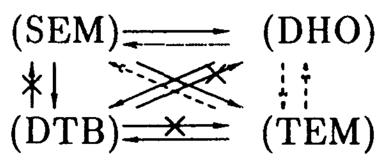
(DTB) N has double tangent balls in M at every point. N is said to have *double tangent balls* in M at a point $p \in N$ if there exist $(n+1)$-balls B_1 and B_2 in M such that $p \in B_i \cap N \subseteq \partial B$ for $i \in \{1, 2\}$, and each B_i lies in the closure of a different component of $M \setminus N$. (See for example [8]).

It is obvious that (SEM) implies (DTB) and (TEM), and by Proposition 3 implies (DHO). Also, (TEM) obviously implies (DTB). Loveland and Wright gave examples in [8] of S^n in $\mathbf{R}^{n+1}$, for $n \geqslant 3$, showing that (DTB) implies neither (SEM) nor (TEM), and the properties of their examples imply that they are not DIFF homogeneous in $\mathbf{R}^{n+1}$, i.e. (DTB) does not imply (DHO) either.

The proof of the following Proposition which shows that (DHO) implies (DTB), is analogous to the proof of Proposition 1 in [2].

PROPOSITION 4. Let N be a closed, codimension-one, DIFF homogeneous TOP submanifold in an orientable DIFF manifold M. Then N has double tangent balls at every point. ■

We do not know if the following implications hold: (DHO) $\Rightarrow$ (SEM), (DHO) $\Rightarrow$ (TEM), (TEM) $\Rightarrow$ (SEM) or (TEM) $\Rightarrow$ (DHO). Schematically:



We conclude by the following two question:

QUESTION. Let $f: S^n \to \mathbb{R}^{n+1}$ be a continuous locally one-to-one map such that $f(S^n)$ is DIFF homogeneous in $\mathbb{R}^{n+1}$. It is known that for $n = 2, f(S^n)$ is tame in $\mathbb{R}^3$ (Proposition 4 and [1], [4]). Is $f(S^n)$ tame in $\mathbb{R}^{n+1}$ also for $n \geq 3$?

QUESTION ([2]). Is every C^∞-homogeneous Jordan curve in the plane $\mathbb{R}^2$ necessarily smooth?

REFERENCES

[1] R. J. DAVERMAN - L. D. LOVELAND, *Wildness and flatness of codimension one spheres having double tangent balls*, Rocky Mountain J. Math., 11 (1981), pp. 113-121.

[2] D. DIMOVSKI - D. REPOVŠ - E. V. ŠČEPIN, *C^∞-homogeneous curves on orientable closed surfaces*, in: *Geometry and Topology*, edited by G. M. RASSIAS and G. M. STRATOPOULOS, World Scientific Publishing Company, Singapore (1989), pp. 100-104.

[3] M. HIRSCH, *Diferential Topology*, Springer-Verlag, New York, Heidelberg, Berlin (1976).

[4] L. D. LOVELAND, *A surface in E^3 is tame if it has round tangent balls*, Trans. Amer. Math. Soc., 152 (1970), pp. 389-397.

[5] L. D. LOVELAND, *Double tangent ball embeddings of curves in* E^3, Pacif. J. Math., 104 (1983), pp. 391-399.

[6] L. D. LOVELAND, *Tangent ball embeddings of sets in* E^3, Rocky Mountain J. Math., 17 (1987), pp. 141-150.

[7] L. D. LOVELAND, *Spheres with continuous tangent planes*, Rocky Mountain J. Math., 17 (1987), pp. 829-844.

[8] L. D. LOVELAND - D. G. WRIGHT, *Codimension one spheres in* $\mathbf{R}^n$ *with double tangent balls*, Topol. Appl., **13** (1982), pp. 311-320.

[9] D. MONTGOMERY - L. ZIPIN, *Topological Transformation Groups*, Interscience Publ., New York (1955).

[10] D. REPOVŠ - A. B. SKOPENKOV - E. V. ŠČEPIN, *A characterization of* C^1*-homogeneous subsets of the plane*, Bollettino U.M.I., (7) 7-A (1993), pp. 437-444.

[11] C. P. ROURKE - B. J. SANDERSON, *Introduction to Piecewise-Linear Topology*, Ergebnisse der Mathematik, **69**, Springer-Verlag, New York, Heidelberg, Berlin (1976).