

On the Browder–Levine–Novikov Embedding Theorems

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Abstract—In this survey, we apply the concepts of complement and neighborhood to embeddings of manifolds into Euclidean space (in codimension at least three). We describe how a combination of these concepts gives a reduction of the embeddability and isotopy problems to algebraic problems. We also present a modern exposition of the Browder–Levine theorem on the realization of normal systems.

INTRODUCTION

According to E.C. Zeeman, the following are the three classical problems of topology:

- (1) *Homeomorphism Problem*: find conditions under which two given spaces are homeomorphic (and also describe the homeomorphism classes of manifolds from a given class, e.g., of given dimension n);
- (2) *Embedding Problem*: find conditions under which a given space embeds into S^m for given m ; and
- (3) *Knotting Problem*: find conditions under which two given embeddings are isotopic (and also describe the isotopy classes of embeddings $N \rightarrow S^m$).

This paper deals with problems (2) and (3). We show how the concepts of complement and neighborhood can be used to study embeddings. Then, we combine these two ideas and formulate the Browder–Levine–Novikov embedding theorems. Such an exposition of these results appears in this paper for the first time. Note that the classification theorem for links [11] can be proved analogously to the Browder embedding theorem [10]. For a survey on a different approach to embeddings, see [26].

Denote by CAT the smooth (DIFF) or piecewise-linear (PL) category. We omit CAT if a definition or a statement holds in both categories. A polyhedron N is said to be *PL embeddable* into S^m if there is a PL injective mapping $f: N \rightarrow S^m$. A smooth manifold N is said to be *DIFF embeddable* into S^m if there is a DIFF injective mapping $f: N \rightarrow S^m$ such that df is a monomorphism at each point. Such a mapping f is called *an embedding* of N into S^m (in the corresponding category).

Two embeddings $f, g: N \rightarrow S^m$ are said to be (*ambient*) *isotopic* if there exists a homeomorphism onto $F: S^m \times I \rightarrow S^m \times I$ such that

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- (i) $F(y, 0) = (y, 0)$ for each $y \in S^m$,
- (ii) $F(f(x), 1) = (g(x), 1)$ for each $x \in N$, and
- (iii) $F(S^m \times \{t\}) = S^m \times \{t\}$ for each $t \in I$.

This homeomorphism F is called an (*ambient*) *isotopy*. An (ambient) isotopy is also a homotopy $S^m \times I \rightarrow S^m$ or a family of maps $F_t: S^m \rightarrow S^m$ generated by the map F in an obvious manner. If any two embeddings $N \rightarrow S^m$ are isotopic, we say that N *unknots* in S^m .

THE CONCEPT OF COMPLEMENT AND OBSTRUCTIONS TO EMBEDDABILITY

In some cases, one can obtain obstructions to embeddability using the following idea that can be traced back to the early works of G. Alexander (about 1910). Considering the complement $S^m - N$ of a polyhedron N embedded into S^m , one can deduce necessary conditions on the polyhedron N itself.

Let us illustrate this idea by the embeddability of graphs in the plane. Suppose that a graph N embeds into the plane. By the Euler formula, $V - E + F = 2$. Since every face has at least three edges in its boundary, we have $3F \leq 2E$. Therefore, $E \leq 3V - 6$. This implies that the graph K_5 is not planar. Analogously, one can prove that the graph $K_{3,3}$ is not planar.

In general, the Euler formula is replaced by its generalization, that is, by the Alexander duality. For example, if a polyhedron N is embedded into S^m , then the Betti numbers satisfy $b^m(N) = b^{-1}(S^m - N) = 0$. This gives a necessary condition for N to be embeddable into S^m . By developing this idea, Hantzsche proved the following result.

Theorem 1 [13, Theorems 1 and 3]. (a) *If a $2l$ -manifold N embeds into S^{2l+1} , then N is orientable and the Euler characteristic of N is even (i.e., the dimension of the free part of $H_l(N, \mathbb{Z})$ or the l th Betti number of N is even).*

(b) *If a $(2l + 1)$ -manifold N embeds into S^{2l+2} , then N is orientable and the torsion part of $H_l(N, \mathbb{Z})$ can be represented as the sum $G \oplus G$ for some group G .*

In particular, Theorem 1(b) implies that $\mathbb{R}P^3$ and the 3-dimensional lens spaces do not embed into S^4 . Under the assumptions of Theorem 1(b), we also have that the *linking form* of N vanishes on each of the summands G [9, Exercise 4.5.12.d].

Similarly, by studying the duality between homology *rings* of a manifold $N \subset S^m$ and the complement $S^m - N$, H. Hopf proved that $\mathbb{R}P^{m-1}$ does not embed into S^m [16].

Theorem 2. *If a manifold N^n embeds into S^{n+1} , then there are groups A^q and B^q for $0 < q < n$ such that $H^q(N) \cong A^q \oplus B^q$ and the \cup -product on N induces an isomorphism $A^q \cong \text{Hom}(B^{n-q}, \mathbb{Z}_2)$ [16; 31, III, 2.1].*

Thom obtained some conditions on the cohomology ring of a closed $(m - 1)$ -manifold that are necessary for the embeddability into S^m [32] (see also [3]). Peterson studied the duality between *cohomology operations* in N and $S^m - N$ and obtained some interesting nonembedding results.

THE CONCEPT OF COMPLEMENT AND OBSTRUCTIONS TO ISOTOPY

The concept of complement is even better applicable to studying the knotting problem. Indeed, if $f, g: N \rightarrow S^m$ are isotopic embeddings, then $S^m - f(N) \cong S^m - g(N)$. Therefore, any invariant of the space $S^m - f(N)$ is an isotopy invariant.

This idea was first applied by Alexander in the 1910s to the knots $S^1 \subset S^3$. Let us illustrate his idea by proving that the trefoil knot $f: S^1 \rightarrow S^3$ is nontrivial. Using the van Kampen theorem on the fundamental group of the union, we obtain that $\pi := \pi_1(S^3 - f(S^1)) = \langle x, y \mid xyx = yxy \rangle$. Evidently, $\pi_1(S^3 - S^1) = \mathbb{Z}$. The first idea to distinguish between \mathbb{Z} and π is to compare \mathbb{Z} and the *abelianization* $\pi/[\pi, \pi]$ of π . But it turns out that $\pi/[\pi, \pi] = \mathbb{Z}$. Perhaps Alexander, while trying

to distinguish knots, observed that $\pi/[\pi, \pi] = \mathbb{Z}$ for the fundamental group π of the complement to *any* knot, which led him to the discovery of his duality theorem. In order to distinguish between a trefoil knot and a trivial knot, one can construct a nontrivial homomorphism $\pi \rightarrow S_3$, defined by $x \rightarrow (12)$ and $y \rightarrow (23)$. Hence, π is not abelian and not isomorphic to \mathbb{Z} .

The exposition of the subsequent development of the concept of complement in the theory of knots $S^1 \subset S^3$ (or, more generally, $S^n \subset S^{n+2}$) is beyond the purposes of this survey. We only formulate sufficient conditions for the completeness of the complement invariant (without proof):

Theorem 3. *For any $n \neq 2$, a DIFF embedding $f: S^n \rightarrow S^{n+2}$ is DIFF unknotted if and only if $S^{n+2} - f(S^n)$ is homotopy equivalent to S^1 [18].*

Note that $S^3 - f(S^1) \simeq S^1$ is equivalent to $\pi_1(S^3 - f(S^1)) \cong \mathbb{Z}$ [24].

An analogue of Theorem 3 also holds in PL locally flat and TOP locally flat categories for $n \neq 2, 3$ [24, 18, 30] (see also [8]). Recall that a (PL or TOP) embedding $S^n \subset S^m$ is said to be (PL or TOP) *locally flat* if every point of S^n has a neighborhood U in S^m such that $(U, U \cap S^n)$ is (PL or TOP) homeomorphic to $(B^n \times B^{m-n}, B^n \times 0)$.

The local flatness assumption in the TOP category is needed to rule out *wild* embeddings, which were first constructed by Antoine in 1920 (the Antoine necklace) and Alexander in 1923 (the Alexander horned sphere) using the same concept of complement [1] (see also [28]).

A motivation for constructing the Alexander horned sphere is the study of the TOP knots in codimension 1, i.e., of embeddings $S^n \rightarrow S^{n+1}$. A short history of this problem is as follows. The well-known Jordan theorem, first proved by Brouwer, states that every S^n contained in S^{n+1} splits S^{n+1} into two components. It is easy to prove the following analogue of Theorem 3: $S^n \subset S^{n+1}$ is unknotted if and only if the closures of these components are balls. In 1912, Schönflies proved that every $S^1 \subset S^2$ is unknotted. Thus, the unknottedness of $S^n \subset S^{n+1}$ is called the Schönflies theorem or problem. In 1921, Alexander announced that he had proved the Schönflies theorem for arbitrary n . However, in 1923, he found a counterexample—the celebrated horned sphere. Nevertheless, he proved that, in the PL category, the Schönflies theorem is true for $n = 2$ [2]. For $n \geq 3$, the PL Schönflies conjecture remains a famous difficult unsolved problem [27]. After the appearance of the Alexander horned sphere, the unknottedness of a *locally flat* embedding $S^n \rightarrow S^{n+1}$ was also referred to as the *Schönflies conjecture*. It was proved only in 1960s.

Theorem 4. *The sphere S^n is smoothly unknotted in S^{n+1} for $n \neq 3$ [29, 4].*

An analogue of Theorem 4 is also true in the PL and TOP locally flat categories (in the PL case, it holds only for $n \neq 3$) [5, 21, 22, 27]. Note that Brown’s elegant short proof [5] of the analogue of Theorem 4 in the topological case gave rise to the theory of cellular sets, which is now an important branch of geometric topology.

THE COMPLEMENT INVARIANT

Now, let us give formal definitions that will be used in the sequel. Denote by $i: S^m \rightarrow S^{m+1}$ the standard inclusion. For an n -space N and a PL or DIFF embedding $f: N \rightarrow S^m$, let

$$C(f) = S^m - f(N) \simeq S^m - Of(N),$$

where $Of(N)$ is a regular neighborhood of $f(N)$ in S^m . The topological and the homotopy type of $C(f)$ is an invariant of f . Note that, for $m - n \geq 3$, the space $C(f)$ is simply connected, and by the Alexander duality, its homology groups do not depend on f . Therefore, the invariant $C(f)$ is relatively weak for $m - n \geq 3$.

Note that

$$C(f) \simeq S^{m-n_1-1} \vee \dots \vee S^{m-n_r-1} \quad \text{for } N = S^{n_1} \sqcup \dots \sqcup S^{n_r}, \quad m - n_i \geq 3,$$

and any embedding $f: N \rightarrow S^m$. But the complement of $S^p \times S^q$ depends on the embedding into S^m even for $m - p - q \geq 3$. Indeed, consider the Hopf fibration $S^3 \rightarrow S^7 \xrightarrow{h} S^4$. Take the standard embedding $S^2 \subset S^4$. Its complement has the homotopy type of S^1 . We have $h^{-1}(S^1) \cong S^1 \times S^3 \subset S^7$, and the complement of this embedding f is

$$C(f) \simeq h^{-1}(S^2) \cong S^2 \times S^3 \not\cong S^2 \vee S^3 \vee S^5 \simeq C(f_0),$$

where $f_0: S^1 \times S^3 \rightarrow S^7$ is the standard embedding. Analogously, one can construct two embeddings $S^3 \times S^7 \subset S^{15}$ whose complements are not homotopy equivalent. This example is due to P. Lambrechts (personal communication).

By general position, $C(f)$ does not depend on f for $m \geq 2n + 2$. This space is denoted by $C_m(N)$. Note that

$$C(i \circ f) \simeq \Sigma C(f); \quad \text{therefore,} \quad \Sigma^{M-m} C(f) \simeq C_M(f) \quad \text{for} \quad M \geq 2n + 2.$$

The *complement necessary condition* for the embeddability of N into S^m is the $(M - m)$ -desuspendability of $C_M(N)$, i.e., the existence of a space C such that $\Sigma^{M-m} C \simeq C_M(f)$. If this condition holds for $M = 2n + 2$, then it automatically holds for any $M \geq 2n + 2$.

THE CONCEPT OF NEIGHBORHOOD

In order to obtain necessary conditions for N to be embeddable into S^m , we can assume that N embeds into S^m and consider relations between $N \subset S^m$ and its neighborhoods in S^m . This method seems to have been first introduced by Whitney in the DIFF category. Whitney created a theory of sphere bundles and introduced the so-called Stiefel–Whitney classes $w^k \in H^k(N, \mathbb{Z}_2)$ and the dual Stiefel–Whitney classes $\bar{w}^k \in H^k(N, \mathbb{Z}_2)$ of a DIFF manifold N , which have played an important role in topology and differential geometry.

Now, let us give formal definitions and statements. Let N be a closed smooth n -manifold. A smooth map $f: N \rightarrow S^m$ is a *smooth immersion* if $df(x)$ is nondegenerate for each $x \in N$. Two smooth immersions $f, g: N \rightarrow S^m$ are *regularly homotopic* if there exists a smooth immersion $F: N \times I \rightarrow S^m \times I$ such that

- (i) $F(x, 0) = (f(x), 0)$ and $F(x, 1) = (g(x), 1)$ for each $x \in N$ and
- (ii) $F(N \times \{t\}) \subset S^m \times \{t\}$ for each $t \in I$.

The normal bundle $\nu(f)$ of f is, up to equivalence, a regular homotopy invariant of f (and an isotopy invariant of f if f is an embedding). This isotopy invariant is not very strong because, e.g., normal bundles of different embeddings are stably equivalent. For an embedding f , the space of $\nu(f)$ and open regular neighborhood $Of(N)$ of $f(N)$ in S^m can be identified by a homeomorphism κ under which the zero section goes to $f(N)$. Note that this homeomorphism κ is not uniquely defined (more precisely, the differences lie in $H^l(N, \pi_l(\text{SO}_{k-1}))$). From now on, denote $m = n + k$.

By general position, $\nu(f)$ does not depend on f for $k \geq n + 2$. This *stable normal bundle* is denoted by $\nu_k(N)$. Note that

$$\nu(i \circ f) = \nu(f) \oplus 1; \quad \text{therefore,} \quad \nu(f) \oplus (K - k) = \nu_K(f) \quad \text{for} \quad K \geq n + 2.$$

The *normal bundle necessary condition* for the immersibility of N into S^{n+k} is the existence of a k -bundle ν over N that is stably equivalent to $\nu_k(f)$. Since $\nu(f) \oplus \tau(N) = n + k$, it follows that the normal bundle condition is equivalent to the stable triviality of $\nu(f) \oplus \tau(N)$ and to the triviality of the latter bundle (by general position, because the dimension of $\nu(f) \oplus \tau(N)$ is greater than n). We have the following theorem [14] (see also [15]):

Theorem 5. *Suppose that N is a closed smooth n -manifold.*

If there exists a k -bundle ν over N such that $\nu \oplus (K - k) \cong \nu_K(N)$, then N immerses in S^{n+k} .

If f and g are immersions such that $df, dg: TN \rightarrow \mathbb{R}^{n+k}$ are homotopic via linear monomorphisms, then f and g are regular homotopic.

NORMAL SYSTEMS AND THE BROWDER–LEVINE EMBEDDING THEOREM

Combining the complement and the neighborhood ideas, Levine, Novikov, and Browder obtained new necessary conditions for the embeddability of manifolds (and for the isotopy of embeddings). The proofs of the sufficiency of these conditions are among the most important applications of *surgery* to topology of manifolds. For closely related concepts of thickenability and thickenings, see [33, 19, 25].

Let us formulate the notion of the attaching invariant and the Browder–Levine theorem. The homotopy class $a(f, \kappa)$ of the inclusion $S\nu(f) \subset C(f)$ is called the *attaching invariant* of the pair (f, κ) . The triple $\mathbb{S}(f, \kappa) = (\nu(f), C(f), a(f, \kappa))$ is called the *normal system* of (f, κ) . In general, a *normal system of codimension k on a manifold N* is a triple $\mathbb{S} = (\nu, C, p)$ of

- (i) a vector k -bundle ν ,
- (ii) a space C , and
- (iii) a continuous mapping $p: S\nu \rightarrow C$.

Two normal systems (ν, C, p) and (ν_1, C_1, p_1) are said to be *equivalent* if there exists a bundle isomorphism $b: \nu \rightarrow \nu_1$ and a homotopy equivalence $r: C \rightarrow C_1$ such that $r \circ p \simeq p_1 \circ Sb$. Clearly, the equivalence class of the normal system (f, κ) does not depend on κ . This class is called the *normal system $\mathbb{S}(f)$ of the embedding f* . The normal system of an embedding is an isotopy invariant. Since every two embeddings $N \rightarrow S^{n+K}$ are isotopic for $K > n + 1$, it follows that $\mathbb{S}(f)$ does not depend on f for $K > n + 1$. This normal system is called the *stable normal system of N* and is denoted by $\mathbb{S}_K(N)$. The *suspension $\Sigma\mathbb{S}$* over a normal system (ν, C, p) is the normal system $(\nu \oplus 1, \Sigma C, p')$, where p' is the suspension of p on each fiber. The *Browder–Levine necessary condition* for the embeddability of N into S^{n+k} is the existence of a normal system \mathbb{S} on N such that $\Sigma^{K-k}\mathbb{S} \sim \mathbb{S}_K(N)$ (clearly, this condition does not depend on K).

Theorem 6 (Browder–Levine). *Suppose that N is a closed smooth n -manifold, $K > n + 1$, $k \geq 3$, and there exists a normal system $\mathbb{S} = (\nu, C, p)$ over N such that $\Sigma^{K-k}\mathbb{S} \sim \mathbb{S}_K(N)$ and $\pi_1(C) = 0$. Then, there exists a smooth embedding $f: N \rightarrow S^{n+k}$ such that $\mathbb{S}(f) \sim \mathbb{S}$.*

In the case when N is a homotopy sphere, Theorem 6 was proved in [18], while in the general case, it was proved in [6] (see also [11, 12, 7]). Theorem 6 easily follows by induction from

Lemma 7 (compression and desuspension). *Suppose that $F: N \rightarrow S^{n+k+1}$ is an embedding of a closed smooth n -manifold, $k \geq 3$, $n + k \geq 5$, and there exists a normal system $\mathbb{S} = (\nu, C, p)$ over N such that $\Sigma\mathbb{S} \sim \mathbb{S}(F)$ and $\pi_1(C) = 0$. Then, there exists a smooth embedding $f: N \rightarrow S^{n+k}$ such that $\mathbb{S}(f) \sim \mathbb{S}$.*

Proof of the compression and desuspension Lemma 7 uses the following notion of normal invariant and is given at the end of the paper.

THE NORMAL INVARIANT AND THE BROWDER–NOVIKOV EMBEDDING THEOREM

The *normal invariant* of the pair (f, κ) is the homotopy class

$$\alpha(f, \kappa) = [S^{n+k} \rightarrow S^{n+k}/C(f) \cong T\nu(f)] \in \pi_{n+k}(T\nu(f)).$$

Let $\alpha(f) = \{\alpha(f, \kappa)\}_\kappa$ be the *set of normal invariants* of the embedding f . This set is an isotopy invariant.

Now, let us state the Browder–Novikov necessary condition for the embeddability of N into S^{n+k} . Since every two embeddings $N \rightarrow S^{n+K}$ are isotopic for $K > n + 1$, it follows that $\alpha(f)$ does not depend on f for $K > n + 1$. This set $\alpha_K(N)$ is called the set of *stable normal invariants* of N . Note that

$$T(\nu \oplus 1) \simeq \Sigma T(\nu) \quad \text{and} \quad \alpha(i \circ f, \kappa \times \text{id}) = \Sigma\alpha(f, \kappa);$$

so, $\Sigma^{K-k}\alpha(f) = \alpha_K(N)$ for $K > n + 1$. The *Browder–Novikov necessary condition* for the embeddability of N into S^{n+k} is the existence of

- (i) a vector bundle ν over N ,
- (ii) a vector bundle isomorphism $b: \nu \oplus (K - k) \rightarrow \nu_K(N)$, and
- (iii) an element $\alpha \in \pi_{n+k}(T\nu)$

such that $b_T \circ \Sigma^{K-k}\alpha \in \alpha_K(N)$ (clearly, this condition does not depend on $K > n + 1$). We denote by b_T the map of Thom spaces corresponding to the bundle map b .

Theorem 8 (Browder–Novikov). *Suppose that N is a closed smooth n -manifold, $K > n + 1$, $k \geq 2$, and the Browder–Novikov necessary condition for the embeddability of N into S^{n+k} is fulfilled. Then, there exists a smooth embedding $f: N \rightarrow S^{n+k+1}$ such that $\nu(f) = \nu \oplus 1$ and $\Sigma\alpha \in \alpha(f)$.*

In the case when N is a homotopy sphere, the Browder–Novikov Theorem 8 was proved in [23], and in the general case, it was proved in [6] (see also [17]). Note that

- the Browder–Novikov necessary condition for the embeddability of N into S^{n+k} implies the Browder–Levine necessary condition for the embeddability of N into S^{n+k+1} .

Indeed, given ν , α , and b from the Browder–Novikov condition, the system

$$\mathbb{S} = (\nu \oplus 1, C, p), \quad \text{where} \quad C = T\nu \bigcup_{\alpha: \partial D^{n+k+1} \rightarrow T\nu} D^{n+k+1}$$

and p is the composition $S(\nu \oplus 1) \rightarrow S(\nu \oplus 1)/\text{the canonical section} \cong T\nu \subset C$,

is the required normal system. In this way, the Browder–Novikov Theorem 8 was deduced from the Browder–Levine Theorem 6 in [6].

Note that

- the Browder–Levine necessary condition for the embeddability of N into S^{n+k} implies the Browder–Novikov necessary condition for the embeddability of N into S^{n+k} .

Indeed, we already have ν and b ; so, it remains to construct α . Since

$$\Sigma^{K-k}\mathbb{S} \sim \mathbb{S}(f), \quad \text{we have} \quad D(\nu \oplus (K - k)) \bigcup_{p^{(K-k)}} \Sigma^{K-k}C \simeq S^{n+K}.$$

Then, since both $D\nu$ and C are simply connected, using the Mayer–Vietoris sequence, one can show that $S^{n+k} \simeq D\nu \cup_p C$. The composition of this homotopy equivalence and collapsing of C to a point represents the required α .

POINCARÉ EMBEDDINGS AND THE BROWDER–WALL EMBEDDING THEOREM

Let us formulate the notion of the Poincaré embedding and the general Browder–Wall embedding theorem. For an embedding $f: N \rightarrow M$ of a smooth manifold N into a smooth manifold M , define $C(f)$, $\nu(f)$, and κ as above. To each pair (f, κ) , there corresponds the inclusion $A(f, \kappa): S\nu(f) \rightarrow C(f)$ and the identity homotopy equivalence $h(f, \kappa): D\nu(f) \cup_A C(f) \rightarrow N$. The *Poincaré embedding* of the pair (f, κ) is the quadruple

$$\mathbb{P}(f, \kappa) = (\nu(f), C(f), A(f, \kappa), h(f, \kappa)).$$

In general, a *Poincaré embedding* of a manifold N into a manifold M is a quadruple $\mathbb{P} = (\nu, C, A, h)$ consisting of

- (i) an S^k -bundle ν over N ,
- (ii) a polyhedron C ,
- (iii) an inclusion $A: S\nu \rightarrow C$, and
- (iv) a homotopy equivalence $h: D\nu \cup_A C \rightarrow M$.

Two Poincaré embeddings $\mathbb{P} = (\nu, C, A, h)$ and $\mathbb{P}_1 = (\nu_1, C_1, A_1, h_1)$ are said to be *equivalent* if there exist a bundle equivalence $b: \nu \rightarrow \nu_1$ and a homotopy equivalence $r: C \rightarrow C_1$ such that $r \circ A \simeq A_1 \circ b$ and $h \simeq h_1 \circ f_{br}$, where the homotopy equivalence $f_{br}: D\nu \cup_A C \rightarrow D\nu_1 \cup_{A_1} C_1$ is constructed from b and r in an obvious fashion. Clearly, the equivalence class of $\mathbb{P}(f, \kappa)$ is invariant under the isotopy of (f, κ) . The Poincaré embedding $\mathbb{P}(f, \kappa)$ can also be defined in locally flat PL and TOP categories [34]. We conclude with the formulation of the following theorem [34, § 11; 20, § 10] (cf. [6, Theorem 2]):

Theorem 9 (Browder–Wall). *Suppose that N and M are closed n - and $(n + k)$ -manifolds, where $k \geq 3$.*

- (a) *If there exists a Poincaré embedding \mathbb{P} of N in M , then there exists an embedding $f: N \rightarrow M$ such that $\mathbb{P}(f) \sim \mathbb{P}$ ($2k + 3 \geq n$ in the DIFF case).*
- (b) *If $f, g: N \rightarrow M$ are embeddings such that $\mathbb{P}(f) \sim \mathbb{P}(g)$, then f and g are isotopic ($2k + 4 \geq n$ in the DIFF case).*

PROOF OF THE COMPRESSION AND DESUSPENSION LEMMA 7

Since $\Sigma S \sim S(F)$, it follows that there is a bundle equivalence $b: \nu \oplus 1 \rightarrow \nu(F)$ and a homotopy equivalence $r: \Sigma C \rightarrow C(F)$ such that $r \circ p' \simeq p_F \circ Sb$.

First, we construct an n -manifold $N' \subset S^{n+k}$ and a map $g: N' \rightarrow N$ of degree 1 such that $\nu' := \nu_{S^{n+k}}(N') = g^*\nu$.

Since $\Sigma S \sim S(F)$, it follows that $D(\nu \oplus 1) \cup_{p'} \Sigma C \simeq S^{n+k+1}$. We have

$$D(\nu \oplus 1) \bigcup_{p'} \Sigma C \simeq \Sigma D\nu \bigcup_{\Sigma p} \Sigma C \simeq \Sigma \left(D\nu \bigcup_p C \right),$$

so that

$$H_{i-1}(D\nu \cup_p C; \mathbb{Z}) \cong H_i(D(\nu \oplus 1) \cup_{p'} \Sigma C; \mathbb{Z}) \cong H_i(S^{n+k+1}; \mathbb{Z}).$$

Since $D\nu$, C , and $S\nu$ are simply connected, using the Seifert–van Kampen theorem, one can show that $D\nu \cup_p C$ is simply connected. Therefore, $D\nu \cup_p C \simeq S^{n+k}$. Let $g: S^{n+k} \rightarrow D(\nu) \cup_p C$ be a homotopy equivalence that is transverse regular on N . Then, $N' = g^{-1}(N) \subset S^{n+k}$ and $g|_{N'}$ are the required objects (see figure).

Second, we construct a cobordism

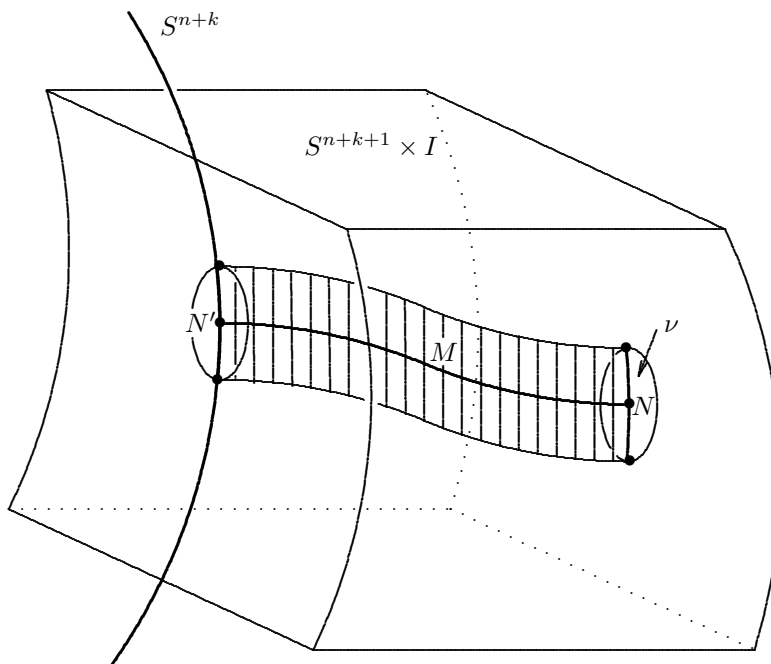
$$M \subset S^{n+k+1} \times I \quad \text{between} \quad N \subset S^{n+k+1} \times 0 \quad \text{and} \quad N' \subset S^{n+k} \times 1 \subset S^{n+k+1} \times 1$$

and a map

$$G: M \rightarrow N \quad \text{such that} \quad G|_N = \text{id}, \quad G|_{N'} = g, \quad \text{and} \quad \mu := \nu_{S^{n+k+1} \times I}(M) = G^*\nu \oplus 1.$$

In order to construct such a cobordism, consider the following diagram:

$$\begin{array}{ccccc} S^{n+k+1} & \xrightarrow{\quad \Sigma g \quad} & \Sigma(D\nu \cup_p C) & & \\ \downarrow \eta' & & \downarrow c & & \\ T(\nu' \oplus 1) & \xrightarrow{\quad (g_* \oplus 1)_T \quad} & T(\nu \oplus 1) & \xrightarrow{\quad b_T \quad} & T\nu(F) \end{array}$$



Figure

where $g: S^{n+k} \rightarrow D\nu \cup_p C$ is the homotopy equivalence constructed at the first step, η' is the collapsing map (representing a normal invariant of N'), and c is the collapsing of ΣC to a point. Clearly, the diagram is commutative. One can check that $b_T \circ c \circ \Sigma g \in \alpha(F)$. Then, there is a homotopy $G: S^{n+k+1} \times I \rightarrow T\nu(F)$, transverse regular on N , between $G_0 = b_T \circ (g_* \oplus 1)_T \circ \eta'$ and a map G_1 representing a normal invariant from $\alpha(F)$. Therefore, $G_1^{-1}(N) = N$, and $G_1|_N = \text{id}$ is covered by a bundle isomorphism of $\nu \oplus 1$ (which we assume to be the identity). So, $M := G^{-1}(N)$ is an $(n+1)$ -dimensional cobordism between $G_0^{-1}(N) = N'$ and $G_1^{-1}(N) = N$. From now on, denote $G|_M: M \rightarrow N$ by G . Clearly, M and G are as required.

Claim. *There exists a manifold X , with boundary $\partial X = S\nu$, that is cobordant to*

$$X_1 := (S^{n+k} - \overset{\circ}{D}\nu') \bigcup_{S\nu'=S(\mu|_{N'})} S\mu$$

modulo the boundary, and a homotopy equivalence $R': X \rightarrow C$ whose restriction to $\partial X = S\nu$ is p .

We prove this claim after completing the proof of the lemma. Let

$$W := X \bigcup_{S(\mu|_N)=S\nu} D\nu \quad \text{and} \quad W_1 := X_1 \bigcup_{S(\mu|_N)=S\nu} D\nu = \partial(D^{n+k+1} \cup D\mu).$$

Observe that W contains N with the normal bundle ν . Since W is cobordant to W_1 , it follows that W is null-cobordant. Also, W is mapped by a homotopy equivalence to $D\nu \cup_p C \simeq S^{n+k}$ (this homotopy equivalence is the “union” of R' and diffeomorphism on $D\nu$). The stable normal bundle of W is trivial (because it is induced from the pullback of the trivial normal bundle of S^{n+k} in S^{n+k+1}). The normal invariant of W is also trivial. Hence, by the Novikov theorem, W is a homotopy sphere, which is the boundary of a parallelizable manifold. Therefore, the connected sum $W \# \Sigma$ with the inverse homotopy sphere (away from $D\nu$) is diffeomorphic to S^{n+k} . We have $N \subset D\nu \subset W \# \Sigma$, and the normal system of N in $W \# \Sigma$ is (ν, C, p) . \square

Proof of the claim. We may assume that $p: S\nu \rightarrow C$ is an inclusion. Define a map

$$R: D\mu \cup S^{n+k} \times I \rightarrow D\nu \cup_p C$$

by

$$\begin{cases} R(x, s) = g(x), & (x, s) \in S^{n+k} \times I, \\ R(y) = G_*(y), & y \in D\mu. \end{cases}$$

We have $R^{-1}(N) = (N' \times I) \cup M$; so,

$$R^{-1}(N) \cap W = N \subset M, \quad R|_M = G, \quad \text{and} \quad R|_N = \text{id}.$$

So, the restriction of the above R is a new map $R: X_1 \rightarrow C$ that extends p from $\partial X_1 = S\nu$. The space $S(\mu \oplus 1)$ can be identified with the boundary of a regular neighborhood of M in $S^{n+k+1} \times I$. Hence, $S\mu \subset S(\mu \oplus 1)$ is embedded into $S^{n+k+1} \times I$. Since $S^{n+k} - D\nu'$ is also embedded in S^{n+k+1} (with trivial normal bundle), it follows that the manifold X_1 can be embedded in $S^{n+k+1} \times I \subset S^{n+k+2}$. We have that $\nu_{S^{n+k+1} \times I}(S\mu) = \nu_{S(\mu \oplus 1)}(S\mu) \oplus G^*1$ is trivial. Therefore, the stable normal bundle of X_1 is also trivial.

Let $W_1 = X_1 \cup_{S(\mu|_N)=S\nu} D\nu$. Clearly, W_1 contains N with normal bundle ν . Recall that W_1 is null-cobordant. Since W_1 satisfies the Poincaré duality, $D\nu \cap C = S\nu$ satisfies the Poincaré duality, and $H_{n+k-1}(D\nu) = 0 = H_{n+k-1}(C)$, it follows that $(C, S\nu)$ is a Poincaré pair in dimension $n+k$. If $n+k = \dim X_1$ is odd, there is no obstruction to changing R by surgery to a map $R': X \rightarrow C$ that is a homotopy equivalence extending p . If $n+k = 2l$, then, by making surgery, we may assume that $R_* = R(X_1)_*$ induces an isomorphism in homology up to dimension $l-1$ and an epimorphism in dimension l . There is however a surgery obstruction $\sigma(R(X_1))$ to make $R(X_1)_*$ a monomorphism in dimension l . There is an analogous obstruction $\sigma(R(W_1))$ to changing by surgery $R(W_1): W_1 \rightarrow S^{2l}$ to a homotopy equivalence with $D\nu \cup_p C$. We have a commutative diagram whose vertical arrows are induced by inclusion:

$$\begin{array}{ccccccc} \text{Ker } R(X_1)_* & \longrightarrow & H_l(X_1) & \xrightarrow{R(X_1)_*} & H_l(C) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow & & \downarrow & & \\ \text{Ker } R(W_1)_* & \xrightarrow{=} & H_l(W_1) & \xrightarrow{R(W_1)_*} & H_l(S^{2l}) & \longrightarrow & 0 \end{array}$$

Hence, by the definition of σ [34], we have $\sigma(R(X_1)) = \sigma(R(W_1))$. However, $\sigma(R(W_1)) = 0$ since W_1 is cobordant to $S^{n+k} \simeq D\nu \cup_p C$. Since $R|_{\partial X_1}$ is a diffeomorphism, we can do surgery in the interior of X_1 and leave ∂X_1 and $R|_{\partial X_1}$ unchanged. \square

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