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A Short Proof of the Twelve-Point Theorem

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Abstract—We present a short elementary proof of the following twelve-point theorem. Let M be a convex polygon with vertices at lattice points, containing a single lattice point in its interior. Denote by m (respectively, m^*) the number of lattice points in the boundary of M (respectively, in the boundary of the dual polygon). Then $m+m^*=12$.

Key words: *lattice, lattice polygon, dual polygon, the Pick formula, toric varieties.*

The twelve-point theorem is an elegant theorem, which is easy to formulate, but no simple proof was available until now. In this paper, we present a short and elementary proof of this result. To state our theorem we need the following definition.

Definition of the dual polygon. In this paper, we assume that a Cartesian coordinate system in the plane is fixed. Let $M = A_1A_2\dots A_n$ be a convex polygon all of whose vertices lie in the lattice of points with integer coordinates (see Fig. 1 on the left). Suppose that O is the only lattice point in the interior of M. Draw the vectors $\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \dots, \overrightarrow{A_nA_1}$ from the point O. Choose on each of the obtained segments the lattice point distinct from O nearest to O. Connecting the n chosen points consecutively, we obtain the polygon M^* *dual* to the original polygon (see Fig. 1 on the right). Denote by m and m^* the number of integer points in the boundary of M and M^* respectively.

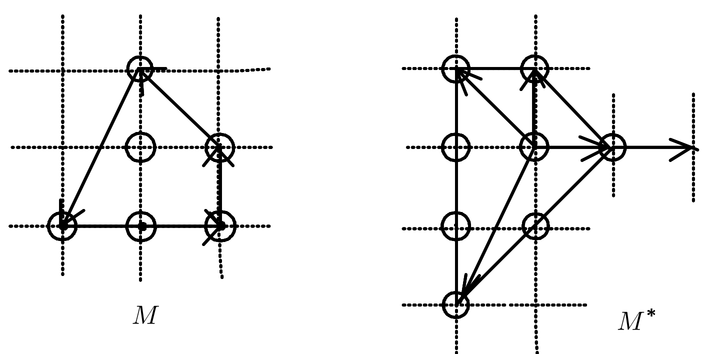

Fig. 1

The Twelve-Point Theorem. *Suppose that M is a convex polygon with integer vertices, containing a single integer point in its interior. Then*

$$m+m^*=12.$$

This theorem appeared in the book [1]. There are some hints to the proof, applying the theory of toric varieties (see also [2]). In the interesting paper [3], entirely dedicated to the twelve-point

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theorem, even four different proofs are discussed. Three of them are rather long and they use toric varieties and modular forms, respectively. There are also two outlined proofs applying only linear algebra. The first of them is by exhaustion (there are 16 different types of polygons M). The idea of the fourth one is very close to the proof of the present paper. (This proof is only outlined in the paper mentioned above, and its straightforward realization is also very cumbersome.)

Our elementary proof is analogous to one of the proofs of the Pick formula. We reduce the twelve-point theorem to the specific case in which M is a parallelogram and $m = 4$. Let us begin with this latter case.

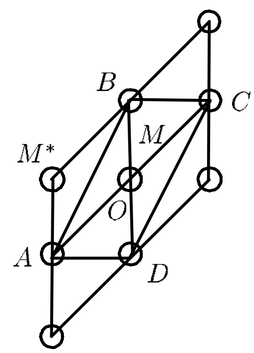

Fig. 2

(1) *If* $M = ABCD$ *is a parallelogram without lattice points in its sides, then* $m + m^* = 12$ (see Fig. 2).

Indeed, in this case $O = AC \cap BD$, because the point symmetric to the point O with respect to $AC \cap BD$ is a lattice point and belongs to the interior of $ABCD$, so it coincides with O. It is easy to show that M^* is a parallelogram with sides obtained from the diagonals AC and BD by parallel translations with vectors $\pm\overrightarrow{OB}$ and $\pm\overrightarrow{OA}$, respectively. Since a unique lattice point O belongs to these diagonals, then any side of the parallelogram M^* contains one lattice point; hence $m + m^* = 4 + 8 = 12$.

Now, suppose that $M = A_1 \dots A_n$. Let us assume that all the lattice boundary points of M are vertices (possibly, with angle $180°$). This does not affect the definition of M^*. Assume that some triangle $A_{i-1}A_iA_{i+1}$ is *simple*, i.e., it contains no lattice points except its vertices (neither in the interior nor in the boundary). An *elementary operation*

$$A_1 \dots A_{i-1}A_iA_{i+1} \dots A_n \to A_1 \dots A_{i-1}A_{i+1} \dots A_n$$

is the cutting off from the polygon M of the triangle $A_{i-1}A_iA_{i+1}$ and the reverse operation. Our reduction is based on the following assertion.

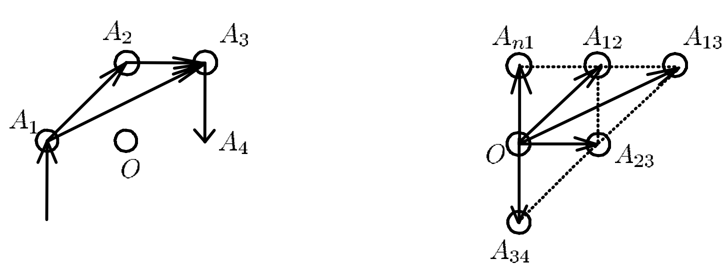

Fig. 3

(2) *The value $m + m^*$ is preserved under an elementary operation.*

It is sufficient to prove that deleting a simple triangle, say $A_1A_2A_3$, from M results in adding a simple triangle $A_{12}A_{13}A_{23}$ to M^* (see Fig. 3). Here by A_{kl} we denote the point such that $\overrightarrow{OA_{kl}} = \overrightarrow{A_kA_l}$. In particular, if $l = k+1$, then A_{kl} is a vertex of the polygon M^*. Delete $A_1A_2A_3$. Then the vertices A_{12} will be deleted from the polygon M^* and A_{23}, and the new vertex A_{13} added to it. The last vertex should be joined by segments with A_{n1} and A_{34}. Let us show that the points A_{12} and A_{23} belong to these segments. Indeed, since O is the only lattice point inside M, it follows that the triangles A_1OA_3, A_2OA_3, A_4OA_3 are simple. By the Pick formula, their areas are equal to $1/2$. Since they have the common base OA_3, it follows that the projections of the vectors $\overrightarrow{A_1A_3}$, $\overrightarrow{A_1A_3}$, and $\overrightarrow{A_1A_3}$ on the direction normal to OA_3 are equal. This implies that the points A_{13}, A_{23}, and A_{34} belong to the same line, and A_{23} lies between the two others, because M is convex. It can be proved similarly that A_{12} belongs to the segment $A_{n1}A_{13}$. Therefore, the transformation of M^* is just adding the triangle $A_{12}A_{13}A_{23}$. Now, note that the triangle $OA_{12}A_{13}$ is obtained from the simple triangle $A_1A_2A_3$ by parallel translation, and $OA_{23}A_{13}$ is obtained from it by central symmetry. So the triangle $A_{12}A_{13}A_{23}$ is simple, and assertion (2) is proved.

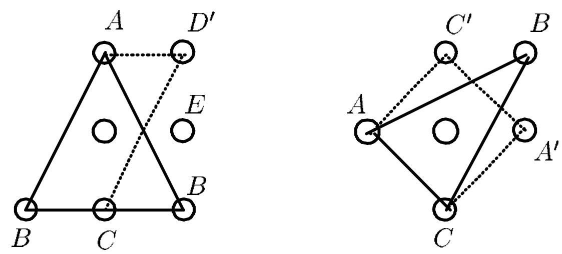

Fig. 4

For the proof of our theorem, it remains to notice the following.

(3) *From any polygon M, one can obtain a parallelogram without lattice points in the sides by a sequence of elementary operations.*

Indeed, first assume that M has a diagonal not passing through O. Cut M along this diagonal and consider the obtained part not containing O. This part necessarily contains a simple triangle of the form $A_{i-1}A_iA_{i+1}$. Deleting it, we decrease the number m. Repeat this operation as long as it is possible. Repetition is impossible only in the following three cases (when such a diagonal does not exist).

(A) $m = 4$, $M = ABCD$, $O = AC \cap BD$; since the segments OA, OB, OC, and OD do not contain lattice points, it follows that $OA = OC$ and $OB = OD$, i.e., $ABCD$ is the required parallelogram;

(B) $m = 4$, $M = ABCD$, and C belongs to the segment BD; in this case let us denote by D' the point symmetric to D with respect to O, and denote by E the midpoint of $D'B$; the required sequence of elementary operations has the form

$$ABCD \to AEBCD \to AD'EBCD \to AD'ECD \to AD'CD$$

(see Fig. 4 on the left);

(C) $m = 3$, $M = ABC$; in this case denote by A' and C' the points symmetric to A and C, respectively, with respect to O; the required sequence of elementary operations has the form

$$ABC \to AC'BC \to AC'BA'C \to AC'A'C$$

(see Fig. 4 on the right).

So, in each case, we obtain the required parallelogram, which completes the proof of our theorem.

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