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On Embeddings of Tori in Euclidean Spaces

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Abstract Using the relation between the set of embeddings of tori into Euclidean spaces modulo ambient isotopies and the homotopy groups of Stiefel manifolds, we prove new results on embeddings of tori into Euclidean spaces.

Keywords Embeddings, Knotted tori, Euclidean space, Stiefel manifolds, Homotopy groups of spheres
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1 Introduction

We shall denote by CAT the category PL of piecewise-linear manifolds or the category DIFF of smooth manifolds (all assumed to be finite dimensional).

Let $\mathrm{Emb}^m_{\mathrm{CAT}}(N^n)$ be the set of all CAT embeddings, $N^n \to \mathbb{R}^m$, of a closed (i.e. compact, connected and without boundary) n-dimensional CAT manifold, N^n, into the m-dimensional Euclidean space $\mathbb{R}^m$, modulo an ambient CAT isotopy. Then the standard inclusion, $i : \mathbb{R}^m \hookrightarrow \mathbb{R}^{m+k}$, induces the map

$$i_* : \mathrm{Emb}^m_{\mathrm{CAT}}(N^n) \to \mathrm{Emb}^{m+k}_{\mathrm{CAT}}(N^n).$$

The study of this map is a classical problem in the topology of manifolds (see [1–6] and [7]). The case when $N^{p+q} = S^p \times S^q$ is known to be of particular interest because it sheds light on the general phenomena. The following summarizes the main known results on this problem:

Theorem 1.1 *The inclusion-induced mapping,*

$$i = i^{m,m+k}_{p,q} : \mathrm{Emb}^m_{\mathrm{CAT}}(S^p \times S^q) \to \mathrm{Emb}^{m+k}_{\mathrm{CAT}}(S^p \times S^q) :$$

(1) *is trivial for* $q > p$, $m = 2q$ *and* $k = p + 1$ (*cf.* [6]; *Theorem* 4);

(2) *is trivial for* $m = p + 2q$, $k = 1$, $2 \le p \le q$ *and* q *even* ≥ 4 (*cf.* [7]);

(3) *is non-trivial for* $q \in \{1, 3, 7\}$, $p = q - 1 = k$ *and* $m = 2q + 1$ (*cf.* [1]);

(4) *is non-trivial for* $k = p \le \rho(q) - 1$, *and* $m = 2q + 1$, *where* $\rho(q) = 2^c + 8d$ *for* $q + 1 = (2a + 1) \cdot 2^{4d+c}$, $c \in \{0, 1, 2, 3\}$ (*cf.* [6]); *and*

(5) *is surjective for* $m = p + 2q$, $k = 1$, $2 \le p \le q$ *and* q *odd* (*cf.* [7]).

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The purpose of this note is to obtain a new result on the embeddings of tori in Euclidean spaces (stated below); more precisely—we obtain new information on the above map i_* for the case of knotted tori. This is done by applying recent results from [8], which relate the set of embeddings of a torus in a Euclidean space, modulo ambient isotopies, with the homotopy groups of Stiefel manifolds.

Theorem 1.2 *Assume that*:

(∗) $1 \leq p \leq q$, $m \geq 2p+q+2$ *and* $2m \geq 3q+2p+4$ *or* $2m \geq 3q+3p+4$ *in the* PL *or* DIFF *category, respectively.*

Let π_*^S *denote the stable homotopy groups of spheres.*

(a) *If* $\pi^S_{2q+1-m} = \pi^S_{2q+1+p-m} = 0$, *then the inclusion-induced map,*

$$i^{m,m+1}_{p,q} : \mathrm{Emb}^m_{\mathrm{CAT}}(S^p \times S^q) \longrightarrow \mathrm{Emb}^{m+1}_{\mathrm{CAT}}(S^p \times S^q),$$

is a monomorphism.

(b) *If* $\pi^S_{2q-m} = \pi^S_{2q+p-m} = 0$, *then the inclusion-induced map,*

$$i^{m,m+1}_{p,q} : \mathrm{Emb}^m_{\mathrm{CAT}}(S^p \times S^q) \longrightarrow \mathrm{Emb}^{m+1}_{\mathrm{CAT}}(S^p \times S^q),$$

is an epimorphism.

By [9], the hypotheses of Theorem 1.2.(a) are fulfilled if, e.g.:

(1) $m = 2q-3$, $p = 1$ and $q \geq 10$ or $q \geq 11$ for CAT=PL or CAT=DIFF, respectively.

(2) $m = 2q-3$, $p = 8$ and $q \geq 26$ or $q \geq 34$ for CAT=PL or CAT=DIFF, respectively.

(3) $m = 2q-4$, $p = 7$ and $q \geq 26$ or $q \geq 33$ for CAT=PL or CAT=DIFF, respectively.

By [9], the hypotheses of Theorem 1.2.(b) are fulfilled if, e.g.:

(1) $m = 2q-4$, $p = 1$ and $q \geq 12$ or $q \geq 13$ for CAT=PL or CAT=DIFF, respectively.

(2) $m = 2q-4$, $p = 8$ and $q \geq 28$ or $q \geq 36$ for CAT=PL or CAT=DIFF, respectively.

(3) $m = 2q-5$, $p = 7$ and $q \geq 28$ or $q \geq 35$ for CAT=PL or CAT=DIFF, respectively.

Theorem 1.3 *The inclusion-induced map,*

$$i = i^{p+2q,p+2q+1}_{p,q} : \mathrm{Emb}^{p+2q}_{\mathrm{CAT}}(S^p \times S^q) \to \mathrm{Emb}^{p+2q+1}_{\mathrm{CAT}}(S^p \times S^q),$$

is trivial for q *even,* $p \geq 1$ *and* $p \leq q-2$ *or* $p \leq q-4$ *in the* PL *or* DIFF *category, respectively.*

Theorem 1.3 is the *non-simply connected* version of Theorem 1.1 (2) above (cited from [7]; note that in [7] a more general situation than that of knotted tori was considered). Our proof of Theorem 1.3 also provides a new proof of Theorem 1.1 (2) for $p \leq q-4$.

2 Proof of Theorems

Proposition 2.1 *Under the hypotheses* (∗) *of Theorem* 1.2 *there is a commutative diagram of groups*

$$\begin{array}{ccc}
\mathrm{Emb}^m_{\mathrm{CAT}}(S^p \times S^q) & \longrightarrow & \mathrm{Emb}^{m+k}_{\mathrm{CAT}}(S^p \times S^q) \\
\cong\downarrow & & \downarrow\cong \\
\pi_q(V_{m-q,p+1}) & \longrightarrow & \pi_q(V_{m-q+k,p+1}),
\end{array}$$

where the lower horizontal map is induced by the natural inclusion between the Stiefel manifolds.

Proof For a manifold N let $\tilde{N} = (N \times N) \setminus \Delta$, where Δ is the diagonal of the product. Then there is a natural $\mathbb{Z}_2$ action on $\tilde{N}$. Let $\pi^m_{eq}(\tilde{N})$ denote the set of all equivariant homotopy classes of the equivariant maps $\tilde{N} \to S^m$. Under the assumptions $(*)$ of Theorem 1.2 there are one-to-one correspondences (cf. [8, 10, 11]):

$$\mathrm{Emb}^m_{\mathrm{CAT}}(S^p \times S^q) \xrightarrow{\cong} \pi^{m-1}_{eq}(\widetilde{S^p \times S^q}) \xrightarrow{\cong} \pi_q(V_{m-q,p+1}),$$

where $V_{k,l}$ denotes the Stiefel manifold of l-frames in $\mathbb{R}^k$ (cf. [12]). Moreover, the first two of the sets above have group structures such that these one-to-one correspondences are isomorphisms (cf. [13]). It is easy to see that the isomorphisms above (under the condition $(*)$) are natural with respect to the inclusions $\mathbb{R}^m \hookrightarrow \mathbb{R}^{m+k}$. This implies the assertion.

Proof of Theorem 1.3 It is easy to check that the conditions $(*)$ are fulfilled, so we can apply Proposition 2.1. Since q is even, $m = 2q+p$ and $p \geq 1$, by [14] it follows that $\pi_q(V_{m-q+1,p+1}) \cong \mathbb{Z}$ and that $\pi_q(V_{m-q,p+1})$ is finite (because $q-1 \neq 2$ for $p+1 = 2$). Thus Theorem 1.3 follows by Proposition 2.1.

Proof of Theorem 1.2 The inclusion $i : V_{m-q,p+1} \hookrightarrow V_{m-q+1,p+1}$ is homotopic to the map which preserves the first p vectors of the $(p+1)$-frame and maps the last vector of the frame to $(0,\ldots,0,1) \in \mathbb{R}^{m-q+1}$. Therefore one obtains the following commutative diagram:

$$\begin{tikzcd}
\pi_q(S^{m-q-p-1}) \arrow[dr] & & & & \pi_q(S^{m-q}) \\
& \pi_q(V_{m-q,p+1}) \arrow[rr] \arrow[dr] & & \pi_q(V_{m-q+1,p+1}) \arrow[ur] & \\
& & \pi_q(V_{m-q,p}) \arrow[ur] \arrow[dr] & & \\
& \pi_{q+1}(S^{m-q}) \arrow[ur] & & \pi_{q-1}(S^{m-q-p-1}) &
\end{tikzcd}$$

Here the NW-SE sequence and the SW-NE sequence are parts of the respective fibration sequences.

Since $2m \geq 3q + 2p + 4$, it follows that all the homotopy groups of spheres in the diagram are stable. Therefore under the assumption of (a) the homomorphism $i_* : \pi_q(V_{m-q,p+1}) \to \pi_q(V_{m-q+1,p+1})$ is a monomorphism, while under the assumption of (b) the homomorphism $i_* : \pi_q(V_{m-q,p+1}) \to \pi_q(V_{m-q+1,p+1})$ is an epimorphism. Hence Theorem 1.2 follows by Proposition 2.1.

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