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On semilinear equations with free boundary conditions on stratified Lie groups

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ABSTRACT

In this paper we establish existence of a solution to a semilinear equation with free boundary conditions on stratified Lie groups. In the process, a monotonicity condition is proved, which is quintessential in establishing the regularity of the solution.

1. Introduction

In this paper we shall prove the existence of solutions to the following *free boundary value* problem:

$$\begin{cases} -\mathcal{L}u = \lambda(u-1)_+^2 f, & \text{in } \Omega \setminus H(u), \\ |\nabla_{\mathbb{G}} u^+|^2 - |\nabla_{\mathbb{G}} u^-|^2 = 2, & \text{in } H(u), \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $\lambda > 0$, $(u-1)_+ = \max\{u-1, 0\}$, and $H(u) = \partial\{u > 1\}$. Also, $\nabla_{\mathbb{G}} u^{\pm}$ are the limits of $\nabla_{\mathbb{G}} u$ for the sets $\{u > 1\}$ and $\{u \leq 1\}^\circ$, respectively. Next, $f \in L^\infty(\Omega)$ is a positive bounded function. The domain $\Omega \subset \mathbb{G}$ is bounded, where $\mathbb{G}$ is a stratified Lie group. Finally, $\mathcal{L}$ is the sub-Laplacian which will be defined in Section 2.

The study of elliptic free boundary value problems (FBVPs) has recently gained momentum, owing to its rich mathematical content besides its physical applications. A naturally occurring free boundary condition

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can be found in the classical problem in fluid dynamics to model a 2-dimensional ideal fluid in terms of its stream function (see Dipierro et al. [11]). Interested readers can also check Batchelor [4,5] for the Prandtl-Batchelor free boundary.

From a mathematical point of view, the problem

$$\begin{cases} -\Delta u = 0, \text{ in } \Omega \setminus G(u), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 2, & \text{on } G(u), \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has been studied by Alt-Caffarelli [2], Alt et al. [3], Caffarelli et al. [6,7], and Weiss [28,29]. Later on, Jerison-Perera in [19,20] considered the problem

$$-\Delta u = (u-1)_+^{p-1}, \text{ in } \Omega \setminus G(u), \tag{1.3}$$

in particular with the same boundary conditions as in (1.2), with $G(u) = \partial\{u > 1\}$, thus pioneering the study of the existence of a mountain pass point at which the associated energy functional has a higher value compared to the global minimum (see [18, Definition 1]). Such a critical point was referred by them as a *higher critical point*. A slightly more general problem was considered by Perera in [24], as follows

$$\begin{cases} -\Delta u = \alpha\chi_{\{u>1\}}(x) f(x, (u-1)_+), \text{ in } \Omega \setminus G(u), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 2, & \text{on } G(u), \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

This problem was also studied by Elcrat-Miller [12] and Jerison-Perera [19] for the case $N = 2$. The main result of [24] is the establishment of a higher critical point. Some of the important works in the Euclidean setting have been documented in Dipierro et al. [11] and Perera [24], and the references therein.

Motivated by the above mentioned works, albeit in the Euclidean setting, we consider (1.1) in the non-Euclidean setup. One key work in this direction is Ferrari-Valdinoci [13], in which a free boundary value problem was studied on the Heisenberg group, and the authors established some density estimates for local minima. The problem which we shall study in this paper is classical, however its consideration over a stratified Lie group is new since the Heisenberg group is also a particular kind of a stratified Lie group.

We now state the main result of this paper pertaining to the existence of solutions to problem (1.1):

Theorem 1.1. *There exists $\lambda_* > 0$ such that for any $0 < \lambda < \lambda_*$, there exists a positive solution u to problem* (1.1) *with the following properties:*

(i) u is a critical point of I;
(ii) u satisfies the free boundary condition in the sense of viscosity.

Remark 1.2. Notice that by a nontrivial solution to (1.1) we mean $u > 0$ on Ω and $u > 1$ on a nonempty open subset of Ω on which $-\mathcal{L}u = \lambda(u-1)^2 f$ holds.

The organization of the paper is as follows. In Section 2 we recall the preliminaries of the stratified Lie group and the space description. In addition, we prepare the necessary tools required to attack problem (1.1). In Section 3 we prove a monotonicity lemma (Lemma 3.1). In Section 4 we prove a convergence lemma (Lemma 4.1). In Section 5 we prove the main result of this paper (Theorem 1.1). Finally, in Section 6 we prove an auxilliary lemma on the Radon measure (Lemma 6.1).

2. Preliminaries

This section includes the necessary tools to study problem (1.1). For all other background information we refer to the comprehensive handbook [23]. We begin by the definition of a homogeneous Lie group.

Definition 2.1. A Lie group $\mathbb{G}$, on $\mathbb{R}^N$ is said to be homogeneous, if for any $\mu > 0$ there exists an automorphism $T_\mu : \mathbb{G} \to \mathbb{G}$ defined by

$$T_\mu(x) = (\mu^{r_1}x_1, \mu^{r_2}x_2, \cdots, \mu^{r_N}x_N), \quad r_i > 0, \quad i = 1, 2, \cdots, N.$$

The map T_μ is called a dilation on $\mathbb{G}$. Here, $x = (x_1, x_2, \cdots, x_N)$.

It is worth noting that N represents the topological dimension of $\mathbb{G}$, whereas $D = r_1 + r_2 + \cdots + r_N$ represents the homogeneous dimension of the homogeneous Lie group $\mathbb{G}$. The symbol dx will serve as our notation for the Haar measure, which is the standard Lebesgue measure on $\mathbb{R}^N$. The following is the definition of a stratified Lie group.

Definition 2.2. A homogeneous Lie group $\mathbb{G} = (\mathbb{R}^N, *)$ is called a stratified Lie group (or a homogeneous Carnot group) if the following two conditions are satisfied:

(i) The decomposition $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_k}$ holds for some natural numbers $N_1, N_2, \cdots, N_k$ such that $N_1 + N_2 + \cdots + N_k = N$. Furthermore, for each $\mu > 0$ there exists a dilation of the form $T_\mu(x) = (\mu x^{(1)}, \mu^2 x^{(2)}, \cdots, \mu^k x^{(k)})$ which is an automorphism of the group $\mathbb{G}$. Here, $x^{(i)} \in \mathbb{R}^{N_i}$ for each $i = 1, 2, \cdots, k$.

(ii) Let N_1 be the same as in the above decomposition of $\mathbb{R}^N$ and let $X_1, X_2, \cdots, X_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_i(0) = \frac{\partial}{\partial x_i}|_0$ for $i = 1, 2, \cdots, N_1$. Then the Hörmander condition $\text{rank}(\text{Lie}\{X_1, X_2, \cdots, X_{N_1}\}) = N$ holds for every $x \in \mathbb{R}^N$. Roughly speaking, the Lie algebra corresponding to the Lie group $\mathbb{G}$ is spanned by the iterated commutators of $X_1, X_2, \cdots, X_{N_1}$.

Here, k is called the *step* of the homogeneous Carnot group. In the case of a stratified Lie group, the homogeneous dimension becomes $D = \sum_{i=1}^k iN_i$. Throughout the paper, we set $N = N_1$ in Definition 2.2. We call a curve $\gamma : [0, 1] \to \mathbb{R}$ admissible if there exists $c_i : [0, 1] \to \mathbb{R}$, for $i = 1, 2, \cdots, N$ such that

$$\gamma'(t) = \sum_{i=1}^N c_i(t)X_i(\gamma(t)), \ \sum_{i=1}^N c_i(t)^2 \leq 1.$$

Here, γ' is the derivative with respect to t. The functions c_i may not be unique since the vector fields X_i may not be linearly independent. For any $x, y \in \mathbb{G}$, the *Carnot-Carathéodory* distance is defined by

$$d_{cc}(x, y) = \inf\{l > 0 : \text{there exists an admissible } \gamma : [0, l] \to \mathbb{G} \text{ such that } \gamma(0) = x, \gamma(l) = y\}.$$

If no such curve exists, $d_{cc}(x, y)$ is set to 0. Although d_{cc} is not a metric in general, the Hörmander condition over the vector fields $X_1, X_2, \cdots, X_{N_1}$ ensures that it is. The space $(\mathbb{G}, d_{cc})$ is then referred to as the *Carnot-Carathéodory* space. The definition of the homogeneous quasi-norm on the homogeneous Carnot group $\mathbb{G}$ is another important entity that will be used in the course of this work. See Ghosh et al. [15, Definition 2.3] for a definition of a homogeneous quasi-norm.

Furthermore, the sub-Laplacian, the horizontal gradient and the horizontal divergence on $\mathbb{G}$ is defined as

$$\mathcal{L} := X_1^2 + X_2^2 + \cdots + X_{N_1}^2, \ \nabla_{\mathbb{G}} := (X_1, X_2, \cdots, X_{N_1}), \ \text{div}_{\mathbb{G}} v := \nabla_{\mathbb{G}} \cdot v,$$

respectively. The sub-Laplacian on the stratified Lie group $\mathbb{G}$ is defined as $\Delta_{\mathbb{G}} u := div_{\mathbb{G}}(\nabla_{\mathbb{G}} u)$.

Now, let S be a Haar measurable subset of $\mathbb{G}$. Then $\mathcal{H}(T_\mu(S)) = \mu^D \mathcal{M}(S)$, where $\mathcal{H}(S)$ is the Haar measure of Ω. A quasi-ball of radius r and centered at $x \in \mathbb{G}$ is defined by $B(x,r) = \{y \in \mathbb{G} : |y^{-1} * x| < r\}$ with respect to the quasi-norm $|\cdot|$.

We define the Sobolev space, which is very essential in order to venture into this problem. For $1 < p < \infty$, the Sobolev space $W^{1,p}(\Omega)$ on a stratified Lie group is defined as

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) : |\nabla_{\mathbb{G}} u| \in L^p(\Omega)\}. \tag{2.1}$$

A norm on this space is given by $\|u\|_{1,p} := \|u\|_p + \|u\|$. Here,

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}, \ \|u\| := \left(\int_\Omega |\nabla_{\mathbb{G}} u(x)|^p dx\right)^{1/p}.$$

We define the space $W_0^{1,p}(\Omega)$ as follows:

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega\},$$

where $u = 0$ on $\partial\Omega$ is in the usual trace sense. We note that $W_0^{1,p}(\Omega)$ is a real separable and uniformly convex Banach space (see [14,26,27,30]). The following embedding result follows from [10, (2.8)], [14], and [17, Theorem 8.1]. We also suggest the reader to check [8, Theorem 2.3].

Lemma 2.3. *Let $\Omega \subset \mathbb{G}$ be a bounded domain with piecewise smooth and simple boundary and assume $1 < p < \nu$. Then $W_0^{1,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for every $q \in [1, \nu^*]$, where $\nu^* = \frac{\nu p}{\nu - p}$. Moreover, the embedding is compact for every $1 \le q < \nu^*$.*

The following proposition, due to Ruzhansky-Suragan [25], will be used on a regular basis. It is an analogue of the divergence theorem in the Euclidean setup.

Proposition 2.4. *Let $f_n \in C^1(\Omega) \cap C(\bar{\Omega})$, $n = 1, 2, \cdots, N_1$. Then for each $n = 1, 2, \cdots, N_1$, we have*

$$\int_\Omega X_n f_n d\nu = \int_{\partial\Omega} f_n \langle X_n, d\nu \rangle.$$

Consequently,

$$\int_\Omega \sum_{n=1}^{N_1} X_n f_n d\nu = \int_{\partial\Omega} \sum_{n=1}^{N_1} f_n \langle X_n, d\nu \rangle.$$

Throughout the paper we shall assume that $\mathcal{H}(\Omega) < \infty$. We define an energy functional associated to problem (1.1) as follows

$$I(u) = \int_\Omega \frac{|\nabla_{\mathbb{G}} u|^2}{2} dx + \int_\Omega \chi_{\{u>1\}}(x) dx - \frac{\lambda}{3} \int_\Omega (u-1)_+^3 f dx.$$

The functional I exhibits the mountain pass geometry. Let

$$\Lambda := \{\psi \in C([0,1]; W_0^{1,2}(\Omega)) : \psi(0) = 0, I(\psi(1)) < 0\}$$

which consists of paths joining $u = 0$ and the set of points $\{u \in W_0^{1,2}(\Omega) : I(u) < 0\}$. We further define

$$c := \inf_{\psi \in \Lambda} \max_{u \in \psi([0,1])} I(u).$$

However, this functional is not even differentiable and hence is an ineligible candidate to fit into the realm of the variational setup. We first define a smooth function $g : \mathbb{R} \to [0, 2]$ as follows

$$g(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \text{a positive function}, & \text{if } 0 < t < 1 \\ 0, & \text{if } t \geq 1 \end{cases}$$

and $\int_0^1 g(t)dt = 1$. We further let $G(t) = \int_0^t g(t)dt$. Clearly, G is smooth and nondecreasing function such that

$$G(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \text{a positive function} < 1, & \text{if } 0 < t < 1 \\ 1, & \text{if } t \geq 1. \end{cases}$$

Finally, inspired by the work of Jerison-Perera [19], we approximate I using the following functionals which vary with respect to a parameter, $\alpha > 0$,

$$I_\alpha(u) = \int_\Omega \frac{|\nabla_{\mathbb{G}} u|^2}{2} dx + \int_\Omega G\left(\frac{u-1}{\alpha}\right) dx - \frac{\lambda}{3} \int_\Omega (u-1)_+^2 f dx.$$

An essential condition in variational techniques which a functional $J : X \to \mathbb{R}$ requires to satisfy is the Palais-Smale (PS) condition. It states that if $J(w_n) \to c$ and $J'(u_n) \to 0$ in X^*, the dual of X, then there exists a subsequence of (w_n) which strongly converges to, say w, in X. We shall prove that both functionals I, I_α defined above satisfy the (PS) condition.

3. Monotonicity lemma

Following the argument in Caffarelli et al. [6, Theorem 5.1], we shall prove an important monotonicity result stated below. We refer to the monograph by Nagel [22, Section 1.2] for the background regarding our proof in a non-Euclidean setup. Most of our modifications are required by the differences from the Euclidean setting.

Lemma 3.1. *Let $u > 0$ be a Lipschitz continuous function on the unit ball $B_1(0) \subset \mathbb{G}$, satisfying the distributional inequalities*

$$\pm \mathcal{L} u \leq \left(\frac{\lambda}{\alpha} \chi_{\{|u-1|<\alpha\}}(x) \mathcal{F}(|\nabla_{\mathbb{G}} u|) + A\right), \tag{3.1}$$

for constants $A > 0$, $0 < \alpha \leq 1$. Suppose further that $\mathcal{F}$ is a continuous function such that $\mathcal{F}(t) = o(t^2)$ near infinity. Then there exist $C = C(N, A) > 0$ and $\int_{B_1(0)} u^2 dx$, but not on α, such that

$$\underset{x \in B_{\frac{1}{2}}(0)}{esssup} \{|\nabla_{\mathbb{G}} u(x)|\} \leq C.$$

Proof. Let u be a Lipschitz continuous function on the unit ball $B_1(0) \subset \mathbb{G}$. Denote

$$v(x) = \frac{15}{\alpha} u\left(\frac{\alpha}{15} x\right), \quad v_1 = v + \max_{B_{1/4}}\{v_-\}.$$

Since the proof is quite technical in nature, before giving the proof we sketch the idea. The primary challenge is to prove that $|\nabla v|$ is bounded on, say $B_{1/32}$. In Step 1 we shall establish the L^∞ bound on v_1, where v_1 is a perturbation of v. Next, we shall show in Step 2 that a uniform bound on $|\nabla v|$ exists and this depends on the bound on v and (3.1). This step is also essential to establish an interior regularity estimate for the semilinear equation independent of the monotonicity theorem. The monotonicity theorem also helps to produce an L^∞ bound on v. A meticulous choice of $\beta > 0$ has to be made so that $\mathcal{F}(t) \leq \beta t^2 + A(\beta)$.
Step 1*:* Since u is a Lipschitz continuous function on the unit ball, it is also bounded on it by a constant say, M_0. By Magnani-Rajala [21, Theorem 1.1], u is also differentiable a.e. on $B_1(0)$. Therefore, $0 \leq v_1 \leq M_1$.
Step 2*:* Let us choose a function $\eta \in C_0^\infty(B_{1/4})$ such that $0 \leq \eta \leq 1$ in $B_{3/4}$ and $\eta = 1$ in $B_{1/2}$. Furthermore, for any $\beta \in (0, 1]$ we have a positive finite number $A(\beta)$ such that

$$\mathcal{F}(t) \leq A(\beta) + \beta t^2. \tag{3.2}$$

Thus by testing with $\eta^2 v_1$, we have

$$\begin{aligned}\int_\Omega \eta^2 |\nabla_{\mathbb{G}} v_1|^2 &= -\int_\Omega (2v_1\eta(\tilde{\nabla} v_1 \eta) + \eta^2 v_1 \mathcal{L} v_1 dx)dx \\ &\leq \frac{1}{2}\int_\Omega \eta^2 |\nabla_{\mathbb{G}} v_1|^2 dx + 2\int_\Omega v_1^2 |\nabla_{\mathbb{G}} \eta|^2 dx + AM_1 \int_\Omega \eta^2\left(A(\beta) + \beta|\nabla_{\mathbb{G}} v_1|^2\right)dx \\ &\leq \frac{1}{2}\int_\Omega \eta^2 |\nabla_{\mathbb{G}} v_1|^2 dx + pM_1^2 \int_\Omega |\nabla_{\mathbb{G}} \eta|^2 dx + M_1 \int_\Omega \eta^2\left(\beta|\nabla_{\mathbb{G}} v_1|^2 + A(\beta)\right)dx.\end{aligned} \tag{3.3}$$

Here, $\tilde{\nabla} v_1 \eta = \sum_{k=1}^{N_1} X_k v_1 X_k$. It is thus established that

$$\frac{1}{2}\int_{B_{1/2}} |\nabla_{\mathbb{G}} v_1|^2 dx \leq M_2. \tag{3.4}$$

We define the maximal operator by

$$\mathfrak{M} f(x) = \sup_{0<r<1/100} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)dy. \tag{3.5}$$

For $\mu > 0$, we further denote

$$S_\mu = \{x \in B_{1/32} : \mathfrak{M}(|\nabla_{\mathbb{G}} v_1|^2)(x) > \mu\}.$$

Claim 3.2. *There exists a constant C_1 such that for any $\epsilon > 0$ there exists a finite positive number μ_0 such that for any $\mu \geq \mu_0$,*

1. *$|S_\mu \cap Q_0| \leq |S_{\mu_0} \cap Q_0| < \epsilon|Q_0|$, where Q_0 is a cube with side length $2^{-10-10N}$ and $Q_0 \cap B_{1/32} \neq \emptyset$.*
2. *If Q is a dyadic subcube of Q_0 for which $|S_{C_1\mu} \cap Q| \geq \epsilon|Q|$, then $Q \subset Q^* \subset S_\mu$, where Q is an immediate dyadic subcube of Q^*.*

Proof. We only sketch the proof of the claim as the ideas are borrowed from [6]. Assertion (1) follows from the argument given in [6].

Suppose now that Assertion (2) fails to hold. Then one can find a cube Q such that $|S_\mu \cap Q| \geq \epsilon|Q|$ and $y \in Q^*$, however $\mathfrak{M}(|\nabla_{\mathbb{G}} v_1|^2)(y) \leq \mu$. Let ρ be 2^{6N} times the length of the side of Q and consider $\mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}} v_1|^2)(0)$, with the supremum taken over $(0, \rho/4)$. Since $\mathfrak{M}(|\nabla_{\mathbb{G}} v_1|^2)(y) \leq \mu$, there exists a constant C_2 such that for any $x \in Q$,

$$\mathfrak{M}(|\nabla_{\mathbb{G}} v_1|^2)(x) \leq \max\{\mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}} v_1|^2)(x), C_2\mu\}. \tag{3.6}$$

Let ϕ be such that

$$\begin{aligned} -\mathcal{L}\phi &= 0 \text{ in } B_\rho(y) \\ \phi &= v_1 \text{ on } \partial B_\rho(y). \end{aligned} \tag{3.7}$$

Since ϕ is a minimizer of the functional $\frac{1}{2}\int_{B_\rho(y)} |\nabla_{\mathbb{G}}\phi|^2 dx$, we have

$$\int\limits_{B_\rho(y)} |\nabla_{\mathbb{G}}\phi|^2 dx \leq \int\limits_{B_\rho(y)} |\nabla_{\mathbb{G}} v_1|^2 dx \leq \mu|B_\rho(y)|. \tag{3.8}$$

Of course, we have the mean value property at our disposal (see Adamowicz-Warhurst [1, Condition 1]) to establish that

$$\sup_{B_{\rho/2}(y)} \{|\nabla_{\mathbb{G}}\phi|^2\} \leq C_3\mu. \tag{3.9}$$

On choosing $C_1 = 15\max\{C_2, C_3\}$ we have

$$\mathcal{A} := \{x \in Q : \mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}} v_1|^2)(x) > C_1\mu\} = \{x \in Q : \mathfrak{M}(|\nabla_{\mathbb{G}} v_1|^2)(x) > C_1\mu\} =: \mathcal{B}. \tag{3.10}$$

If $x \in \mathcal{A}$, then it is easy see that $x \in \mathcal{B}$. Thus $\mathcal{A} \subset \mathcal{B}$. Suppose that $x \in \mathcal{B}$. Then $\mathfrak{M}(|\nabla_{\mathbb{G}} v_1|^2)(x) > C_1\mu$. However, by (3.6) and by the choice of C_1 we have that $x \in \mathcal{A}$.

Also observe that

$$\{x \in Q : \mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}}\phi|^2)(x) > C_1\mu/4\} = \emptyset.$$

For if not, then there exists $x \in Q$ such that $\mathfrak{M}(|\nabla_{\mathbb{G}}\phi|^2)(x) > C_\mu/4$. One can thus produce $r \in (0, \rho/4)$ such that

$$\frac{C_1\mu}{4} < \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |\nabla_{\mathbb{G}}\phi|^2 dy \leq \frac{C_1\mu}{15}.$$

This is a contradiction since this leads to an absurdity $4 > 15$. Therefore,

$$\begin{aligned} &|\{x \in Q : \mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}} v_1|^2) > C_1\mu\}| \\ &\quad\leq |\{x \in Q : \mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}}(v_1 - \phi)|^2) + \mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}}\phi|^2) > C_1\mu/2\}| \\ &\quad\leq |\{x \in Q : \mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}}(v_1 - \phi)|^2) > C_1\mu/4\}| + |\{\mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}}\phi|^2) > C_1\mu/4\}| \\ &\quad= |\{x \in Q : \mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}}(v_1 - \phi)|^2) > C_1\mu/4\}|. \end{aligned} \tag{3.11}$$

Thus there exists a constant C_4, which follows by the weak $(1,1)$ inequality for $\mathfrak{M}$, such that

$$C_4\mu^{-1}\int\limits_{B_\rho(y)}|\nabla_{\mathbb{G}}(v_1-\phi)|^2dx \geq |\{x\in Q:\mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}}(v_1-\phi)|^2)>C_1\mu/4\}|. \tag{3.12}$$

Furthermore, by the maximum principle we have $|v_1-\phi|\leq C$ on the ball $B_\rho(y)$. By the weak formulation of problem (3.7), we have

$$0=\int\limits_{B_\rho(y)}\tilde{\nabla}\phi(v_1-\phi)dx. \tag{3.13}$$

Furthermore,

$$-\int\limits_{B_\rho(y)}\mathcal{L}v_1(v_1-\phi)dx=-\int\limits_{B_\rho(y)}(\mathcal{L}v_1-\mathcal{L}\phi)(v_1-\phi)dx. \tag{3.14}$$

Thus we have

$$\begin{aligned}C_5\int\limits_{B_\rho(y)}|\nabla_{\mathbb{G}}(v_1-\phi)|^2dx&=\int\limits_{B_\rho(y)}\tilde{\nabla}(v_1-\phi)(v_1-\phi)dx\\&\leq-\int\limits_{B_\rho(y)}(\mathcal{L}v_1-\mathcal{L}\phi)(v_1-\phi)dx=-\int\limits_{B_\rho(y)}(\mathcal{L}v_1)(v_1-\phi)dx\\&\leq\int\limits_{B_\rho(y)}C\left(\beta|v_1|^2+A(\beta)\right)dx.\end{aligned} \tag{3.15}$$

Using inequality (3.12), we get

$$|\{x\in Q:\mathfrak{M}_{\rho/4}(|\nabla_{\mathbb{G}}v_1|^2)>C_1\mu\}|\leq C_6\left(\beta+\frac{A(\beta)}{\mu}\right)|Q|. \tag{3.16}$$

Thus, for a sufficiently small $\delta>0$ and large $\mu>0$, we have

$$C_6\delta<\epsilon/3\quad C_6A(\beta)/\mu<\epsilon/3.$$

Therefore

$$\{x\in Q:\mathfrak{M}(|\nabla_{\mathbb{G}}v_1|^2)>C_1\mu\}<\epsilon|Q|,$$

which indeed is a contradiction to the hypothesis. Therefore, assertion (2) indeed holds. □

One can now follow verbatim [6] to conclude that assertion (2) leads to

$$|S_{C_1^k\mu}\cap Q_0|\leq\epsilon^{k+1}|Q_0|. \tag{3.17}$$

We further note from (3.17) that for any $1<\theta<\infty$, a sufficiently small $\epsilon>0$ can be chosen so that $\mathfrak{M}(|\nabla_{\mathbb{G}}v_1|^2)$ is bounded in $L^\theta(B_{1/16})$, i.e.

$$\int_{B_{1/16}} |\nabla_{\mathbb{G}} v_1|^{\theta} dx \leq C_7, \tag{3.18}$$

where C_7 is a uniform constant that depends on $\theta, A, \mathcal{F}$. On choosing $\theta = N$, we have $2\theta > N$. Hence we obtain from (3.2)

$$\sup_{B_{1/32}} \{|\nabla_{\mathbb{G}} v_1|\} \leq C_8. \tag{3.19}$$

Reverting back to the variables in terms of u, we get

$$\sup_{B_{\alpha/320}(x)} \{|\nabla_{\mathbb{G}} u|\} \leq C_8 \text{ for any } x \in B_{1/4} \text{ such that } |\{u(x) < \alpha\}|, \tag{3.20}$$

and in order to finally arrive at the conclusion

$$\sup_{B_{r/4}(0)} \{|\nabla_{\mathbb{G}} u|\} \leq C_9, \tag{3.21}$$

we follow the proof of [6] again, however with the choice of

$$w(x) = A_0 r(r^{N-2}|x|^{2-N} - 1) + A(|x|^2 - r^2) + O(\alpha).$$

Therefore $\sup_{B_{r/2}} \{|\nabla_{\mathbb{G}} u|\} < \infty$. $\square$

Remark 3.3. The above monotonicity bound of the type (3.1) implies uniform Lipschitz continuity of a family of solutions to a class of semilinear equations with free boundary conditions. In fact, a very important component in the passage to the limit in the proof of Theorem 1.1 in Section 5 will be the uniform Lipschitz continuity result derived in the next section.

4. Convergence lemma

Before proving Theorem 1.1, we shall prove the following convergence result, which is also of independent interest. It helps us to conclude that the obtained solution is nontrivial in the sense of Remark 1.2.

Lemma 4.1. *Let (α_j) be a sequence of positive numbers such that $\alpha_j \to 0$, as $j \to \infty$, and let u_j be a critical point of I_{α_j}. Suppose that (u_j) is bounded in $W_0^{1,2}(\Omega) \cap L^{\infty}(\Omega)$. Then there exists a Lipschitz continuous function u on $\bar{\Omega}$ such that $u \in W_0^{1,2}(\Omega) \cap C^2(\bar{\Omega} \setminus H(u))$, and for a renamed subsequence the following holds:*

(i) $u_j \to u$ *uniformly over* $\bar{\Omega}$;
(ii) $u_j \to u$ *locally in* $C^1(\bar{\Omega} \setminus \{u = 1\})$;
(iii) $u_j \to u$ *strongly in* $W_0^{1,2}(\Omega)$; *and*
(iv) $I(u) \leq \liminf I_{\alpha_j}(u_j) \leq \limsup I_{\alpha_j}(u_j) \leq I(u) + |\{u = 1\}|$.

In other words, u is a nontrivial function if $\liminf I_{\alpha_j}(u_j) < 0$ or $\limsup I_{\alpha_j}(u_j) > 0$. Furthermore, u satisfies $-\mathcal{L}u(x) = \lambda(u(x) - 1)_+^2 f(x)$ classically in $\Omega \setminus H(u)$ and u satisfies the free boundary condition in the generalized sense and vanishes continuously on $\partial\Omega$. In the case when u is nontrivial, the set $\{u > 1\}$ is nonempty.

Proof. Let $0 < \alpha_j < 1$. Consider the sequence of problems (P_j)

$$\begin{aligned} -\mathcal{L}u_j &= -\frac{1}{\alpha_j} g\left(\frac{(u_j-1)_+}{\alpha_j}\right) + \lambda(u-1)_+^2 f \text{ in } \Omega, \\ u_j &> 0 \text{ in } \Omega, \\ u_j &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.1}$$

The nature of the problem allows us to conclude by an iterative technique that the sequence (u_j) is bounded in $L^\infty(\Omega)$. Hence, there exists C_0 such that $0 \leq (u_j - 1)_+^2 f \leq C_0$. Let φ_0 be a solution of

$$\begin{aligned} -\mathcal{L}\varphi_0 &= \lambda C_0 \text{ in } \Omega \\ \varphi_0 &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.2}$$

Now, since $g \geq 0$, we have that $-\mathcal{L}u_j \leq \lambda C_0 = \mathcal{L}\varphi_0$ in Ω. Therefore by the maximum principle,

$$0 \leq u_j(x) \leq \varphi_0(x) \text{ for all } x \in \Omega. \tag{4.3}$$

Since $\{u_j \geq 1\} \subset \{\varphi_0 \geq 1\}$, it follows that φ_0 gives a uniform lower bound, say d_0, on the distance from the set $\{u_j \geq 1\}$ to $\partial\Omega$. Thus (u_j) is bounded with respect to the $C^{2,a}$ norm. Therefore it has a convergent subsequence in the C^2-norm on $\frac{d_0}{2}$-neighbourhood of the boundary $\partial\Omega$. Obviously, $0 \leq g \leq 2\chi_{(-1,1)}$ and hence

$$\begin{aligned} \pm\mathcal{L}u_j &= \pm\frac{1}{\alpha_j} g\left(\frac{(u_j-1)_+}{\alpha_j}\right) \mp \lambda(u_j-1)_+^2 f \\ &\leq \frac{2}{\alpha_j}\chi_{\{|u_j-1|<\alpha_j\}}(x) + \lambda C_0. \end{aligned} \tag{4.4}$$

Since, (u_j) is bounded in $L^2(\Omega)$, there exists by Lemma 3.1, $A > 0$ such that

$$\operatorname*{esssup}_{x \in B_{\frac{r}{2}}(x_0)} \{|\nabla_{\mathbb{G}} u_j(x)|\} \leq \frac{A}{r}, \tag{4.5}$$

for a suitable $r > 0$ for which $B_r(0) \subset \Omega$. However, since (u_j) is a sequence of Lipschitz continuous functions which incidentally are also C^1 functions a.e., we have

$$\sup_{x \in B_{\frac{r}{2}}(x_0)} \{|\nabla_{\mathbb{G}} u_j(x)|\} \leq \frac{A}{r}. \tag{4.6}$$

Therefore (u_j) is a sequence of uniformly Lipschitz continuous functions on the compact subsets, say K, of Ω such that $d(K, \partial\Omega) \geq \frac{d_0}{2}$. By the Ascoli-Arzela theorem applied to (u_j), we get a subsequence, still referred to by the same name, that converges uniformly to a Lipschitz continuous function u in Ω which vanishes on the boundary $\partial\Omega$. The convergence is strong in C^2 on a $\frac{d_0}{2}$-neighbourhood of $\partial\Omega$. By the Banach-Alaoglu theorem we can conclude that $u_j \rightharpoonup u$ in $W_0^{1,2}(\Omega)$.

We now prove that u satisfies

$$-\mathcal{L}u = \lambda(u-1)_+^2 f \tag{4.7}$$

on the set $\{u \neq 1\}$. Let $\varphi \in C_0^\infty(\{u > 1\})$. Thus $u \geq 1 + 2\delta$ on the support of φ for some $\delta > 0$. By using the convergence of u_j to u uniformly on Ω, we conclude that $|u_j - u| < \delta$. Thus for any sufficiently large j with $\delta_j < \delta$ we have $u_j \geq 1 + \delta_j$ on the support of φ. Testing (4.7) with φ yields

$$\int_\Omega \tilde{\nabla} u_j \varphi dx = \lambda \int_\Omega (u_j - 1)_+^2 f \varphi dx. \tag{4.8}$$

By passing the limit $j \to \infty$ to (4.7), we obtain

$$\int_\Omega \tilde{\nabla} u \varphi dx = \lambda \int_\Omega (u - 1)_+^2 f \varphi dx. \tag{4.9}$$

In order to obtain (4.9) we have used the weak and uniform convergence of u_j to u in $W_0^{1,2}(\Omega)$ and Ω, respectively. Therefore u is a weak solution of $-\mathcal{L}u = \lambda f$ in $\{u > 1\}$. Similarly, by choosing $\varphi \in C_0^\infty(\{u < 1\})$, we can similarly find a $\delta > 0$ such that $u \leq 1 - 2\delta$ due to which $u_j < 1 - \delta$.

We now analyze the nature of u on the set $\{u \leq 1\}^\circ$. Testing (4.7) with any nonnegative function, passing to the limit $j \to \infty$ and using the fact that $g \geq 0$, $G \leq 1$, it can be shown that u satisfies

$$\mathcal{L}u \leq \lambda (u - 1)_+^2 f \text{ in } \Omega \tag{4.10}$$

in the sense of distribution. Furthermore, $\mu = \mathcal{L}(u-1)_-$ is a positive Radon measure supported on $\Omega \cap \partial\{u < 1\}$ (the reader can refer to Lemma 6.1 in Section 5). From (4.10), $\mu > 0$ and the usage of the regularity result by Gilbarg-Trudinger [16, Section 9.4] we establish that $u \in W_{\text{loc}}^{2,2}(\{u \leq 1\}^\circ)$. Hence $\mathcal{M}$ is supported on $\Omega \cap \partial\{u < 1\} \cap \partial\{u > 1\}$ and u satisfies $\mathcal{L}u = 0$ on the set $\{u \leq 1\}^\circ$.

To prove (ii), we shall show that $u_j \to u$ locally in $C^1(\Omega \setminus \{u = 1\})$. We have already proved that $u_j \to u$ with respect to the C^2 norm in a neighbourhood of $\partial\Omega$ of $\bar{\Omega}$. Let $M \subset\subset \{u > 1\}$. In this set M we have $u \geq 1 + 2\delta$ for some $\delta > 0$. Hence, for sufficiently large j, with $\delta_j < \delta$, we have $|u_j - u| < \delta$ in Ω and hence $u_j \geq 1 + \delta_j$ in M. From (4.1) we have

$$\mathcal{L}u_j = \lambda (u - 1)^2 f \text{ in } M.$$

This analysis says something more stronger - *since* $\mathcal{L}u_j = \lambda(u - 1)^2 f$ *in* M, *we have that* $u_j \to u$ *in* $W^{2,2}(M)$. By the embedding $W^{2,2}(M) \hookrightarrow C^1(M)$ for $p > 2$, we have $u_j \to u$ in $C^1(M)$. This proves that $u_j \to u$ in $C^1(\{u > 1\})$. Similarly, we can also show that $u_j \to u$ in $C^1(\{u < 1\})$.

We shall now prove (iii). Since $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, we have by the weak lower semicontinuity of the norm $\|\cdot\|$,

$$\|u\| \leq \liminf \|u_j\|.$$

It suffices to prove that $\limsup \|u_j\| \leq \|u\|$. To this end, we multiply (4.1) with $(u_j - 1)$ and then integrate by parts. We shall also use that $tg\left(\frac{t}{\delta_j}\right) \geq 0$ for any $t \in \mathbb{R}$. This yields

$$\begin{aligned} \int_\Omega |\nabla_{\mathbb{G}} u_j|^2 dx &\leq \lambda \int_\Omega (u_j - 1)_+^2 f dx - \int_{\partial\Omega} u_j \langle X_i, dn \rangle dS \\ &\to \lambda \int_\Omega (u - 1)_+^2 f dx - \int_{\partial\Omega} u \langle X_i, dn \rangle dS \end{aligned} \tag{4.11}$$

as $j \to \infty$.

We choose $\vec{\varphi} \in C_0^1(\Omega, \mathbb{G})$ such that $u \neq 1$ a.e. on the support of $\vec{\varphi}$. Multiplying by $\sum_{k=1}^N \varphi_k X_k u_n$ the weak formulation of (4.1) and integrating over the set $\{1-\epsilon^- < u_n < 1+\epsilon^+\}$, we get

$$\begin{aligned}\int\limits_{\{1-\epsilon^-<u_n<1+\epsilon^+\}} \left[-\Delta_{\mathbb{G}} u_n + \frac{1}{\alpha_n} g\left(\frac{u_n-1}{\alpha_n}\right)\right] \sum_{k=1}^N \varphi_k X_k u_n dx \\ = \lambda \int\limits_{\{1-\epsilon^-<u_n<1+\epsilon^+\}} (u_n-1)_+^2 f \sum_{k=1}^N \varphi_k X_k u_n dx.\end{aligned} \tag{4.12}$$

The term on the left hand side of (4.12) can be expressed as follows:

$$\begin{aligned}\nabla_{\mathbb{G}} \cdot \left(\frac{1}{2}|\nabla_{\mathbb{G}} u_n|^2 \vec{\varphi} - (\sum_{k=1}^N X_k u_n \varphi_k) \nabla_{\mathbb{G}} u_n\right) + \sum_{k=1}^N \sum_{l=1}^N X_l \varphi_k X_l u_n X_k u_n \\ - \frac{1}{2}|\nabla_{\mathbb{G}} u_n|^2 \nabla_{\mathbb{G}} \cdot \vec{\varphi} + \sum_{k=1}^N \varphi_k X_k G\left(\frac{u_n-1}{\alpha_n}\right).\end{aligned} \tag{4.13}$$

Using (4.13) and on integrating by parts, we obtain

$$\begin{aligned}&\int\limits_{\{u_n=1+\epsilon^+\}\cup\{u_n=1-\epsilon^-\}} \left[\frac{1}{2}|\nabla_{\mathbb{G}} u_n|^2 \sum_{k=1}^N \varphi_k \langle X_k, dn\rangle - (\sum_{k=1}^N X_k u_n \varphi_k) \sum_{l=1}^N X_l u_n \langle X_l, dn\rangle \right. \\ &\qquad \left. + G\left(\frac{u_n-1}{\alpha_j}\right) \sum_{k=1}^N \varphi_k \langle X_k, dn\rangle\right] \\ &= \int\limits_{\{1-\epsilon^-<u_n<1+\epsilon^+\}} \left(\frac{1}{2}|\nabla_{\mathbb{G}} u_n|^2 \sum_{k=1}^N X_k \varphi_k - \sum_{k=1}^N \sum_{l=1}^N X_k \varphi_l X_l u_n X_k u_n\right) dx \\ &+ \int\limits_{\{1-\epsilon^-<u_n<1+\epsilon^+\}} \left[G\left(\frac{u_n-1}{\alpha_n}\right) \sum_{k=1}^N X_k \varphi_k + \lambda (u_n-1)_+^2 f \sum_{k=1}^N X_k \varphi_k\right] dx.\end{aligned} \tag{4.14}$$

The integral on the left of equation (4.14) converges to

$$\begin{aligned}&\int\limits_{\{u=1+\epsilon^+\}\cup\{u=1-\epsilon^-\}} \left[\frac{1}{2}|\nabla_{\mathbb{G}} u|^2 \sum_{k=1}^N \varphi_k \langle X_k, dn\rangle - (\sum_{k=1}^N X_k u \varphi_k) \sum_{l=1}^N X_l u \langle X_l, dn\rangle \right. \\ &\left. + \int\limits_{\{u=1+\epsilon^+\}} \sum_{k=1}^N \varphi_k \langle X_k, dn\rangle\right] \\ &= \int\limits_{\{u=1+\epsilon^+\}\cup\{u=1-\epsilon^-\}} \left[\left(1 - \frac{1}{2}|\nabla_{\mathbb{G}} u|^2\right) \sum_{k=1}^N \varphi_k \langle X_k, dn\rangle - \sum_{k\neq l; 1\le k,l\le N} \varphi_k X_l u X_k u \langle X_l, dn\rangle\right] \\ &= \int\limits_{\{u=1+\epsilon^+\}} \left[\left(1 - \frac{1}{2}|\nabla_{\mathbb{G}} u|^2\right) \sum_{k=1}^N \varphi_k \langle X_k, dn\rangle - \sum_{k\neq l; 1\le k,l\le N} \varphi_k X_l u X_k u \langle X_l, dn\rangle\right]\end{aligned} \tag{4.15}$$

$$
\begin{aligned}
&- \int\limits_{\{u=1-\epsilon^-\}} \left[\left(\frac{1}{2} |\nabla_{\mathbb{G}} u|^2 \right) \sum_{k=1}^{N} \varphi_k \langle X_k, dn \rangle - \sum_{k \neq l; 1 \leq k,l \leq N} \varphi_k X_l u X_k u \langle X_l, dn \rangle \right] \\
&= \int\limits_{\{1-\epsilon^- < u < 1+\epsilon^+\}} \left(\frac{1}{2} |\nabla_{\mathbb{G}} u|^2 \sum_{k=1}^{N} X_k \varphi_k - \sum_{k=1}^{N} \sum_{l=1}^{N} X_k \varphi_l X_l u X_k u \right) dx \\
&+ \int\limits_{\{1-\epsilon^- < u < 1+\epsilon^+\}} \left[\sum_{k=1}^{N} X_k \varphi_k + \lambda (u_n - 1)_+^2 f \sum_{k=1}^{N} X_k \varphi_k \right] dx,
\end{aligned}
$$

as $n \to \infty$.

Note that the normal vector at the point P on the set $\{u = 1 + \epsilon^+\} \cup \{u = 1 - \epsilon^-\}$ is $n = \pm \frac{\nabla_{\mathbb{G}} u(P)}{|\nabla_{\mathbb{G}} u(P)|}$. Thus equation (4.15) under the limit $\epsilon \to 0$ becomes

$$
\begin{aligned}
0 = &\lim_{\epsilon \to 0} \int\limits_{\{u=1+\epsilon^+\}} \left[\left(1 - \frac{1}{2} |\nabla_{\mathbb{G}} u|^2 \right) \sum_{k=1}^{N} \varphi_k \langle X_k, dn \rangle \right] \\
&- \lim_{\epsilon \to 0} \int\limits_{\{u=1-\epsilon^-\}} \left[\left(\frac{1}{2} |\nabla_{\mathbb{G}} u|^2 \right) \sum_{k=1}^{N} \varphi_k \langle X_k, dn \rangle \right].
\end{aligned} \tag{4.16}
$$

This proves that u satisfies the free boundary condition in the sense of *viscosity*. The solution cannot be trivial since $u \in C^1(\{u > 1\})$ and it satisfies the free boundary condition. $\square$

Remark 4.2. Notice that I_α satisfies the Palais-Smale (PS) condition. To prove this, we define

$$
u_n^+(x) := \max\{u_n(x), 0\}, u^+ + u^- := (u-1)_+ + [1 - (u-1)_-] = u.
$$

Notice that

$$
\begin{aligned}
I_\alpha(u_n) &\geq 2^{-1} \|u_n\|^2 - \frac{\lambda}{3} \int\limits_\Omega f (u_n^+)^3 dx \\
\langle I_\alpha'(u_n), u_n \rangle &\leq \|u_n\|^2 - \lambda \int\limits_\Omega f (u_n^+)^3 dx + \frac{2}{\alpha} |\Omega|.
\end{aligned} \tag{4.17}
$$

Let $c \in \mathbb{R}$ and consider

$$
c + \sigma \|u_n\| + o(1) \geq I_\alpha(u_n) - \frac{1}{3} \langle I_\alpha'(u_n), u_n \rangle \geq 6^{-1} \|u_n\|^2 - \frac{2}{\alpha} |\Omega|. \tag{4.18}
$$

This implies that (u_n) is bounded in $W_0^{1,2}(\Omega)$. This implies that there exists a subsequence of (u_n) such that $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$, $u_n \to u$ in $L^3(\Omega)$ and $u_n(x) \to u(x)$ a.e. in Ω. Since $\langle I_\alpha'(u_n), v \rangle \to 0$ as $n \to \infty$ we have

$$
\lim_{n \to \infty} \int\limits_\Omega \tilde{\nabla} u_n v dx = \lim_{n \to \infty} \left[\int\limits_\Omega \frac{1}{\alpha} g\left(\frac{u_n - 1}{\alpha} \right) v dx + \lambda \int\limits_\Omega (u_n - 1)_+^2 v f dx \right] \text{ for all } v \in W_0^{1,2}(\Omega). \tag{4.19}
$$

We choose $v = u_n - u$ in (4.19) to obtain

$$\lim_{n\to\infty}\int_\Omega \tilde{\nabla} u_n(u_n-u)dx = \lim_{n\to\infty}\left[\int_\Omega \frac{1}{\alpha}g\left(\frac{u_n-1}{\alpha}\right)(u_n-u)dx + \lambda\int_\Omega (u_n-1)_+^2(u_n-u)f dx\right] \\ = 0. \tag{4.20}$$

This implies that $u_n \to u$ in $W_0^{1,2}(\Omega)$. Hence I_α satisfies the (PS) condition.

5. Proof of the main theorem

Before we prove the existence of a solution to the problem (1.1), we develop a few tools which will be used in the proof. We observe that

$$I_\alpha(u) \le I(u) \text{ in } W_0^{1,2}(\Omega).$$

Furthermore, we have

$$\begin{aligned} I_\alpha(u) &\ge \frac{1}{2}\|u\|^2 - \frac{\lambda}{3}\int_\Omega |u|^3 f dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{C\lambda}{3}\|f\|_\infty\|u\|^3 \end{aligned} \tag{5.1}$$

by Lemma 2.3. Therefore, there exists $r_0 = r_0(\nu, \lambda, \|f\|_\infty) > 0$ such that

$$I_\alpha(u) \ge \frac{1}{4}\|u\|^2 \tag{5.2}$$

for $\|u\| \le r_0$. Furthermore, for a fixed nonzero u we have $I_\alpha(tu) \to -\infty$ as $t \to \infty$ and hence there exists a function v_0 such that $I_\alpha(v_0) < 0 = I_\alpha(0)$. This indicates that the set

$$\Lambda_\alpha := \{\psi \in C([0,1]; W_0^{1,2}(\Omega)) : \psi(0) = 0, I_\alpha(\psi(1)) < 0\}$$

is nonempty. Hence by the Mountain pass theorem we have

$$c_\alpha := \inf_{\psi\in\Lambda_\alpha}\max_{u\in\psi([0,1])} I_\alpha(u). \tag{5.3}$$

By the definition of the set Λ_α we have $\Lambda \subset \Lambda_\alpha$ and

$$c_\alpha \le \max_{u\in\psi([0,1])} I_\alpha(u) \le \max_{u\in\psi([0,1])} I(u) \tag{5.4}$$

for all $\psi \in \Lambda$. This implies that $c_\alpha \le c$.

Remark 5.1. Let ϕ_1 be the first eigenfunction pertaining to the first eigen value λ_1 (see Proposition 3.1 [9]). Notice that

$$I(t\phi) \to -\infty \text{ as } t \to \infty. \tag{5.5}$$

Thus there exists $t_* > 0$ such that $I(t_*\phi_1) < 0$. Consider the path which is defined by $\psi(t) = t\phi_1$ for $t \in [0, t_*]$. Then ψ yields a path from Λ on which

$$I(t\phi_1) \leq \mathcal{C} := \sup_{t\geq 0} \int\limits_{\Omega} \left(\frac{\lambda_1}{2}t^2\phi_1 + 1\right) dx. \tag{5.6}$$

Therefore $c \leq \mathcal{C}$.

Proof of Theorem 1.1. From Remark 5.1 we conclude that $c_\alpha \leq c \leq \mathcal{C}$. Since I_α obeys the (PS) condition, it follows that a limit of the (PS) sequence, say u_α, can be shown to be a critical point of I_α. Hence we have $I_\alpha(u_\alpha) = c_\alpha$.

Now consider a sequence α_n which converges to zero and name u_{α_n} as u_n and c_{α_n} as c_n. By Lemma 4.1 $(i) - (ii)$, we know that a subsequence of (u_n), still denoted by the same name, converges uniformly in $\bar{\Omega}$, locally in $C^1(\bar{\Omega} \setminus \{u = 1\})$, and strongly in $W_0^{1,2}(\Omega)$, to a locally Lipschitz function $u \in W_0^{1,2}(\Omega) \cap C^2(\bar{\Omega} \setminus H(u))$. Moreover, by (5.2) in Remark 4.2 we have $\limsup I_{\alpha_n}(u_n) = \limsup c_n \geq \frac{r_0}{4} > 0$. This indicates that one of the limit conditions $\limsup I_{\alpha_n}(u_n) > 0$ or $\liminf I_{\alpha_n}(u_n) < 0$ in Lemma 4.1 indeed holds.

Hence by the paragraph after Lemma 4.1 (iv), we can conclude that u is nontrivial. Furthermore, by Lemma 4.1, u is a classical solution of $\mathcal{L}u = \lambda(u-1)_+^2 f$ in $\Omega \setminus \partial\{u > 1\}$ and the free boundary condition $|\nabla_{\mathbb{G}} u^+|^2 - |\nabla_{\mathbb{G}} u^-|^2 = 2$ in the sense of (4.16), plus it vanishes on the boundary $\partial\Omega$. $\square$

Remark 5.2. We note that the limiting conditions in Lemma 4.1 are still an open problem, which is sublinear in its nature.

6. Appendix: Radon measure lemma

Lemma 6.1. *$u \in W_{loc}^{1,p}(\Omega)$ and the Radon measure $\mathcal{M} = \mathcal{L}u$ is nonnegative and supported on $\Omega \cap \{u < 1\}$.*

Proof. We follow the idea of the proof in Alt-Caffarelli [2]. Choose $\delta > 0$ and a test function $\varphi^p \chi_{\{u<1-\delta\}}$, where $\varphi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned} 0 &= \int\limits_{\Omega} \tilde{\nabla} u \nabla(\varphi^2 \min\{u - 1 + \delta, 0\}) dx \\ &= \int\limits_{\Omega\cap\{u<1-\delta\}} \tilde{\nabla} u(\varphi^2 \min\{u - 1 + \delta, 0\}) dx \\ &= \int\limits_{\Omega\cap\{u<1-\delta\}} |\nabla_{\mathbb{G}} u|^2 \varphi^2 dx + \int\limits_{\Omega\cap\{u<1-\delta\}} \varphi(u - 1 + \delta)\tilde{\nabla} u \varphi dx, \end{aligned} \tag{6.1}$$

and so by the Caccioppoli like estimate, we have

$$\begin{aligned} \int\limits_{\Omega\cap\{u<1-\delta\}} |\nabla_{\mathbb{G}} u|^2 \varphi^2 dx &= -2 \int\limits_{\Omega\cap\{u<1-\delta\}} \varphi(u - 1 + \delta)\tilde{\nabla} u \varphi dx \\ &\leq c \int\limits_{\Omega} u^2 |\nabla_{\mathbb{G}} \varphi|^2 dx. \end{aligned} \tag{6.2}$$

Since $\int_\Omega |u|^2 dx < \infty$, by passing the limit $\delta \to 0$, we can conclude that $u \in W_{\mathrm{loc}}^{1,2}(\Omega)$. Furthermore, for a nonnegative $\zeta \in C_0^\infty(\Omega)$ we have

$$
\begin{aligned}
-\int_{\Omega} \tilde{\nabla} u \zeta dx &= \left(\int_{\Omega\cap\{0<u<1-2\delta\}} + \int_{\Omega\cap\{1-2\delta<u<1-\epsilon\}} + \int_{\Omega\cap\{1-\delta<u<1\}} + \int_{\Omega\cap\{u>1\}} \right) \\
&\quad \left[\tilde{\nabla} u \left(\zeta \max\left\{ \min\left\{ 2 - \frac{1-u}{\delta}, 1 \right\}, 0 \right\} \right) \right] dx \\
&\geq \int_{\Omega\cap\{1-2\delta<u<1-\delta\}} \left[\tilde{\nabla} u \left(2 - \frac{1-u}{\delta} \right) \zeta + \frac{\zeta}{\delta} |\nabla_{\mathbb{G}} u|^2 \right] dx \geq 0.
\end{aligned} \tag{6.3}
$$

On passing to the limit $\delta \to 0$, we obtain $\mathcal{L}(u-1)_- \geq 0$ in the distribution sense. Therefore there exists a Radon measure, say $\mathcal{M}$, such that $\mathcal{M} = \mathcal{L}(u-1)_- \geq 0$. □

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