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On the Alexandroff–Borsuk problem

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A B S T R A C T

We investigate the classical Alexandroff–Borsuk problem in the category of non-triangulable manifolds: Given an n-dimensional compact non-triangulable manifold M^n and $\varepsilon > 0$, does there exist an ε-map of M^n onto an n-dimensional finite polyhedron which induces a homotopy equivalence?

1. Introduction

In 1928 Alexandroff [1] proved the following important theorem:

Alexandroff Theorem. *Every n-dimensional compact metric space X has the following properties:*

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- *for every $\varepsilon > 0$, X admits an ε-map $f : X^n \to P^n$ onto some n-dimensional finite polyhedron P^n; and*
- *for some $\mu > 0$, X does not admit any μ-map $g : X^n \to Q^k$ of X^n onto a polyhedron Q^k of dimension $k < n$.*

Recall that for a metric space X and $\varepsilon > 0$, a continuous map $f : X \to P$ is called an ε-map if the preimage $f^{-1}(p)$ of every point $p \in P$ has diameter $< \varepsilon$.

The condition of compactness in Alexandroff's theorem above is essential since in 1953 Sitnikov [8,25] constructed an example of a 2-dimensional subspace of R^3 which can be ε-mapped onto a 1-dimensional polyhedron for arbitrarily small $\varepsilon > 0$.

In 1954 Borsuk [2] asked whether every compact absolute neighborhood retract (ANR) is homotopy equivalent to a finite polyhedron. This difficult question was answered in the affirmative in 1977 by West [27].

It follows by Wall's obstruction theory [26] that for every $n > 2$, every n-dimensional compact ANR is homotopy equivalent to an n-dimensional polyhedron, having the structure of a finite simplicial complex (cf. [27]).

The following natural problem has been opened for a very long time (compare with [17]):

Alexandroff–Borsuk ANR Problem. *Given any compact n-dimensional absolute neighborhood retract X^n and any $\varepsilon > 0$, does there exist an ε-map $f : X^n \to P^n$ of X^n onto a finite n-dimensional polyhedron P^n which is a homotopy equivalence?*

In this paper we shall consider the Alexandroff–Borsuk problem for the category of non-triangulable manifolds. Since every topological n-manifold is a separable metric locally Euclidean space of dimension n (cf. [19]), it is a locally contractible finite-dimensional space and therefore it is an ANR (cf. [3]).

It follows by the West Theorem mentioned above that every compact manifold has the homotopy type of a finite polyhedron. This fact was first proved in 1969 by Kirby and Siebenmann [18]. So we have the following natural special case of the Alexandroff–Borsuk problem:

Alexandroff–Borsuk Manifold Problem. *Given a compact n-dimensional manifold M^n and $\varepsilon > 0$, does there exist an ε-map of M^n onto a finite n-dimensional polyhedron P^n which is a homotopy equivalence?*

Recall that for every $n \geq 4$, there exists a closed n-dimensional manifold which is not a polyhedron. Such manifolds were first constructed by Freedman [10] in dimension 4, by Galewski and Stern [12] in dimension 5, and by Manolescu [20] in dimensions $n \geq 6$.

The following is the main result of our paper:

Main Theorem. *The Alexandroff–Borsuk Manifold Problem has an affirmative solution for the non-triangulable closed manifolds of Freedman, Galewski and Stern, and Manolescu.*

2. Preliminaries

We shall work with the categories of separable metric spaces, CW complexes and continuous maps. In these categories all classical definitions of dimension coincide: $\dim X = \operatorname{ind} X = \operatorname{Ind} X$ (cf. [8]).

We list some definitions and theorems which we shall need in the sequel:

Theorem 2.1. (Cellular Approximation Theorem [28, p. 77]). *Let (X, A) and (Y, B) be relative CW complexes and let $f : (X, A) \to (Y, B)$ be a continuous map. Then f is homotopic (rel A) to a cellular map.*

Recall that a map $f\colon X \to Y$ between CW complexes X and Y is said to be *cellular* if $f(X_n) \subset Y_n$ for every n. The Simplicial Approximation Theorem is a special case of the Cellular Approximation Theorem, cf. [28, p. 76] for details.

Definition 2.2. A manifold M is said to be almost-smooth if it admits a smooth structure in the complement $M \setminus \{p\}$ of any point $p \in M$.

Theorem 2.3. (Cairns–Whitehead [4,5,16,29]). *Every smooth manifold can be given a simplicial structure.*

Freedman has proved the following important theorem (cf. [10, Corollary 1.6]):

Theorem 2.4. *There is a closed connected almost-smooth 4-manifold $|E_8|$ with the intersection matrix E_8.*

Freedman also established the following surprising fact:

Theorem 2.5. *Either $|E_8|$ is the first example in any dimension of a manifold which is not homeomorphic to a polyhedron, or the 3-dimensional Poincaré conjecture is false.*

Since Perel'man has proved the Poincaré conjecture (cf. e.g. [21]), it follows that the 4-manifold $|E_8|$ is not homeomorphic to any polyhedron. However, the complement of any point in $|E_8|$ can be given a polyhedral structure, since it admits a smooth structure on the complement of any point (cf. [11, Theorem on p. 116]) and by the Cairns–Whitehead Theorem 2.3 every smooth manifold is triangulable.

We briefly recall the Kirby–Siebenmann and the Galewski–Stern obstructions and some of their theorems (cf. [13,22]).

Definition 2.6. (cf. [22, pp. xii–xiii, 78]). $BTOP$ is the classifying space for stable topological bundles and χ is some element of the 4-dimensional cohomology group $H^4(BTOP;\mathbb{Z}_2)$ which is called the universal Kirby–Siebenmann class.

The construction of the CW complex $BTOP$ and the element χ is given e.g. in [22, pp. xii–xiii, 78].

Definition 2.7. (cf. e.g. [9, pp. 403–404]). Let M^n be a topological manifold. Then the topological tangent bundle τ_M^n of M^n is a neighborhood U of the diagonal $\Delta \subset M^n \times M^n$ such that the projection $p_1\colon U \to M^n$ is a topological $\mathbb{R}^n$-bundle. Here, $p_1(x,y) = x$.

Let M be a topological manifold, and let $f\colon M \to BTOP$ classify the stable tangent bundle of M. The class

$$f^4(\chi) \in H^4(M;\mathbb{Z}_2)$$

is a well-defined invariant of M since f is unique up to homotopy. The Kirby–Siebenmann class $\Delta(M)$ of M is by definition the element $f^4(\chi)$.

If $U \subset M$, $i\colon U \to M$ is the inclusion, and f_M, f_U classify the corresponding stable tangent bundles then $f_M \circ i \simeq f_U$ and we have $i^4(\Delta(M)) = \Delta(U)$.

Theorem 2.8. (cf. e.g. [22, p. 78]). *Let M be a topological manifold. If M admits a PL structure, in particular if it admits a smooth structure, then $\Delta(M) = 0$. For dim $M \geq 5$ the converse also holds: if $\Delta(M) = 0$, then M admits a PL structure.*

Theorem 2.9. (cf. [7,13,23]). *Let M^n be a topological manifold, $n \geq 5$. Then the Galewski–Stern obstruction for a manifold M^n to having a simplicial triangulation is the image*

$$\beta(\Delta(M^n)) \in H^5(M^n; \text{Ker } \mu)$$

of the Kirby–Siebenmann class $\Delta(M^n)$ by the Bockstein homomorphism β for the exact sequence of coefficients:

$$0 \longrightarrow \text{Ker } \mu \longrightarrow \theta_3^H \xrightarrow{\mu} \mathbb{Z}_2 \longrightarrow 0.$$

The homomorphism $\mu : \theta_3^H \to \mathbb{Z}_2$ is the Rokhlin invariant homomorphism of the abelian group θ_3^H of homology cobordism classes of oriented PL homology 3-spheres with the operation of connected sum, cf. e.g. [7,23]. For the purposes of our paper it suffices to know that Ker μ is some nontrivial abelian group.

Theorem 2.10. (cf. [15, for $n = 3$], [9, for $n = 4$], and [6, for $n \geq 5$]). *Let α be an open cover of an n-manifold M. Then there exists an open cover β of M such that any proper β-map $g\colon M \to N$ onto any n-manifold N is homotopic through α-maps to a homeomorphism.*

For uniformity we denote the manifolds of Freedman, Galewski and Stern, and Manolescu by M^4_F, M^5_{GS} and M^{5+n}_M, respectively.

3. Proof of the Main Theorem

First consider the manifold M^4_F. According to Theorems 2.3 and 2.4, the complement $M^4_F \setminus \{p\}$ of a point $p \in M^4_F$ is a polyhedron. Let $\varepsilon > 0$ be any positive number. Let $U \subset M^4_F$ be an open topological ball in M^4_F with center at p and with diameter less than ε. Consider a triangulation T of $M^4_F \setminus \{p\}$ and let K be the polyhedron which is the union of all simplices of T which intersect with $M^4_F \setminus U$ and let L be the compactum which is the union of all the remaining simplices in U and the point p.

Obviously, L is a locally contractible 4-dimensional space and therefore by the Borsuk theorem [3] it is a compact ANR. By the above mentioned theorems of West and Wall it follows that L is homotopy equivalent to a finite 4-dimensional polyhedron, call it B. We have a homotopy equivalence $f\colon L \to B$. Consider the closed star $st(f(p))$ of the point $f(p)$ in the finite polyhedron B, that is the union of all simplices of B containing the point $f(p)$. This is a closed neighborhood of the point p. Consider the preimage of $st(f(p))$. We get a closed neighborhood of the point p in L.

Consider the second barycentric subdivision of $st(f(p))$ and the closed star of the point $f(p)$ in it. Call it $st_2(f(p))$. Obviously,

$$f^{-1}(st_2(f(p))) \subset \text{Int } f^{-1}(st(f(p))).$$

Let d be the minimal distance between the points of the boundary $\partial(f^{-1}(st(f(p))))$ and the points of compactum $f^{-1}(st_2(f(p)))$. Clearly, $d > 0$.

Consider small triangulation of T such that the diameters of all of its simplices are less than d and let Q be the union of all simplices of this triangulation which intersect $f^{-1}(st_2(f(p)))$ and the point p. We get a relative CW complex (L, Q) and the mapping

$$f\colon (L, Q) \to (B, st(f(p))).$$

By the Cellular Approximation Theorem 2.1, the map f is homotopy equivalent to a map g such that its restriction to $L \cap K$ is a simplicial map (in our case the relative CW complexes are obviously relative simplicial complexes).

Since M^4_F and L are ANR's, the pair (M^4_F, L) has the homotopy extension property with respect to any space [14, p. 120], [28]. Therefore M^4_F is homotopy equivalent to the quotient space $M^4_F \cup_g B$, cf. [28, p. 26, Corollary (5.12)].

The space $M^4_F \cup_g B$ is the union of two finite polyhedra intersection of which is $g(L \cap K)$, a subpolyhedron of both of these polyhedra and therefore $M^4_F \cup_g B$ is a finite polyhedron. Obviously, the quotient map of M^4_F onto $M^4_F \cup_g B$ is an ε-map.

Consider now the Galewski–Stern manifold M^5_{GS}. The obstruction to triangulability of any manifold M^n for $n \geq 5$ is the obstruction

$$\beta(\Delta)(M^n) \in H^5(M^n; \text{Ker } \mu)$$

(cf. Theorem 2.9). Manolescu [20] has showed that $\beta(\Delta)(M^5_{GS})$ is nontrivial and therefore the manifold M^5_{GS} is non-triangulable.

However, for any connected closed 5-dimensional manifold,

$$H^5(M^5 \setminus \{p\}; \text{Ker } \mu) = 0$$

and therefore the Galewski–Stern obstruction $\beta(\Delta)(M^5_{GS} \setminus \{p\})$ is trivial and $M^5_{GS} \setminus \{p\}$ is an infinite polyhedron. Using the arguments similar to those used in the proof of the theorem for the Freedman manifold it follows that for every $\varepsilon > 0$ there exists an ε-map of M^5_{GS} onto a 5-dimensional polyhedron P^5 which is a homotopy equivalence.

The Manolescu non-triangulable manifold M^{5+n}_M is the product of M^5_{GS} with the torus T^n. Since M^5_{GS} admits an ε-map onto P^5 which is a homotopy equivalence for an arbitrarily small $\varepsilon > 0$, it follows that obviously,

$$M^{5+n}_M = M^5_{GS} \times T^n$$

also admits an ε-map which is a homotopy equivalence onto $P^5 \times T^n$ for arbitrarily small $\varepsilon > 0$.

4. Some complementary results and remarks

Theorem 4.1. *There do not exist any non-triangulable almost-smooth manifolds M^n for any $n > 4$.*

Proof. Since M^n is non-triangulable it does not have a PL structure and therefore the Kirby–Siebenmann obstruction

$$\Delta(M^n) \in H^4(M^n; Z_2)$$

is nonzero, by Theorem 2.8. It follows from the exact sequence of the pair $(M^n, M^n \setminus \{p\})$:

$$0 = H^4(M^n, M^n \setminus \{p\}; \mathbb{Z}_2) \to H^4(M^n; \mathbb{Z}_2) \to H^4(M^n \setminus \{p\}; \mathbb{Z}_2)$$

that $\Delta(M^n \setminus \{p\})$ is nonzero for $n > 4$. Therefore $M^n \setminus \{p\}$ does not have a PL structure and is thus not smooth. □

Theorem 4.2. *A non-triangulable connected manifold of dimension $n > 4$ has an infinite simplicial complex structure in the complement of a point if and only if $n = 5$.*

Proof. The obstruction class $\beta(\Delta)(M \setminus \{p\})$ is obtained as the image of $\beta(\Delta)(M)$ by the restriction

$$H^5(M; \text{Ker } \mu) \to H^5(M \setminus \{p\}; \text{Ker } \mu).$$

When $n = 5$, we have

$$H^5(M \setminus \{p\}; \text{Ker } \mu) = 0$$

since M is a connected manifold, hence the obstruction vanishes. When $n > 5$, we see that by the exact sequence of the pair $(M, M \setminus \{p\})$ we get

$$0 = H^5(M, M \setminus \{p\}; \text{Ker } \mu) \to H^5(M; \text{Ker } \mu) \to H^5(M \setminus \{p\}; \text{Ker } \mu).$$

The restriction is injective, so the image of $\beta(\Delta)(M)$ is non-zero and $M \setminus \{p\}$ is non-triangulable as simplicial complex, by Theorem 2.9. □

Remark 4.3. For non-triangulable manifolds there do not exist ε-maps onto a triangulable manifold for small enough $\varepsilon > 0$. This follows from deep results of Chapman and Ferry, Ferry and Weinberger, and Jakobsche (cf. Theorem 2.10).

5. Epilogue

The non-triangulable closed 4-manifolds of Freedman [10,24], 5-manifolds of Galewski and Stern [7,12], and n-manifolds of Manolescu for $n \geq 6$ [20, p. 148] have nice geometric descriptions. All of them are homotopy equivalent to polyhedra of the corresponding dimension. We have polyhedral homotopy representatives PH^4_F, PH^5_{GS}, PH^{5+n}_M of the above mentioned non-triangulable manifolds.

Problem 5.1. *Find a geometric description of the polyhedra PH^4_F, PH^5_{GS} and PH^{5+n}_M.*

The Alexandroff–Borsuk problem is solved for the special class of non-triangulable manifolds and is still open for general non-triangulable manifolds. The following problems seem to be of interest:

Problem 5.2. *Let M^n be any non-triangulable manifold. Does there exist any polyhedron P embeddable in M^n, such that $M^n \setminus P$ is also a polyhedron?*

According to our Main Theorem, for every positive number ε and for the manifolds of Freedman, Galewski and Stern, and Manolescu there exist ε-maps of these manifolds onto some polyhedra PH^4_F, PH^5_{GS}, PH^{5+n}_M, respectively. The following version of the Alexandroff–Borsuk Manifold Problem remains open:

Problem 5.3. *Does there exist for every compact n-dimensional manifold M^n, a finite n-dimensional polyhedron P^n such that for an arbitrarily small $\varepsilon > 0$ there exists an ε-map $f : M^n \to P^n$ which is a homotopy equivalence?*

The answer for the corresponding version of Alexandroff–Borsuk ANR Problem is negative, even for 1-dimensional compact absolute retracts (AR), i.e. for the dendrites.

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