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Recent Results on Topology of Three-Manifolds.

To the memory of Mario Pezzana

Sunto. – *In questo articolo si espongono numerosi risultati, ottenuti dagli autori e dai loro collaboratori, sulla classificazione delle strutture topologiche e geometriche di varietà tridimensionali. In particolare, si descrivono varie famiglie notevoli di 3-varietà iperboliche e si ottengono loro rappresentazioni combinatorie come rivestimenti ramificati della 3-sfera e mediante chirurgia su nodi e links con coefficienti. Infine, si determinano i gruppi di isometrie, i volumi e i principali invarianti algebrici (gruppo fondamentale, omologia, genere di Heegaard etc.) delle suddette famiglie, e si fornisce una panoramica su alcuni recenti risultati riguardanti la costruzione di 3-varietà compatte non omeomorfe che possiedono spine isomorfe.*

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1. - Hyperbolic 3-manifolds.

This survey paper covers various areas of topology of 3-manifolds with emphasis on those problems which were studied by the authors and their coauthors in recent years. We present key results and state several open problems on the following subjects: topology and geometry of hyperbolic 3-manifolds, manifolds with cyclic symmetry, manifolds obtained by Dehn surgery and knot spaces.

Let \mathbb{H}^3 be the three-dimensional Lobachevsky (or, hyperbolic) space, and let $\Gamma < \text{Isom}(\mathbb{H}^3)$ be a discrete group of isometries acting without fixed points. The quotient space $M^3 = \mathbb{H}^3/\Gamma$ is said to be a *hyperbolic 3-manifold* (or in other terminology, a Clifford-Klein space-form). The question about the existence of hyperbolic Clifford-Klein space-forms was formulated in 1880 by Killing. The first example of a *noncompact nonorientable* hyperbolic 3-manifold was obtained in 1912 by Gieseking (see [42], pp. 153-156), from the ideal (with all vertices at infinity) regular tetrahedron in \mathbb{H}^3 with all dihedral angles equal to $\pi/3$. The first example of a *compact orientable* hyperbolic 3-manifold was obtained in 1931 by Löbell [40] by gluing eight copies of the right-angled polyhedron in \mathbb{H}^3 with 14 faces, similar to the dodecahedron (it has hexagonal top and bottom and twelve pentagons on the lateral surface; see Figure 1.1).

Further examples were obtained by Seifert and Weber [53], Best [5], and others. Recall that by the Mostow Rigidity theorem (see for example [4], Ch. C) if M_1 and M_2 are finite volume complete connected hyperbolic n -manifolds, $n \geq 3$, and if there is a group isomorphism from $\pi_1(M_1)$ to $\pi_1(M_2)$, then there exists an isometry from M_1 to M_2 . Hence the *hyperbolic volume* is a good topological invariant for hyperbolic 3-manifolds.

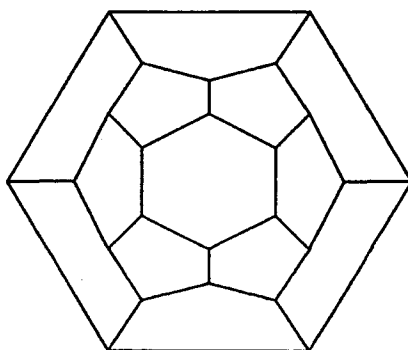


Figure 1.1. - The Löbell polyhedron.

In the paper, we will always consider hyperbolic manifolds of finite volume.

We recall that by the Gauss–Bonnet formula, the volume of an even-dimensional manifold is given by

$$\text{vol}_{2m}(M^{2m}) = (-1)^m \frac{1}{2} \text{vol}_{2m}(S^{2m}) \chi(M^{2m}).$$

Therefore, the set of volumes of hyperbolic manifolds of a given even dimension is discrete. Some calculations of volumes of hyperbolic manifolds in higher dimensions were done by Kellerhals in [35].

The structure of the set of volumes of hyperbolic 3-manifolds is described by the Thurston–Jørgensen theorem (see, for example, [4], Ch. E, and [28]):

THEOREM 1.1. *The set of volumes of all hyperbolic 3-manifolds (with finite volume) is a closed non-discrete well-ordered subset of the real line, whose ordinal type is ω^ω . Moreover, the function $M \rightarrow \text{vol}(M)$ is finite-to-one, i.e., there are only finitely many distinct hyperbolic 3-manifolds with a given volume.*

In particular, the following questions arise in a natural way:

1.1) It is known that for any given value v_0 , the set of manifolds whose volume equals v_0 , is finite. Can this finite number be arbitrarily large?

1.2) Volumes of noncompact manifolds correspond to limit ordinals. It was asked in [56] whether there exist a compact and a noncompact manifold with the same volume?

1.3) It follows from Theorem 1.1 that there exists a hyperbolic 3-manifold with the smallest volume. What are the smallest manifold and the initial part of the set of volumes of hyperbolic 3-manifolds?

1.4) As seen above, hyperbolic 3-manifolds can be ordered by their volumes. Is it possible to estimate the volume of a hyperbolic 3-manifold and the order of its isometry group in terms of its Heegaard genus?

Hereafter, we will discuss answers to the questions above.

First, we present some properties of volumes and, in particular, some results on the number of hyperbolic 3-manifolds of the same volume.

For the case of noncompact manifolds, the affirmative answer to question 1.1 was given by Wielenberg [61] and Adams [1]: for any natural number N there exist at least N pairwise nonhomeomorphic *noncompact* hyperbolic 3-manifolds with equal volumes. For the case of compact manifolds, the affirmative answer to question 1.1 was also given, by examples constructed by Apanasov and Gutsul [3], by Zimmermann [62], and by one of the authors [58]: for any natural number N there exist at least N pairwise nonhomeomorphic *compact* hyperbolic 3-manifolds with equal volumes.

We now give the sketch of examples from [58]. The construction is based on the following four-coloring approach [57]. Let R be a right-angled polyhedron in \mathbb{H}^3 . Then each 4-coloring of its faces defines an orientable hyperbolic 3-manifold constructed from eight copies of R . Each 5-, 6-, or 7-coloring defines a nonorientable manifold. In [57] infinite series of compact orientable and compact nonorientable manifolds \mathcal{L}_n , $n \geq 5$, were constructed. These manifolds are generalizations of the first Löbell example constructed in 1931. Each of manifolds \mathcal{L}_n is obtained from eight copies of a right-angled $(2n + 2)$ -hedron in \mathbb{H}^3 with n -gonal top and bottom and $2n$ pentagons on the lateral surface similar to the polyhedron shown in Figure 1.1. The manifolds \mathcal{L}_n are referred to as *Löbell manifolds*. The Löbell example, constructed in [40], corresponds to case $n=6$.

The Löbell manifolds give examples which answer in affirmative question 1.1. More precisely, we have the following result [58].

THEOREM 1.2. *For any natural number N , there exists a right-angle polyhedron in the hyperbolic 3-space \mathbb{H}^3 which is the fundamental polyhedron for at least N pairwise nonhomeomorphic closed hyperbolic 3-manifolds.*

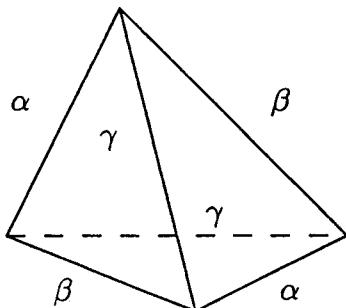
Recall that volumes of hyperbolic polyhedra and hyperbolic manifolds can be expressed in terms of the *Lobachevsky function*:

$$A(x) = - \int_0^x \ln |2 \sin \zeta| d\zeta .$$

For example, let $T(\alpha, \beta, \gamma)$ be an ideal (with all vertices at infinity) tetrahedron with dihedral angles α , β , and γ such that $\alpha + \beta + \gamma = \pi$ (see Figure 1.2).

Then we have

$$\text{vol } T(\alpha, \beta, \gamma) = A(\alpha) + A(\beta) + A(\gamma) .$$

Figure 1.2. - The tetrahedron $T(\alpha, \beta, \gamma)$.

The maximal possible volume of a tetrahedron in \mathbb{H}^3 is

$$v_0 = \text{vol } T(\pi/3, \pi/3, \pi/3) = 3\mathcal{A}(\pi/3) = 1.014 \dots$$

For practical calculations, it is convenient to use the following series

$$\mathcal{A}(x) = x \left(1 - \ln|2x| + \sum \frac{B_n(2x)^{2n}}{2n(2n+1)} \right)$$

which converges for $|x| \leq \pi$ and where B_n is the n -th Bernoulli number.

The volumes of Löbell manifolds were calculated in [59].

THEOREM 1.3. *For any natural number N , there exists n such that at least N pairwise nonhomeomorphic closed hyperbolic 3-manifolds have the volume:*

$$\text{vol}_n = 4n \left(2\mathcal{A}(\zeta) + \mathcal{A}\left(\zeta + \frac{\pi}{n}\right) + \mathcal{A}\left(\zeta - \frac{\pi}{n}\right) - \mathcal{A}\left(2\zeta - \frac{\pi}{2}\right) \right),$$

where $\zeta = \frac{\pi}{2} - \arccos \frac{1}{2 \cos(\pi/n)}$.

2. - Manifolds with cyclic symmetry.

Let F_n be the free group on free generators x_0, \dots, x_{n-1} . Let $\theta : F_n \rightarrow F_n$ be the automorphism defined by $\theta(x_i) = x_{i+1}$ (where the indices are taken mod n).

For any reduced word $w = w(x_0, \dots, x_{n-1})$ in F_n , let us consider the factor group $G_n(w) = F_n/R$, where R is the normal closure in F_n of the set

$$\{w, \theta(w), \dots, \theta^{n-1}(w)\}.$$

A group G is said to have a *cyclic presentation* if G is isomorphic to $G_n(w)$ for some w and n . Of course, θ induces an automorphism of $G_n(w)$ which determines an action of the cyclic group $Z_n = \langle \theta : \theta^n = 1 \rangle$ on $G_n(w)$.

The *split extension group* of $G_n(w)$ by Z_n will be denoted by $H_n(v)$. It has a 2-generator presentation of the following form

$$H_n(v) = \langle \theta, x : \theta^n = 1, v(\theta, x) = 1 \rangle,$$

where

$$v(\theta, x) = w(x, \theta^{-1}x\theta, \dots, \theta^{-(n-1)}x\theta^{n-1}).$$

The *polynomial* associated with $G_n(w)$ is defined to be

$$f_w(t) = \sum_{i=0}^{n-1} a_i t^i,$$

where a_i is the exponent sum of x_i in w .

EXAMPLE 2.1. Sieradski groups [54] are defined by the following cyclic presentations

$$S(n) = \langle x_0, \dots, x_{n-1} : x_i x_{i+2} = x_{i+1} \text{ (indices mod } n) \rangle.$$

The defining word is $w = x_0 x_2 x_1^{-1}$, and the polynomial associated with this presentation is $f_w(t) = 1 - t + t^2$. So the split extension group is presented by

$$H_n(v) = \langle \theta, x : \theta^n = 1, x\theta^{-2}x\theta x^{-1}\theta = 1 \rangle.$$

Setting $x = \theta\lambda^{-1}$ yields

$$\begin{aligned} H_n(v) &= \langle \theta, \lambda : \theta^n = 1, \theta\lambda^{-1}\theta^{-1}\lambda^{-1}\theta\lambda = 1 \rangle \\ &= \langle \theta, \lambda : \theta^n = 1, \theta\lambda\theta = \lambda\theta\lambda \rangle. \end{aligned}$$

Since λ is conjugate with θ (use the second relation), it follows that $\lambda^n = 1$. Thus we have

$$H_n(v) = \langle \theta, \lambda : \theta^n = \lambda^n = 1, \theta\lambda\theta = \lambda\theta\lambda \rangle.$$

We remark that the group presented by $\langle \theta, \lambda : \theta\lambda\theta = \lambda\theta\lambda \rangle$ is the funda-

mental group of the trefoil knot, and the generator θ represents a meridian of it. Hence $H_n(v)$ is the fundamental group of the orbifold $\mathcal{O}_n(3_1)$ whose underlying space is the standard 3-sphere S^3 , and whose singular set is the trefoil knot 3_1 with branching index n . Then the Sieradski group $S(n)$ is the fundamental group of the n -fold cyclic branched covering of the trefoil knot (see [13] and [14] for more details). Finally, the polynomial associated with $S(n)$ is $f_w(t) = 1 - t + t^2$, which coincides with the Alexander polynomial of the trefoil knot.

This example suggests in a natural way the following questions:

2.1) Does $G_n(w)$ (resp. $H_n(v)$) correspond to the fundamental group of some closed orientable 3-manifold (resp. 3-orbifold) $M_n(w)$ (resp. $\mathcal{O}_n(K)$, K being a knot or a link)?

2.2) Which of the manifolds $M_n(w)$ admits a hyperbolic structure? In this case, determine the Heegaard genus, the volume, and the isometry group of $M_n(w)$;

2.3) Are the manifolds $M_n(w)$ homeomorphic to the cyclic branched coverings of some (hyperbolic) knots or links? In the first case, does $f_w(t)$ coincide (up to the sign) with the Alexander polynomial of the knot?

These questions have nice solutions for the cyclic branched coverings of 2-bridge knots [19] [20] (here we are principally interested in hyperbolic case, and refer to [13] for 2-bridge torus knots). A *2-bridge knot* is determined by a pair of coprime integers (α, β) satisfying $0 < \beta < \alpha$, and α odd. Following [7], let us denote by α/β the 2-bridge knot determined by (α, β) . It is well-known that two such knots α/β and α'/β' belong to the same knot type if and only if $\alpha = \alpha'$ and $\beta^{\pm 1} \equiv \beta' \pmod{\alpha}$. In Figure 2.1 we show the trefoil knot $3/1 \approx 3/2$ and the figure-eight knot $5/3 \approx 5/2$ depicted by using the normal Schubert form for 2-bridge knots.

The knot group of α/β has the 2-generator presentation

$$\langle u, v : ww(\alpha, \beta) = w(\alpha, \beta) v \rangle,$$

where

$$w(\alpha, \beta) = v^{\varepsilon_1} u^{\varepsilon_2} \dots u^{\varepsilon_{\alpha-3}} v^{\varepsilon_{\alpha-2}} u^{\varepsilon_{\alpha-1}},$$

and ε_i is the sign (± 1) of $i\beta$ reduced mod 2α in the interval $(-\alpha, \alpha)$. If $1 < \beta < \alpha - 1$, then α/β is *hyperbolic*, i.e. its complement in the 3-sphere S^3 admits a complete hyperbolic structure of finite volume. The condition $1 < \beta < \alpha - 1$ implies that α/β is not a torus knot (see [56]). Let $\mathcal{O}_n(\alpha/\beta)$ be the orbifold whose underlying space is S^3 , and its singular set is α/β

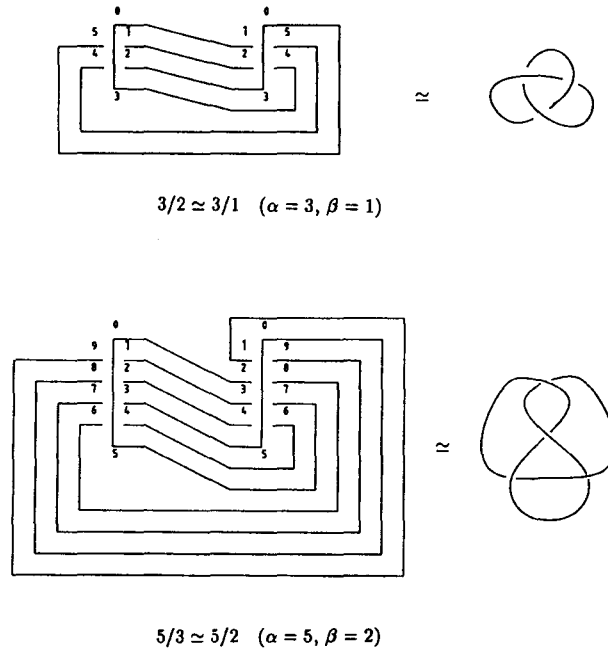


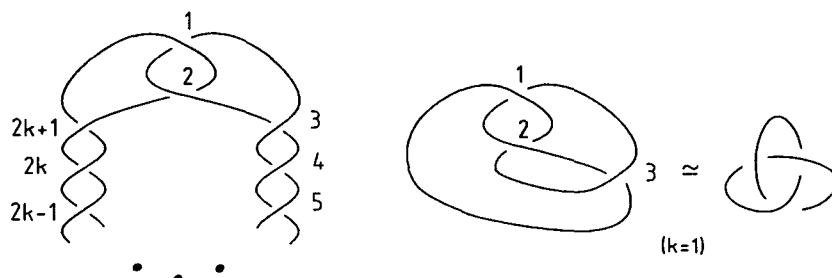
Figure 2.1. - The trefoil knot $3/1$ and the figure-eight knot $5/2$.

with branching index $n \geq 1$. The geometric structures of the orbifolds $\mathcal{O}_n(\alpha/\beta)$ are well-known [56] (see also [24] and [32]). Finding a geometric structure for $\mathcal{O}_n(\alpha/\beta)$ immediately implies that the n -fold cyclic covering $M_n(\alpha/\beta)$ of the 3-sphere branched over α/β has a structure modelled on the same geometry [32].

THEOREM 2.1. *Assume $1 < \beta < \alpha - 1$. The manifolds $M_n(\alpha/\beta)$ are hyperbolic when $\alpha = 5$ and $n \geq 4$ or $\alpha \neq 5$ and $n \geq 3$. Furthermore, $M_2(\alpha/\beta)$ is homeomorphic to the lens space $L(\alpha, \beta)$ for any α , while $M_3(5/3)$ (i.e. the 3-fold cyclic branched covering over the figure-eight knot) is Euclidean.*

The following answers affirmatively the questions above for 2-bridge knots [19].

THEOREM 2.2. *The fundamental group of $M_n(\alpha/\beta)$ admits a cyclic presentation $G_n(\alpha/\beta)$ whose split extension group $H_n(\alpha/\beta)$ is isomorphic to the fundamental group of the orbifold $\mathcal{O}_n(\alpha/\beta)$. Furthermore,*

Figure 2.2. - The 2-bridge knot $(4k-1)/2$.

the polynomial associated with $G_n(\alpha/\beta)$ coincides (up to the sign) with the Alexander polynomial of α/β .

EXAMPLE 2.2. The finite cyclic presentation ($k \geq 1$)

$$\mathcal{P}(n, k) = \langle x_0, \dots, x_{n-1} : x_{i+1}^{-1} x_i x_{i+2} (x_{i+1}^{-1} x_i x_{i+1}^{-1} x_{i+2})^{k-1} = 1 \rangle$$

(indices mod n)

defines the fundamental group of the n -fold cyclic covering of the 3-sphere branched over the 2-bridge knot $\alpha/\beta = (4k-1)/2$, shown in Figure 2.2. For $k=1$ we obtain again the Sieradski groups considered above. The defining word of $\mathcal{P}(n, k)$ is

$$w = x_1^{-1} x_0 x_2 (x_1^{-1} x_0 x_1^{-1} x_2)^{k-1}$$

so the polynomial associated with the presentation is

$$f_w(t) = kt^2 - (2k-1)t + k$$

which is in fact the Alexander polynomial of the knot $(4k-1)/2$.

EXAMPLE 2.3. The Fibonacci group defined by the finite presentation

$$F(2, 2n) = \langle x_0, \dots, x_{2n-1} : x_i x_{i+1} x_{i+2}^{-1} = 1 \rangle$$

(indices mod n)

is isomorphic to the fundamental group of the n -fold cyclic covering M_n of the 3-sphere branched over the figure-eight knot $5/2$ (or, equivalently 4_1 , according to the Rolfsen notation [51]). For $n \geq 4$, Helling, Kim, and Mennicke [29] proved that $F(2, 2n)$ can be realized as a discrete cocompact subgroup of $\text{PSL}(2; \mathbb{C})$ (i.e. the group of orientation preserving iso-

metries of the hyperbolic 3-space \mathbb{H}^3). Hence the manifold $M_n \cong \mathbb{H}^3/F(2, 2n)$ is hyperbolic for any $n \geq 4$, while M_2 is the lens space $L(5, 3)$ (so, it is spherical), and M_3 is the Euclidean Hantzsche–Wendt manifold [63].

The defining word of $F(2, 2n)$ is

$$w = x_0 x_1 x_2^{-1}$$

and the associated polynomial is

$$f_w(t) = 1 + t - t^2$$

which cannot be the Alexander polynomial of any knot. So this presentation does not satisfy Theorem 2.2. But there must be another one which works well.

In fact [37], set $y_i = x_{2i+1}$ for any $i = 0, \dots, n-1$.

From Fibonacci relations

$$x_{2i-1} x_{2i} x_{2i+1}^{-1} = 1$$

or equivalently,

$$x_{2i} = x_{2i-1}^{-1} x_{2i+1}$$

we get

$$x_{2i} = y_{i-1}^{-1} y_i,$$

so we can eliminate the generators with even indices.

Now Fibonacci relations

$$x_{2i} x_{2i+1} x_{2i+2}^{-1} = 1$$

become

$$y_{i-1}^{-1} y_i^2 y_{i+1}^{-1} y_i = 1.$$

Thus $F(2, 2n)$ also admits the following cyclic presentation

$$F(2, 2n) = \langle y_0, \dots, y_{n-1} : y_{i-1}^{-1} y_i^2 y_{i+1}^{-1} y_i = 1 \\ \text{(indices mod } n) \rangle.$$

The defining word of the new presentation is

$$w = y_0^{-1} y_1^2 y_2^{-1} y_1$$

and the associated polynomial is

$$f_w(t) = -1 + 3t - t^2$$

which coincides (up to the sign) with the Alexander polynomial of the figure-eight knot. In this case, the split extension group is presented by

$$H_n(v) = \langle \theta, y : \theta^n = 1, v(\theta, y) = 1 \rangle$$

where

$$\begin{aligned} v(\theta, y) &= w(y, \theta^{-1}y\theta, \theta^{-2}y\theta^2) \\ &= y^{-1}(\theta^{-1}y^2\theta)(\theta^{-2}y^{-1}\theta^2)(\theta^{-1}y\theta) \\ &= y^{-1}\theta^{-1}y^2\theta^{-1}y^{-1}\theta y\theta. \end{aligned}$$

Setting $y = \theta\lambda$ yields

$$\begin{aligned} H_n(v) &= \langle \theta, \lambda : \theta^n = 1, \lambda^{-1}\theta^{-1}\lambda\theta\lambda\theta^{-1}\lambda^{-1}\theta\lambda\theta = 1 \rangle \\ &= \langle \theta, \lambda : \theta^n = 1, [\lambda, \theta]\lambda = \theta^{-1}[\lambda, \theta] \rangle \end{aligned}$$

where $[\lambda, \theta] = \lambda^{-1}\theta^{-1}\lambda\theta$. Since λ is conjugate with θ^{-1} (use the second relation), it follows that $\lambda^n = 1$. Thus we have

$$H_n(v) = \langle \theta, \lambda : \theta^n = \lambda^n = 1, [\lambda, \theta]\lambda = \theta^{-1}[\lambda, \theta] \rangle.$$

We recall that the group presented by $\langle \theta, \lambda : [\lambda, \theta]\lambda = \theta^{-1}[\lambda, \theta] \rangle$ is the fundamental group of the figure-eight knot 4_1 , where the generator θ represents a meridian of it. Thus the group $H_n(v)$ is the fundamental group of the orbifold $\mathcal{O}_n(4_1)$, whose underlying space is the standard 3-sphere, and whose singular set is 4_1 with branching index n .

Moreover, the class of Fibonacci manifolds can be obtained as 2-fold coverings of the 3-sphere branched along specified links. Let us denote by Th_n , $n \geq 1$, the closed 3-strings braid $(\sigma_1\sigma_2^{-1})^n$. It is obvious that Th_n is a 3-component link if n is divisible by 3, and it is a knot otherwise. The initial members of the family Th_n , called *Turk's head links*, are well-known, and we show them in Figure 2.3. More precisely, we have that Th_1 is the trivial knot, Th_2 is the *figure-eight knot*, Th_3 is the 3-component link 6_3^2 known as the *Borromean rings*, Th_4 is *Turk's head knot* 8_{18} , and Th_5 is the knot 10_{123} (compare with the Rolfsen notation [51]).

Mednykh and Vesnin proved the following result in [44].

THEOREM 2.3. *The Fibonacci manifold M_n , $n \geq 2$, is the 2-fold covering of the 3-sphere branched over the closed 3-strings braid $(\sigma_1\sigma_2^{-1})^n$,*

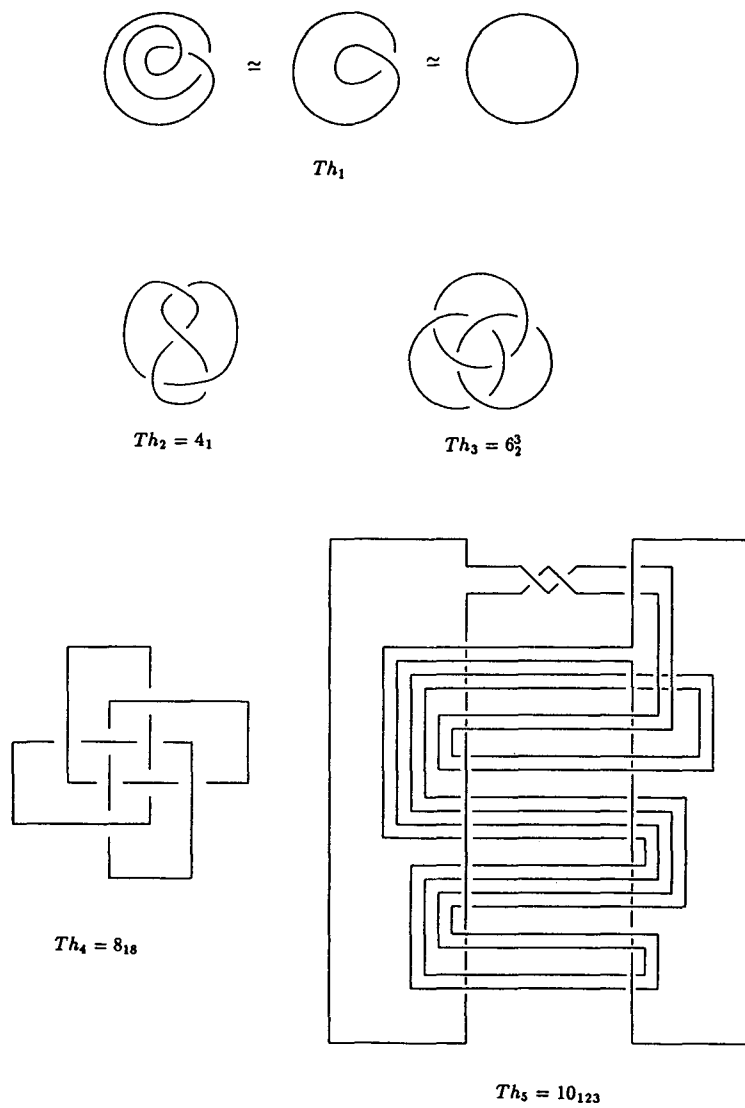


Figure 2.3. - Turk's head links 4_1 , 6_2^3 , 8_{18} , and 10_{123} .

i.e. the Turk head link Th_n . In particular, the Heegaard genus of M_n equals 2, for any $n \geq 3$.

In [43] the volumes of the Fibonacci manifolds were calculated in terms of the Lobachevsky function.

THEOREM 2.4. *For any $n \geq 4$, the hyperbolic volume of the Fibonacci manifold M_n is given by*

$$\text{vol}(M_n) = 2n(A(\delta + \gamma) + A(\delta - \gamma)),$$

where $\gamma = \frac{\pi}{n}$, and $\delta = \frac{1}{2} \arccos\left(\cos(2\gamma) - \frac{1}{2}\right)$.

Since M_n is hyperbolic for any $n \geq 4$, and its volume goes to infinity as n gets larger, Theorems 2.3 and 2.4 imply that there exists an infinite class of Heegaard genus 2 hyperbolic 3-manifolds with an arbitrary large volume. This situation is completely different from that of dimension 2. In fact, if S_g is the aspherical (hyperbolic) closed surface of genus g ($g \geq 2$), then its volume equals $4\pi(g-1)$ by the Gauss-Bonnet theorem. In particular, the volume of the connected sum of two copies of the torus (i.e. $g=2$) is 4π . By the Hurwitz theorem, the order of the isometry group of S_g does not exceed $84(g-1)$. Now the class of Fibonacci manifolds shows that the 3-dimensional analogue of the Hurwitz theorem does not exist at least in terms of the Heegaard genus (which represents the natural extension of the 2-dimensional genus). More precisely, the following was proved by Maclachlan and Reid [41] (for $n = 4, 5, 6, 8$ and 12) and by Rasskazov and Vesnin [49] (for $n \geq 6$).

THEOREM 2.5. *Let M_n be the (hyperbolic) Fibonacci manifold for $n \geq 4$. Then the isometry group of M_n admits the finite presentation*

$$\text{Isom}(M_n) = \langle x, y : x^{2n} = y^4 = (yx)^2 = (y^{-1}x)^2 = 1 \rangle.$$

In particular, the order of $\text{Isom}(M_n)$ is $8n$.

It follows that there exist infinitely many closed (hyperbolic) Heegaard genus 2 3-manifolds with arbitrary large isometry groups.

We can now discuss examples of compact and noncompact manifolds with the same volume. The affirmative answer to question 1.2 was given in [43] by finding infinite series of compact and noncompact manifolds whose volumes were calculated exactly in terms of the Lobachevsky function.

It was shown by Thurston [56] that for $n \geq 2$, noncompact manifolds, obtained as complements $S^3 \setminus \text{Th}_n$, are hyperbolic and

$$\text{vol}(S^3 \setminus \text{Th}_n) = 4n(\mathcal{A}(\delta + \gamma) + \mathcal{A}(\delta - \gamma)),$$

where $\gamma = \frac{\pi}{2n}$ and $\delta = \frac{1}{2} \arccos\left(\cos(2\gamma) - \frac{1}{2}\right)$. In particular, we have the asymptotic formula for volumes:

$$\text{vol}(S^3 \setminus \text{Th}_n) \sim 8n\mathcal{A}(\pi/6)$$

for $n \rightarrow \infty$. So there exist hyperbolic knots with an arbitrary large volume of their complements.

By comparing this result with the formula in Theorem 2.4, we get [43]:

COROLLARY 2.6. *For any $n \geq 2$, the compact Fibonacci manifold M_{2n} and the noncompact Turk's head link complement $S^3 \setminus \text{Th}_n$ have the same hyperbolic volume, i.e.*

$$\text{vol}(M_{2n}) = \text{vol}(S^3 \setminus \text{Th}_n).$$

In particular, the volume of the Fibonacci manifold M_4 (resp. M_6) is equal to the volume of the (open) complement in S^3 of the figure-eight knot (resp. of the Borromean rings).

We remark that topological properties of the Fibonacci manifolds are well-studied. So there are different descriptions of these manifolds. It was shown by Hilden, Lozano and Montesinos [31] that for any n the Fibonacci manifold M_n can be obtained as the n -fold cyclic covering of the 3-sphere, branched over the figure-eight knot. It was shown in [44] that for any $n \geq 2$ the Fibonacci manifold M_n can be obtained as the 2-fold covering of the 3-sphere, branched over Turk's head link Th_n (see Theorem 2.3). It was shown by Cavicchioli and Spaggiari [21] that for any n the Fibonacci manifold M_n can be obtained by Dehn surgeries with parameters $(1, 1)$ and $(-1, 1)$ on components of the link that is the chain of $2n$ unknotted circles. So these manifolds belong to a series of *Takahashi manifolds*, studied in [37] and [52], and considered in the next section.

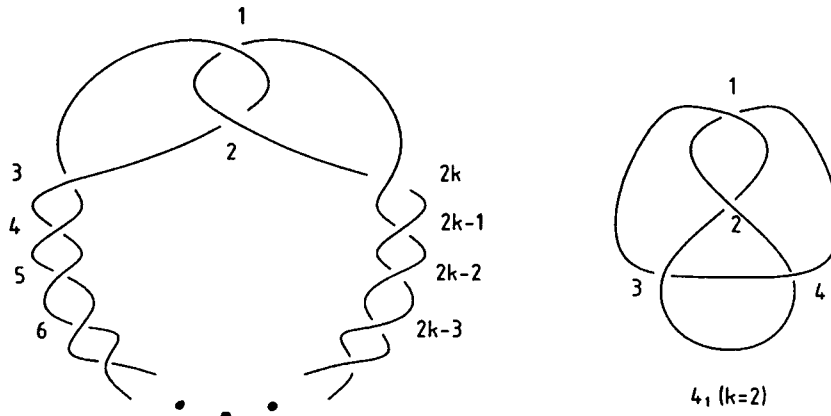


Figure 2.4. - The knot $(2k)_1 = (4k - 3)/2$.

The following extends the cyclic presentation obtained for Fibonacci groups [19].

THEOREM 2.7. *The cyclically presented group*

$$G(n, k) = \langle x_0, \dots, x_{n-1} : x_i^{-1} x_{i+1}^2 (x_{i+2}^{-1} x_{i+1})^{k-1} (x_i^{-1} x_{i+1})^{k-2} = 1$$

(indices mod n)

is isomorphic to the fundamental group of the n -fold cyclic covering of the 3-sphere branched over the 2-bridge knot $(4k - 3)/2$ (or, equivalently $(2k)_1$, according to Rolfsen notation), $k \geq 2$, depicted in Figure 2.4. The polynomial associated with $G(n, k)$ is the Alexander polynomial of the knot $(2k)_1$, i.e. $(k - 1)t^2 - (2k - 1)t + k - 1$.

Other interesting families of cyclically presented groups, which include the Fibonacci groups and the Sieradski groups, were studied in [37]. These groups were proved to encode the cyclic branched coverings of the 2-bridge knots $(4ml \pm 1)/2l$ by using Dehn surgery on a certain chain of linked circles.

Let us consider the cyclically presented groups $G_n(m, l; \varepsilon)$ with n generators x_i , and n relations ($\varepsilon = \pm 1$)

$$(x_i^{-l} x_{i+1}^l)^m x_{i+1}^\varepsilon = (x_{i+1}^{-l} x_{i+2}^l)^m \quad (\text{indices mod } n).$$

The following result is due to [37] (in [19] one can find an alternative proof obtained by algebraic methods from combinatorial group theory):

THEOREM 2.8. *Let $G_n(m, l; \varepsilon)$ be the cyclically presented group with n generators x_i , and n relations ($\varepsilon = \pm 1$)*

$$(x_i^{-l} x_{i+1}^l)^m x_{i+1}^\varepsilon = (x_{i+1}^{-l} x_{i+2}^l)^m \quad (\text{indices mod } n).$$

Then $G_n(m, l; \varepsilon)$ is isomorphic to the fundamental group of the n -fold cyclic covering $M_n((4ml + \varepsilon)/2l)$ of the 3-sphere branched over the 2-bridge knot $(4ml + \varepsilon)/2l$, where $\varepsilon = \pm 1$. The polynomial associated with $G_n(m, l; \varepsilon)$ is the Alexander polynomial

$$\Delta_{m,l}(t) = mlt^2 - (2ml + \varepsilon)t + ml$$

of $(4ml + \varepsilon)/2l$. The manifolds $M_n((4ml + \varepsilon)/2l)$ are hyperbolic for all $n \geq 3$ if $m \geq 2$ or $l \geq 2$, and $M_n(5/2)$ are hyperbolic for all $n \geq 4$. In these cases, $G_n(m, l; \varepsilon)$ are hyperbolic groups (hence infinite) which encode the corresponding manifolds.

Since the rational number $(4ml - 1)/2l$ can be expressed by the continued fraction

$$\frac{4ml - 1}{2l} = 2m - 1 + \frac{1}{1 + \frac{1}{2l - 1}},$$

the corresponding 2-bridge knot admits the Conway normal form $C(2l - 1, 1, 2m - 1)$, and hence it has the same knot type as the pretzel knot $P(-1, 2l - 1, 2m - 1)$ (see for example Theorem 2.3.1 of [34]). By [39],

p. 57, a Seifert matrix for such a knot is

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} l & l \\ l-1 & m+l-1 \end{pmatrix}.$$

We can now apply the procedure described in [26] to determine the homology characters of the cyclic branched coverings of our knots $(4ml - 1)/2l$. Since $\det V = ml$, and

$$\begin{aligned} w &= \text{g.c.d.} (v_{11}, v_{12} + v_{21}, v_{22}) \\ &= \text{g.c.d.} (l, 2l-1, m+l-1) = 1, \end{aligned}$$

the homology groups of $M_n((4ml + \varepsilon)/2l)$ can be completely computed from [26] (for $\varepsilon = 1$ see the remark above).

The following can be found in [19].

THEOREM 2.9. *Let $M_n = M_n((4ml + \varepsilon)/2l)$ be the n -fold cyclic covering of the 3-sphere branched over the 2-bridge knot $(4ml + \varepsilon)/2l$, where $\varepsilon = \pm 1$. Then the first integral homology group of M_n is*

$$H_1(M_n) \cong \begin{cases} \mathbb{Z}_{|(4ml + \varepsilon)a_n|} \oplus \mathbb{Z}_{|a_n|} & n \text{ even} \\ \mathbb{Z}_{|b_n|} \oplus \mathbb{Z}_{|b_n|} & n \text{ odd} \end{cases}$$

where

$$\begin{aligned} a_1 = a_2 = 1, \quad a_{n+2} &= a_{n+1} + \varepsilon m l a_n \\ b_1 = 1, \quad b_2 &= 1 + 2\varepsilon m l, \quad b_{n+2} = b_{n+1} + \varepsilon m l b_n. \end{aligned}$$

To complete this section we discuss some series of hyperbolic manifolds which are cyclic branched coverings of (hyperbolic) links with two components. Let $L = K_1 \cup K_2$ be an oriented link in the 3-sphere. Let us denote by $\mathcal{O}_n(L)$ the 3-orbifold whose underlying space is S^3 and whose singular set is the link L of branching index n . The fundamental group of the orbifold $\mathcal{O}_n(L)$ is isomorphic to the factor group of $\pi_1(S^3 \setminus L)$ over the subgroup generated by m_1^n and m_2^n , where m_i is the homotopy class of an oriented meridian of K_i , $i = 1, 2$. The abelianized group of $\pi_1(\mathcal{O}_n(L))$ is isomorphic to $\mathbb{Z}_n \oplus \mathbb{Z}_n$ (generated by the images of the classes m_1 and

m_2). For any integer $k \neq 0$ with $(k, n) = 1$, let us consider the homomorphism

$$\psi_{n,k}: \pi_1(\mathcal{O}_n(L)) \rightarrow \pi_1(\mathcal{O}_n(L))^{\text{ab}} \cong \mathbb{Z}_n \oplus \mathbb{Z}_n \rightarrow \mathbb{Z}_n$$

defined by $\psi_{n,k}(m_1) = \bar{1}$, and $\psi_{n,k}(m_2) = \bar{k}$. Let $M_{n,k}(L)$ denote the closed orientable 3-manifold which is the (strongly) cyclic branched covering of L corresponding to the kernel of $\psi_{n,k}$. Suppose now that L is a *hyperbolic link*, i.e. the (open) complement of L in S^3 is a hyperbolic 3-manifold of finite volume. By Thurston's hyperbolic surgery theorem, the orbifold $\mathcal{O}_n(L)$ and the manifold $M_{n,k}(L)$ are hyperbolic for sufficiently large n . Moreover, the isometry group of $\mathcal{O}_n(L)$ and that of $M_{n,k}(L)$ are finite groups.

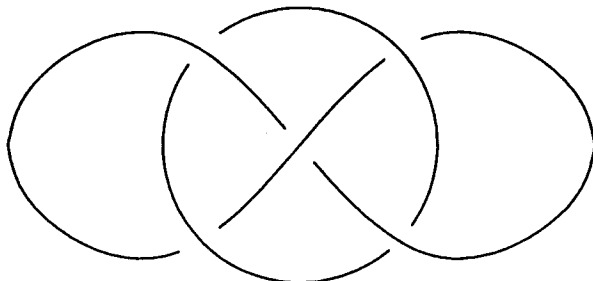
The following was proved in [64].

THEOREM 2.10. *Under the notation above, suppose that n does not divide the order of the isometry group of $\mathcal{O}_n(L)$. Then $M_{n,k}(L)$ is isometric (homeomorphic) to $M_{n,k'}(L)$ if and only if there exist isometries $\varphi: M_{n,k}(L) \rightarrow M_{n,k'}(L)$ and $\Phi: \mathcal{O}_n(L) \rightarrow \mathcal{O}_n(L)$ such that the diagram*

$$\begin{array}{ccc} M_{n,k}(L) & \xrightarrow{\varphi} & M_{n,k'}(L) \\ \downarrow & & \downarrow \\ \mathcal{O}_n(L) & \xrightarrow{\Phi} & \mathcal{O}_n(L) \end{array}$$

commutes. In particular, $M_{n,k}(L) \cong M_{n,k'}(L)$ if and only if $k \equiv \pm k' \pmod{n}$ or $kk' \equiv \pm 1 \pmod{n}$.

This result extends in some sense the classification of lens spaces which are the cyclic branched coverings of the Hopf link (which is not hyperbolic however) [51]. Theorem 2.10 applies for example to classify, up to isometry, the cyclic branched coverings of the Whitehead link \mathcal{W} (which is hyperbolic) shown in Figure 2.5. In this case, the orbifolds $\mathcal{O}_n(\mathcal{W})$ and the n -fold cyclic coverings $M_{n,k}(\mathcal{W})$ branched over \mathcal{W} are hyperbolic for any $n \geq 3$. Furthermore, the order of the isometry group of $\mathcal{O}_n(\mathcal{W})$ equals 8. Explicit constructions by polyhedral schemata of the strongly branched coverings of the Whitehead link were given in [30]. In particular, nice presentations for their fundamental groups were obtained as follows.

Figure 2.5. - The Whitehead link \mathcal{W} .

THEOREM 2.11. Let $\Gamma_{n,k}$ be the finite group presentation

$$\Gamma_{n,k} = \langle x_0, \dots, x_{n-1}, y : x_0 x_k x_{2k} \dots x_{(n-1)k} = 1$$

$$x_{i-k-1}^{-1} x_{i-k} x_{i-1} x_i^{-1} = y$$

$$(\text{indices mod } n) \rangle,$$

where $(n, k) = 1$, and $n \geq 3$. Then $\Gamma_{n,k}$ corresponds to a spine of the n -fold strongly cyclic covering $M_{n,k}(\mathcal{W})$ of the 3-sphere branched over the Whitehead link \mathcal{W} . Moreover, $M_{n,k}(\mathcal{W})$ is homeomorphic to $M_{n,k'}(\mathcal{W})$ if and only if $k \equiv \pm k' \pmod{n}$ or $kk' \equiv \pm 1 \pmod{n}$.

It is a routine matter to compute $\Gamma_{n,k}^{\text{ab}} \cong H_1(M_{n,k}(\mathcal{W}))$ from the presentation above. So we obtain the following result.

THEOREM 2.12. Let $n \geq 3$ and let $k \pmod{n}$ satisfy conditions $(k, n) = 1$ and $1 \leq k \leq [(n-1)/2]$. Then the manifold $M_{n,k}(\mathcal{W})$ has the homology:

$$H_1(M_{n,k}(\mathcal{W})) \cong \begin{cases} \mathbb{Z}_{n/6} \oplus \mathbb{Z}_{n/2} \oplus \mathbb{Z}_{12n} & n \equiv 0 \pmod{6} \\ \mathbb{Z}_{n/2} \oplus \mathbb{Z}_{n/2} \oplus \mathbb{Z}_{4n} & n \equiv \pm 2 \pmod{6} \\ \mathbb{Z}_{n/3} \oplus \mathbb{Z}_n \oplus \mathbb{Z}_{3n} & n \equiv 3 \pmod{6} \\ \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n & (n, 6) = 1. \end{cases}$$

The classification of the geometric and topological structures of all cyclic (also non strongly) coverings of the 3-sphere branched over the Whitehead link was obtained by Cavicchioli and Paoluzzi in [16].

3. - Manifolds obtained by Dehn surgery.

Consider the link L_{2n} with $2n$ components in the oriented 3-sphere ($n \geq 2$) shown in Figure 3.1. Each component is unknotted, oriented and linked with exactly two adjacent components. Let us denote by $M(p_1/q_1, \dots, p_n/q_n; r_1/s_1, \dots, r_n/s_n)$, or briefly $M(p_i/q_i; r_i/s_i)$, the

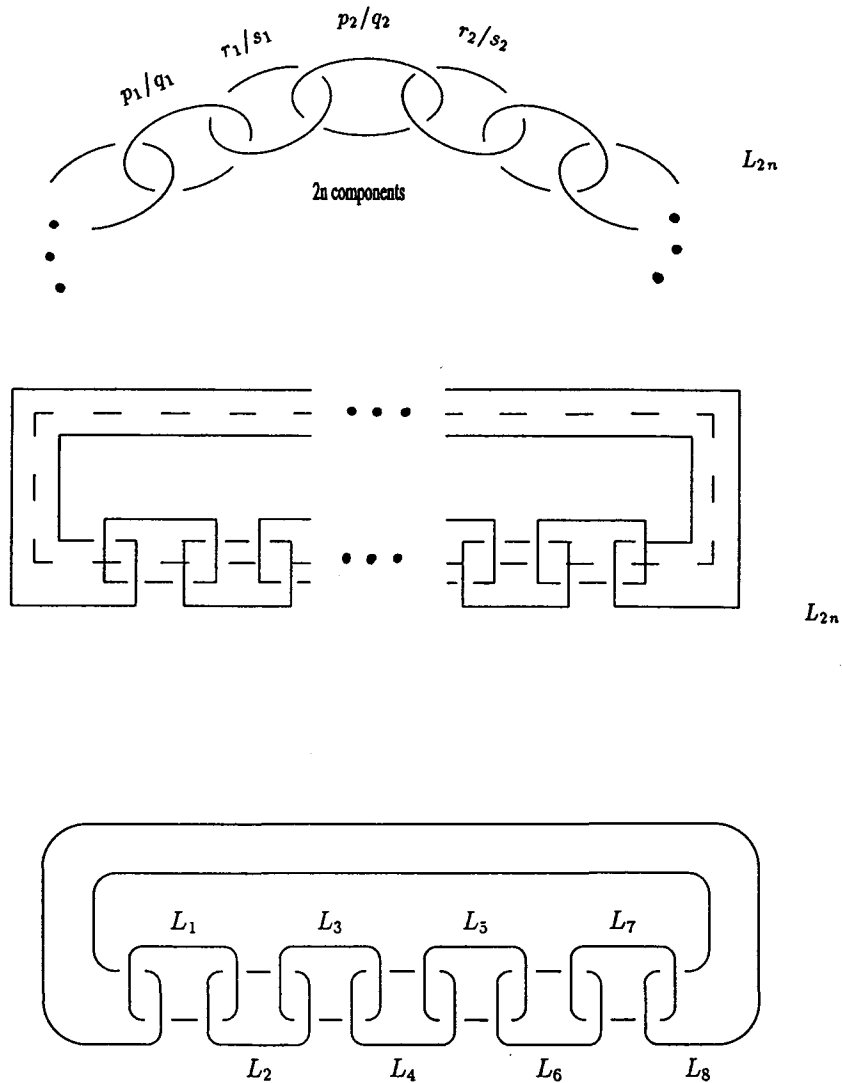


Figure 3.1. - The link L_{2n} .

closed connected orientable 3-manifold obtained by Dehn surgery along the link L_{2n} with surgery coefficients p_i/q_i and r_i/s_i , for any $i = 1, \dots, n$, according to Figure 3.1. These manifolds were first considered by Takahashi in [55], so, for convenience, we refer to them as the *Takahashi manifolds*.

The following was proved in [55].

THEOREM 3.1. *The finite presentation*

$$\begin{aligned} \langle x_1, \dots, x_{2n}: & \quad x_{2i}^{s_i} x_{2i+1}^{p_i+1} = x_{2i+2}^{s_i+1} \\ & \quad x_{2i-1}^{q_i} x_{2i}^{-r_i} = x_{2i+1}^{q_i+1} \\ & \quad (\text{indices mod } 2n) \rangle \end{aligned}$$

corresponds to a spine of the Takahashi manifold $M(p_i/q_i; r_i/s_i)$.

In particular, if $p_i/r_i = 1$ and $r_i/s_i = -1$ for any $i = 1, \dots, n$, we have the Fibonacci manifolds. If $p_i/r_i = k/l$ and $r_i/s_i = -k/l$, we obtain the *Fractional Fibonacci manifolds* studied in [36]. Furthermore, by using the Kirby-Rolfsen calculus, one can directly verify that Sieradski manifolds are also included in Takahashi manifolds (see [37] and [52]).

We observe that the link L_{2n} is strongly invertible. By using a well-known theorem of Montesinos, the following result was proved independently in [37] and [52].

THEOREM 3.2. *The Takahashi manifold $M(p_i/q_i; r_i/s_i)$ is the 2-fold cyclic covering of the 3-sphere branched along the closure of the rational 3-string braid*

$$\sigma_1^{p_1/q_1} \sigma_2^{r_1/s_1} \dots \sigma_1^{p_n/q_n} \sigma_2^{r_n/s_n}$$

which is depicted in Figure 3.2 (here any rectangular box stands for the rational tangle defined by the ratio inside it).

In particular, the 2-fold branched coverings of closed 3-string braids are Takahashi manifolds.

Since L_{2n} is also hyperbolic, we can apply the Thurston-Jørgensen theory of hyperbolic surgery to state that for almost all the pairs (p_i, q_i) and (r_i, s_i) the Takahashi manifolds $M(p_i/q_i; r_i/s_i)$ are hyperbolic.

We now show that the closed orientable 3-manifolds obtained by Dehn surgery along the Whitehead link form a special subclass of the Takahashi manifolds. This class is of particular interest since it contains the ten smallest volume hyperbolic 3-manifolds. More precisely, the structure of the initial segment of the set of volumes was firstly con-

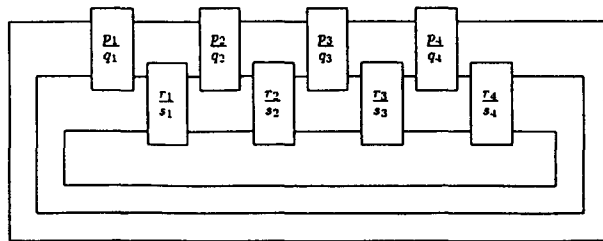
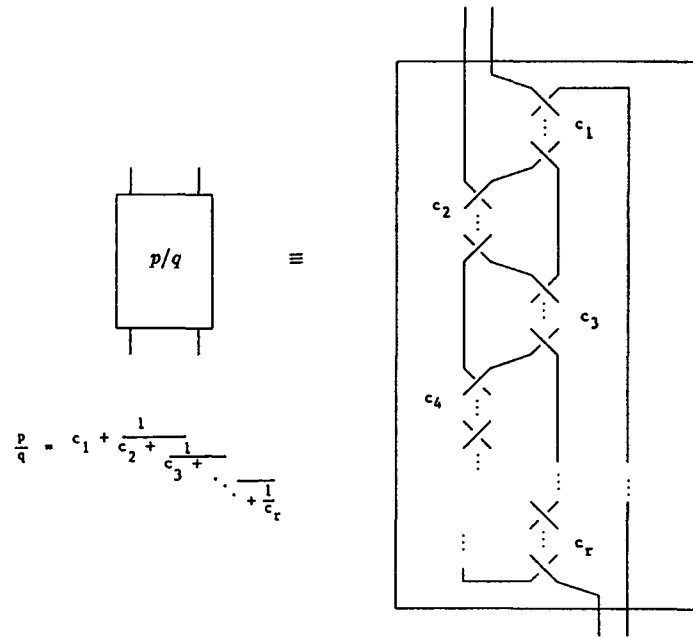
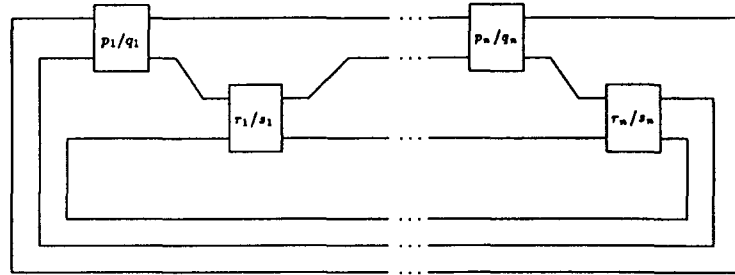
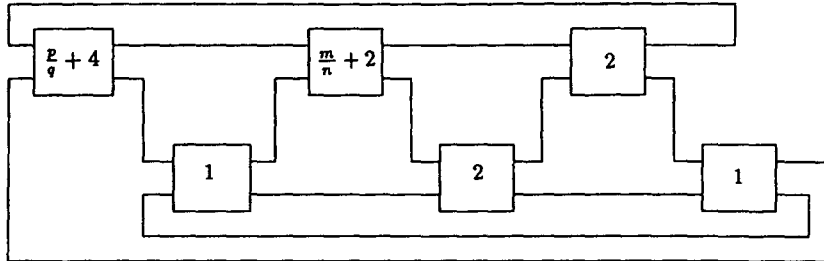


Figure 3.2. - The rational braid $\sigma_1^{p_1/q_1} \sigma_2^{r_1/s_1} \dots \sigma_1^{p_n/q_n} \sigma_2^{r_n/s_n}$.



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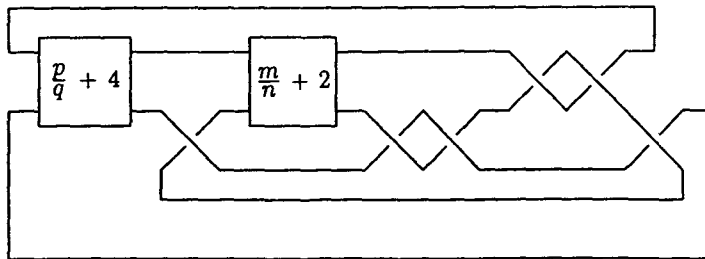


Figure 3.3. - The link $\mathcal{L}(m/n; p/q)$.

tured by Fomenko and Matveev [25]. Hodgson and Weeks [33] described the ten smallest known manifolds in term of Dehn surgery on some links. We are going to discuss covering properties of these manifolds.

Let us denote by $\mathcal{W}(m/n; p/q)$ the closed orientable 3-manifold obtained by m/n and p/q Dehn surgeries on the Whitehead link \mathcal{W} . Consider a series of links $\mathcal{L}(m/n; p/q)$ pictured in Figure 3.3 (which are a special case of the link presented in the statement of Theorem 3.2).

The following result gives the connection between these links and manifolds $\mathcal{W}(m/n; p/q)$ [45]:

THEOREM 3.3. *Let $M = \mathcal{W}(m/n; p/q)$ be the closed orientable 3-manifold obtained by m/n and p/q Dehn surgeries on the Whitehead link \mathcal{W} . Then M is the 2-fold covering of S^3 branched over the link $\mathcal{L}(m/n; p/q)$.*

In particular, we get the description of the ten smallest volume hyperbolic 3-manifolds:

	$\text{vol}(\mathcal{M}_i)$	$\text{vol}(S^3 \setminus \mathcal{L}_i)$	\mathcal{L}_i
$\mathcal{M}_1 = \mathcal{W}\left(-\frac{5}{2}; -5\right)$	0.9427...	9.4270...	9_{49}
$\mathcal{M}_2 = \mathcal{W}(1; -5)$	0.9813...	5.6387...	10_{161}
$\mathcal{M}_3 = \mathcal{W}\left(-\frac{3}{2}; -6\right)$	1.0149...	8.1195...	10_{138}^2
$\mathcal{M}_4 = \mathcal{W}(-5; -5)$	1.2637...	9.2505...	10_{155}
$\mathcal{M}_5 = \mathcal{W}(1; -6)$	1.2844...	5.8430...	11_7^2
$\mathcal{M}_6 = \mathcal{W}\left(1; -\frac{1}{2}\right)$	1.3986...	5.8296...	14_7
$\mathcal{M}_7 = \mathcal{W}\left(-\frac{1}{2}; -6\right)$	1.4140...	5.9782...	11_7^2
$\mathcal{M}_8 = \mathcal{W}(2; -5)$	1.4140...	7.7948...	11_7^2
$\mathcal{M}_9 = \mathcal{W}\left(-\frac{7}{3}; -5\right)$	1.4236...	10.6933...	10_{162}
$\mathcal{M}_{10} = \mathcal{W}\left(1; -\frac{3}{2}\right)$	1.4406...	7.1180...	13_7

Here «?» in the suffices means that corresponding knots or links are of too big order and they are not contained in the standard tables of knots and links known from the literature.

Observe that by [47] each of manifolds $\mathcal{W}(1/n; 1/q)$, $\mathcal{W}(1/n; 2/q)$ with q odd, and $\mathcal{W}(2/n; 2/q)$ with n and q odd, has three hyperelliptic involutions.

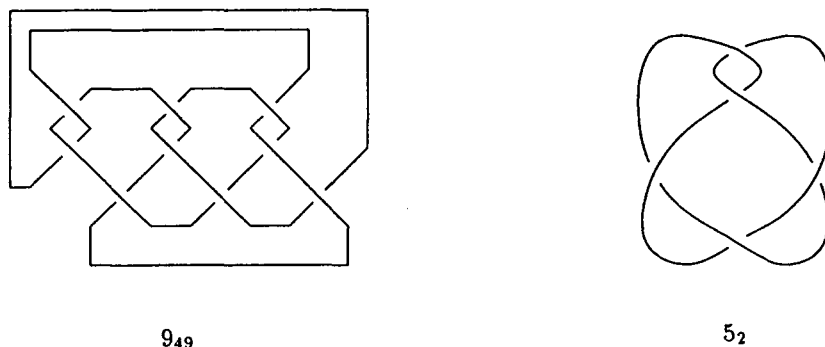
The smallest known closed hyperbolic 3-manifold \mathcal{M}_1 was obtained by Fomenko and Matveev [25] and by Weeks [60]. The *lattice of the action* of the isometry group $\text{Isom}(\mathcal{M}_1) \cong D_6$ on the *Fomenko-Matveev-Weeks manifold* \mathcal{M}_1 was described in [46]. In particular, we get the following descriptions of this manifold [46].

COROLLARY 3.4. *The Fomenko-Matveev-Weeks manifold \mathcal{M}_1 can be obtained as the 2-fold covering of S^3 branched over the knot 9_{49} , and as the 3-fold covering of S^3 branched over the knot 5_2 .*

The knots in Corollary 3.4 are depicted in Figure 3.4.

Recall that \mathcal{M}_1 is an arithmetic manifold, and moreover it is the smallest volume arithmetic manifold.

There are nice estimates of volumes of manifolds as follows. In virtue

Figure 3.4. - The knots 9_{49} and 5_2 .

of [27] for any closed hyperbolic 3-manifold M , we have that $\text{vol}(M) \geq 0.00115$. If the first integral Betti number of M is at least 3, then a result due to Culler and Shalen [23] implies that $\text{vol}(M) \geq 0.92$.

Analogously, to study the initial segment of volumes of compact orientable 3-manifolds, it is interesting to consider the initial segment of volumes of hyperbolic knots. Their volumes were tabulated in [2]. In particular, it was shown in [8] and [9] that the five smallest volume knots have the following volumes:

$$\begin{aligned} \text{vol}(S^3 \setminus 4_1) &= 2.029 \dots & \text{vol}(S^3 \setminus (-2; 3; 7)\text{-pretzel knot}) &= 2.828 \dots \\ \text{vol}(S^3 \setminus 5_2) &= 2.828 \dots & \text{vol}(S^3 \setminus 6_1) &= 3.163 \dots & \text{vol}(S^3 \setminus 7_2) &= 3.331 \dots \end{aligned}$$

Finally, we complete the section with a list of some open problems:

3.1) [28] Do there exist hyperbolic 3-manifolds M and N such that the ratio $\text{vol}(M)/\text{vol}(N)$ is irrational?

3.2) [38], Problem 3.94. The smallest volume Fomenko-Matveev-Weeks manifold \mathcal{M}_1 is of Heegaard genus 2. What is the smallest volume manifold for a given Heegaard genus $g \geq 3$?

3.3) Let K be a knot in S^3 and $\mathcal{O}_n(K)$ the orbifold with the underlying space S^3 whose singular set is K with singularity of order n . If $K = 4_1$ is the figure-eight knot and $n = 4$, then it follows from results on volumes of [43] that $4 \text{vol}(\mathcal{O}_4(4_1)) = \text{vol}(S^3 \setminus 4_1)$. Does there exist another K and n such that $n \text{vol}(\mathcal{O}_n(K)) = \text{vol}(S^3 \setminus K)$?

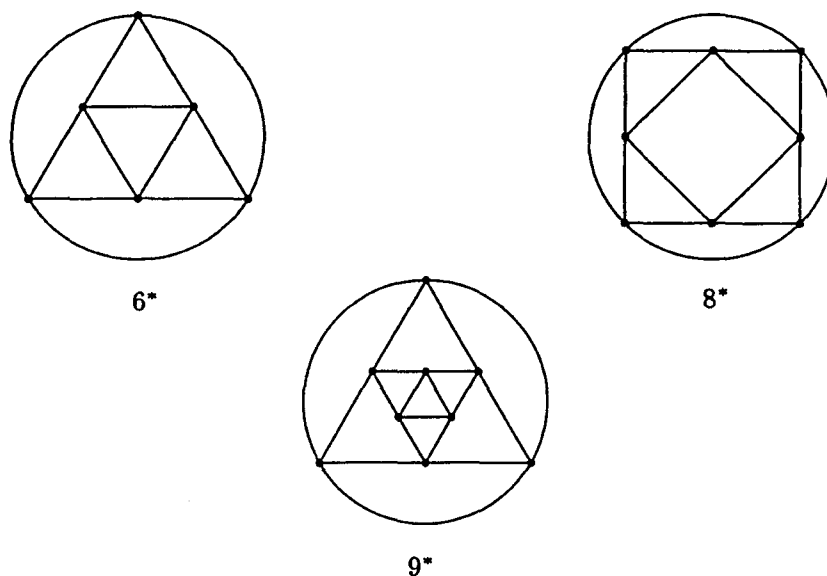


Figure 3.5. - Conway's polyhedra 6*, 8* and 9*.

3.4) Do there exist non-compact 3-orbifolds $T_n \in \mathbb{S}^3$ such that $\text{vol}(M_n) = \text{vol}(T_n)$, where M_n is the Fibonacci manifold and n is odd?

3.5) For a given Conway basic polyhedron [22], what is the smallest volume π -orbifold whose singular set is a link corresponding to this polyhedron? (If Conway basic polyhedron is 6*, i.e. the octahedron, then the smallest volume π -orbifold has singular set the knot 9_{49}). In particular, what are the smallest π -orbifolds corresponding to Conway polyhedra 8* and 9* shown in Figure 3.5?

3.6) What is the smallest volume of a hyperbolic manifold with a given number N of hyperelliptic involutions? We conjecture from [47] that the smallest volumes are:

$$N = 1 : \text{vol}(\mathcal{M}_1 = \mathcal{W}(5, 5/2)) = 0.94 \dots$$

$$N = 2 : \text{vol}(\mathcal{M}_2 = \mathcal{W}(1, -5)) = 0.98 \dots$$

$$N = 3 : \text{vol}(\mathcal{M}_6 = \mathcal{W}(1, -1/2)) = 1.39 \dots$$

4. - Knot spaces.

The following is a central problem of (Texas) geometric topology school (R. H. Bing and his circle):

Let X be a topological (unknown) space. Find conditions under which there exists a continuous surjection $f : M \rightarrow X$ where M is a topological manifold. We assume to know the topological properties of M and f , but it is unknown the topological properties of X , e.g. Is X metrizable resp. separable, finite dimensional, an ANR, a homology manifold, a topological manifold, homeomorphic to M ?

A typical method of investigation is to study fibers (point-inverses) of the map f and try to find a better map f' which is close to f . In conclusion, it is important to understand topological properties of the family $\{f^{-1}(x) : x \in X\}$.

For a motivation in dimension 3, consider a classical example given by the Bing dogbone space shown in Figure 4.1. Let Ω be the set of connected components of the intersection

$$T_0 \cap (T_{00} \cup T_{01} \cup T_{10} \cup T_{11}) \cap \dots$$

Then we have a map from \mathbb{R}^3 onto the quotient $X = \mathbb{R}^3/\Omega$. The following was proved by Bing in [6]:

THEOREM 4.1. *Under the above notation, we have:*

- (1) X is not a topological 3-manifold;
- (2) the product $X \times \mathbb{R}$ is homeomorphic to \mathbb{R}^4 .

To understand the idea of the proof of (1), look at the picture in Figure-

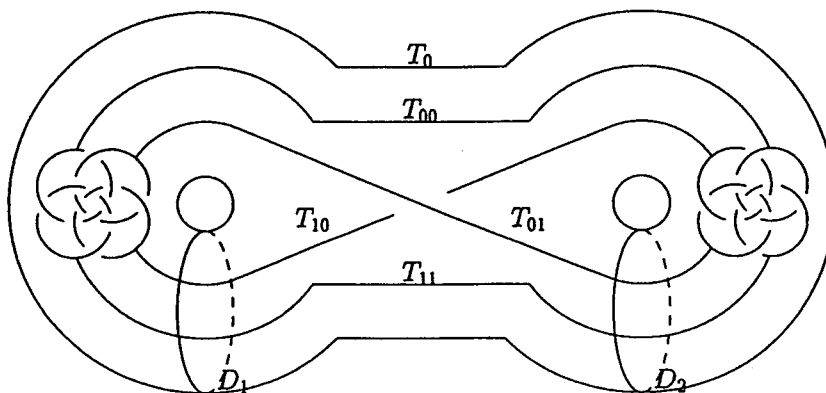


Figure 4.1. - Bing's dogbone space.

re 4.1, and consider the disks D_1 and D_2 . Their boundaries can not span any pair of disks which would not intersect in the quotient space. To see (2) recall that links in \mathbb{R}^3 can be unlinked in \mathbb{R}^4 . So the entire pattern of Ω can be unlinked.

So Bing topologists study the pre-images $f^{-1}(x)$'s, and, in particular, we shall be interested in regular neighborhoods of compacta embedded in piecewise-linear (PL) manifolds of dimension ≥ 3 .

PROBLEM 4.1. Given a compact 3-manifold M^3 , a compact polyhedron K , and homotopic PL embeddings

$$f_1 \simeq f_2: K \rightarrow M^3$$

when are the regular neighborhoods N_1 and N_2 of $f_1(K)$ and $f_2(K)$, respectively, in M^3 homeomorphic, i.e. PL isomorphic?

EXAMPLE 4.1. Let M^3 be the 3-sphere S^3 , and K the wedge $S^2 \vee S^1 \vee S^1$. Clearly, the regular neighborhoods of the compacta, embedded in S^3 as shown in Figure 4.2, are not homeomorphic. However, ∂N_1 and ∂N_2 have (at least) something in common, i.e. the number of connected components and the Euler characteristics. More precisely, we have the following result:

THEOREM 4.2. Let M^3 be a compact 3-manifold, K a compact polyhedron, and $f_1, f_2: K \rightarrow \text{int } M^3$ PL embeddings such that

$$(f_1)_* = (f_2)_*: H_*(K; \mathbb{Z}_2) \rightarrow H_*(M; \mathbb{Z}_2).$$

If N_i is a regular neighborhood of $f_i(K)$ in $\text{int } M$, then we have

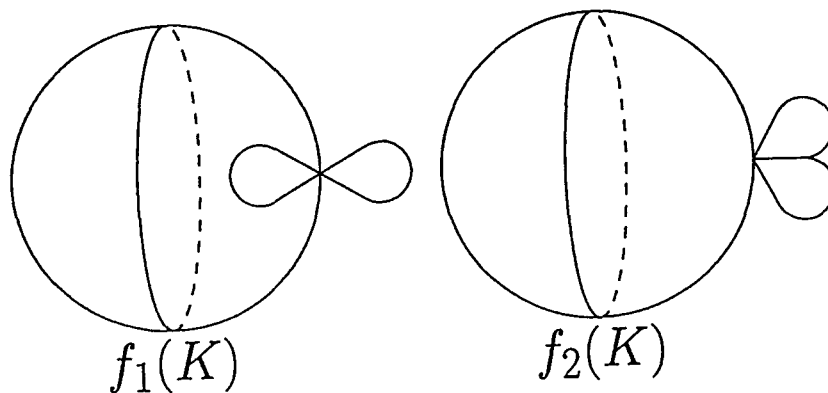


Figure 4.2. - Example: $M^3 = S^3$ and $K = S^2 \vee S^1 \vee S^1$.

$b_n(\partial N_1; \mathbb{Z}_2) = b_n(\partial N_2; \mathbb{Z}_2)$ for every n , where $b_n(\cdot)$ denotes the n -th Betti number.

The proof is based on diagram chase (the homology is taken with coefficients in \mathbb{Z}_2):

$$\begin{array}{ccccccc}
 0 & \rightarrow & H_3(M) & \rightarrow & H_3(M, M \setminus f_1(K)) & \rightarrow & H_2(M \setminus f_1(K)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \check{H}^0(M) & \rightarrow & \check{H}^0(f_1(K)) & \rightarrow & \check{H}^0(M, f_1(K)) \\
 & & & & & & \\
 & & H_2(M \setminus f_1(K)) & \rightarrow & H_2(M) & \rightarrow & H_2(M, M \setminus f_1(K)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \check{H}^0(M, f_1(K)) & \rightarrow & \check{H}^1(M) & \rightarrow & \check{H}^1(f_1(K))
 \end{array}$$

By using standard 3-manifold topology, the following result was proved in [50].

THEOREM 4.3. *Let K be any compact 2-polyhedron, and $f_1, f_2: K \rightarrow M$ homotopic PL embeddings into the interior of a compact 3-manifold M . Suppose that any one of the following conditions holds: either $\dim K \leq 1$, or K is a surface with nonvoid boundary, or K is the 2-sphere (resp. the real projective plane). Then every two regular neighborhoods N_1 and N_2 of $f_1(K)$ and $f_2(K)$, respectively, in $\text{int} M$ are PL isomorphic.*

CONJECTURE 4.1. Let K be any compact polyhedron such that $H_2(K; \mathbb{Z}_2) \cong 0$, and let $f_1, f_2: K \rightarrow M$ be any homotopic PL embeddings in the interior of a compact 3-manifold M . Suppose also that for any regular neighborhood N_i of $f_i(K)$ in M , $i = 1, 2$, the genus of ∂N_i is zero. Then N_1 is homeomorphic to N_2 .

Note that if Conjecture 4.1 is true, then the Poincaré conjecture is equivalent to the following statement: *Every homotopy 3-cell has a spine which PL embeds in \mathbb{R}^3 .*

The proof in the «only if» direction is obvious. For the converse, we need Conjecture 4.1. Let F be a homotopy 3-cell, and C be a tame 3-cell embedded in the interior of F as shown in Figure 4.3. By hypothesis, a PL embedding $f: K \rightarrow \text{int} C$ exists. Now one can use Conjecture 4.1 to conclude that $N_1 \cong N_2$. Hence N_1 is a genuine 3-cell.

The genus zero case turns out to be the only case with the possibility of an affirmative answer to our question, because as we shall demonstrate, for every genus ≥ 1 , there exists a counterexample.

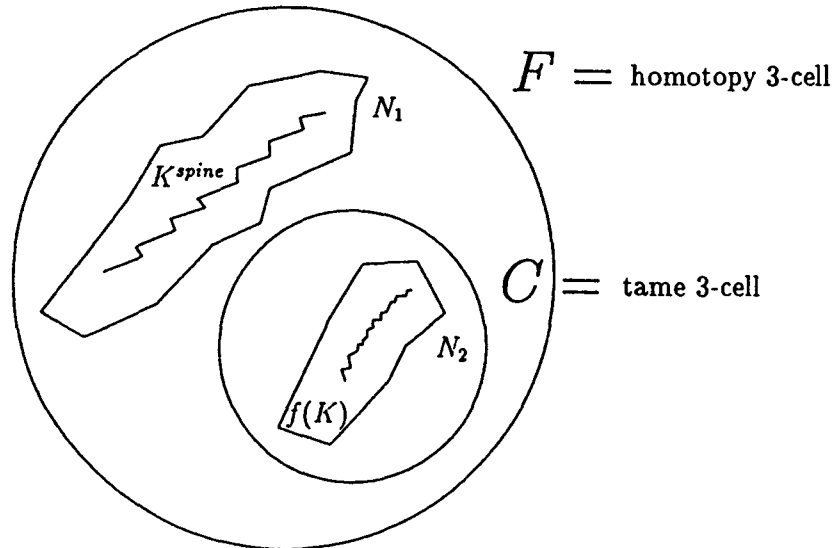


Figure 4.3. - The homotopy 3-cell F and the tame 3-cell C embedded in $\text{int}(F)$.

We now come to knot theory. Let K_1 be the square knot, and K_2 the granny knot, depicted in Figure 4.4. Let us consider the knot spaces N_1

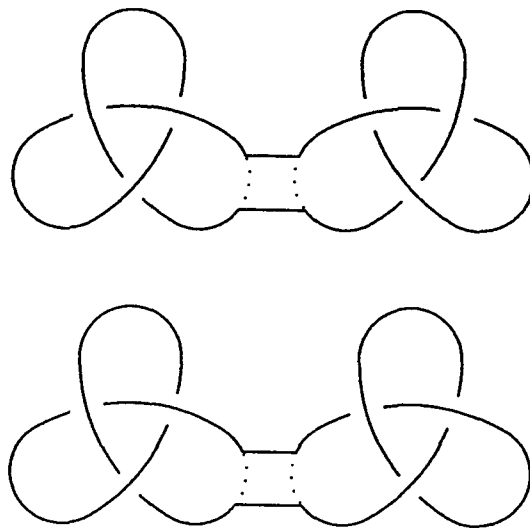


Figure 4.4. - The square knot and the granny knot.

and N_2 of these two knots, i.e. N_i is the complement in S^3 of an open tubular neighborhood of K_i , for $i = 1, 2$. It is well-known that N_1 can not be homeomorphic to N_2 . However, it was proved independently, using completely different techniques, by Mitchell-Przytycki-Repovš [48] and Cavicchioli [10] that N_1 and N_2 possess homeomorphic spines. This yields a counterexample for the genus 1 case.

The following more general result is true [48]:

THEOREM 4.4. *Let $K(p, q)$ be the (p, q) -torus knot in the standard 3-sphere, where p and q are any relatively prime numbers ≥ 2 . For any knot L in S^3 , let N_1 be the knot space of the composite knot $K(p, q) \# L$, and N_2 the knot space of $K(p, -q) \# L$. Then there exist a 2-dimensional compact polyhedron P and PL embeddings $\varphi_i: P \rightarrow \text{int} N_i$ such that $\varphi_i(P)$ is a spine of N_i , i.e. N_i collapses onto $\varphi_i(P)$.*

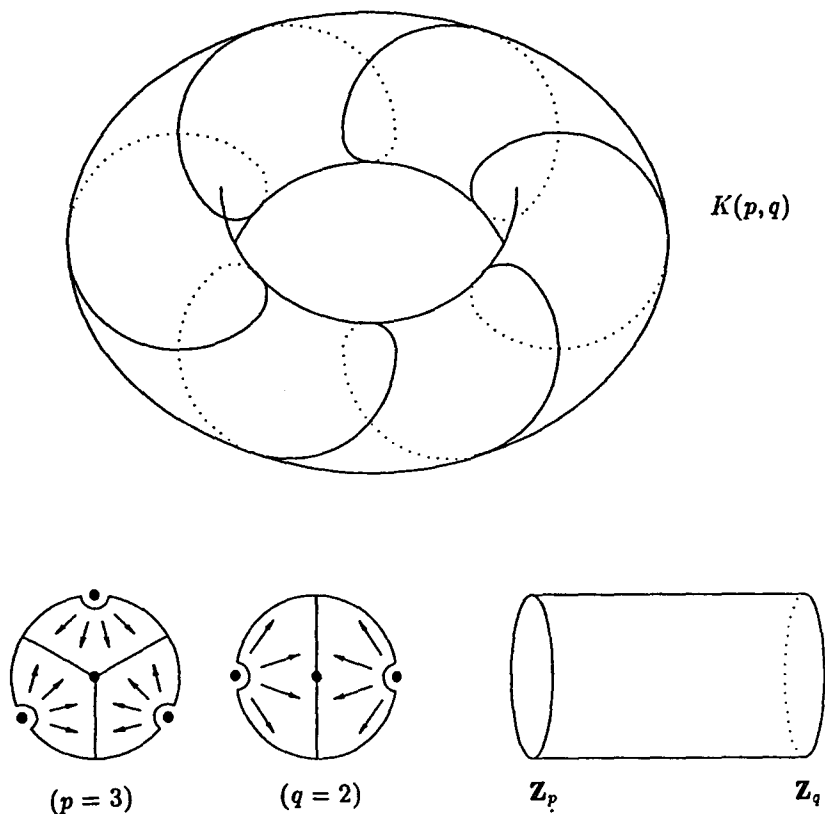


Figure 4.5. - The spine of a torus knot space.

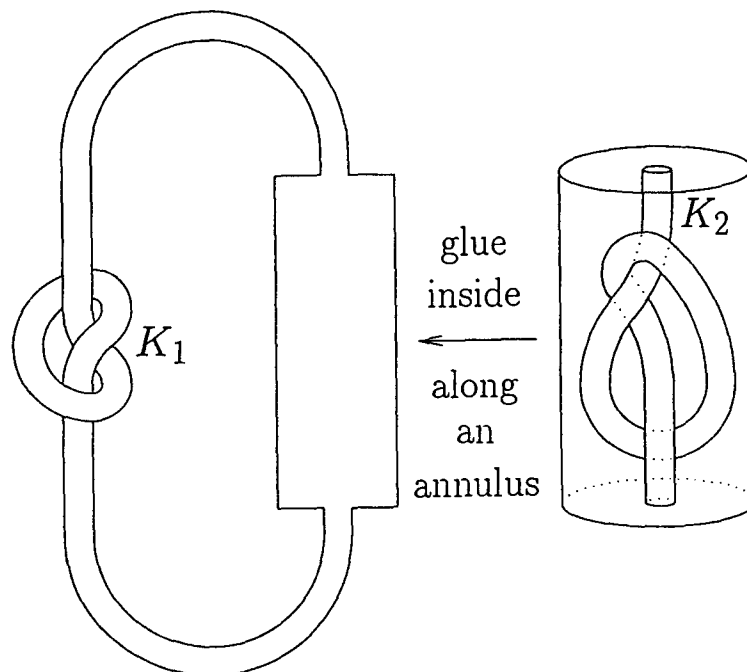
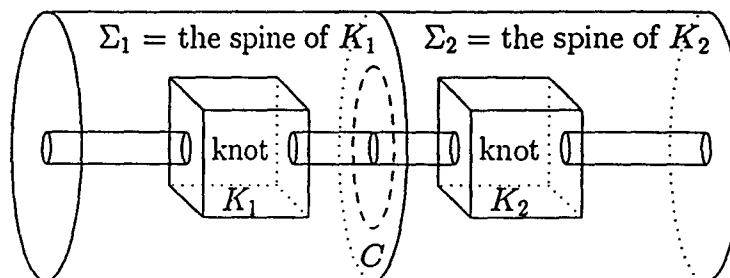


Figure 4.6. – The spine of the knot space of the composite knot $K_1 \# K_2$.

In particular, the respective knot spaces of the sums of knots $K(p, q) \# K(p, q)$ and $K(p, q) \# K(p, -q)$ are not homeomorphic but they do possess homeomorphic spines. For this, it is necessary to first understand the spine of a torus knot space (see Figure 4.5). Secondly, under-



$C =$ the circle along which Σ_1 intersects Σ_2

Figure 4.7. – The spine Σ of $K_1 \# K_2$ is given by the gluing $\Sigma_1 \cup_C \Sigma_2$.

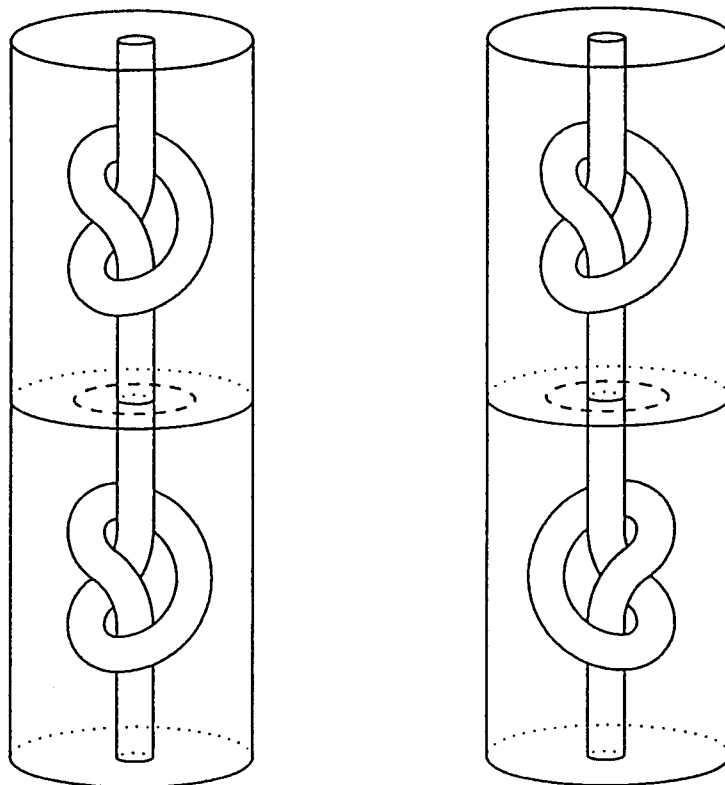


Figure 4.8. – The granny knot space and the square knot space.

stand the spine of the knot space of a connected sum of knots (see Figure 4.6). Therefore, the spine Σ of the knot space of $K_1 \# K_2$ is given by the glueing $\Sigma_1 \cup_C \Sigma_2$, where Σ_i is the spine of the knot space of K_i , and C is the circle along which Σ_1 intersects Σ_2 (see Figure 4.7). In Figure 4.8 we show the above constructions for the granny knot space and the square knot space.

CAVEAT. The compact 3-manifolds, shown in Figure 4.9, are not homeomorphic, but however, after drilling a hole in each one of them, they become homeomorphic (see Figure 4.10). So a simple idea of getting higher genera examples by drilling holes into the nonhomeomorphic knot spaces described above does not work since they may no longer be different 3-manifolds after the drilling is over. Stronger tools are needed and we shall discuss them below.

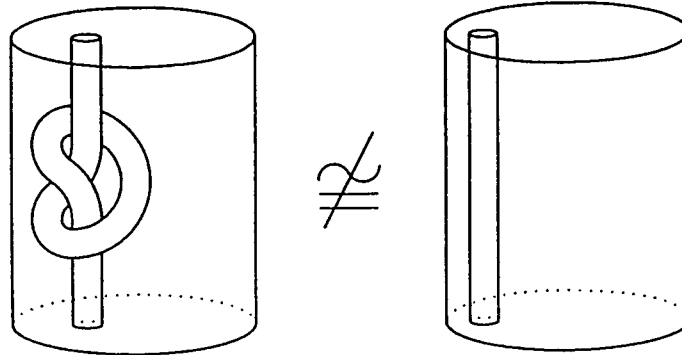


Figure 4.9. – Two non-homeomorphic 3-manifolds.

Now we recall some standard definitions of knot theory. Let K be an oriented knot in the right-hand oriented 3-sphere S^3 . Then \bar{K} denotes the image of K under an orientation reversing homeomorphism of S^3 , i.e. \bar{K} can be seen as the mirror image of K . The *inverted knot*, written rK , is the knot K with the reversed orientation. The knot K is called *invertible* if $K = rK$ (here $=$ denotes the *equivalence* of knots). The knot K is called *amphicheiral* if $K = \bar{K}$. The knot K is called *simple* if the knot space X of K is atoroidal, i.e. every incompressible torus is boundary parallel. For example, the trefoil knot is invertible but it is not amphicheiral (see Figure 4.11).

The following result on spines of knot manifolds was proved in [12].

THEOREM 4.5. *Suppose that η is a Wirtinger presentation (with deficiency one) of the knot group of K . Then the two-dimensional complex*

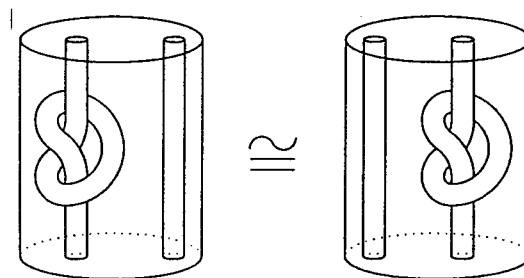


Figure 4.10. – The homeomorphic 3-manifolds obtained by drilling a hole in each one of manifolds in Figure 4.9.

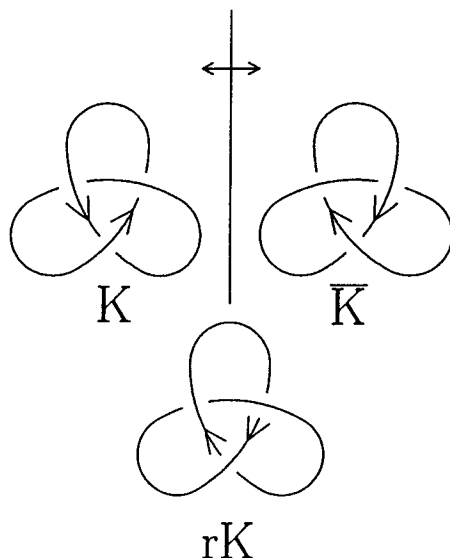


Figure 4.11. – The trefoil knot is invertible but not amphicheiral.

K_η (with one vertex), canonically associated with η , is a spine of the knot manifold of K .

Theorem 4.5 directly implies the existence of many examples of nonhomeomorphic compact 3-manifolds $M_1, M_2 \subset S^3$ with $\partial M_i \cong S^1 \times S^1$ which admit the same spine. The following result, due to Cavicchioli and Hegenbarth [12], extends earlier constructions on connected sums of torus knots, given in [48].

THEOREM 4.6. *Let $K_1 \subset S^3$ be an invertible nonamphicheiral knot and let $K_2 \subset S^3$ be an arbitrary knot. Then the knot manifolds of $K_1 \# K_2$ and $\bar{K}_1 \# K_2$ (\bar{K}_1 being the mirror-image of K_1) have a common spine. If K_2 is also nonamphicheiral, then these knot manifolds are not homeomorphic.*

In particular, for any invertible nonamphicheiral knot K , the knot manifolds of $K \# K$ and $\bar{K} \# K$ are not homeomorphic but they do possess a common spine.

To construct examples with boundary genus greater than one, we need the concept of θ -manifold, introduced in [15]. Let $\theta(K_1, K_2, K_3)$ be the oriented θ -curve, embedded in S^3 , and formed by two points joined

with three arcs knotted according to the oriented knots $K_1, K_2,$ and $K_3,$ respectively. Let $M(K_1, K_2, K_3)$ be the closure of the complement of a regular neighborhood of this graph in S^3 . Then $M = M(K_1, K_2, K_3)$ is an irreducible 3-manifold, with boundary of genus two, called a θ -manifold. It is also oriented with orientation induced by the one on S^3 . Obviously, this construction can be generalized to give manifolds with higher boundary genus.

From Jaco–Shalen–Johannson theory of *characteristic* varieties we apply the *Torus Decomposition Theorem* for compact irreducible 3-manifolds: there exists a collection of incompressible tori in M^3 which separate M into atoroidal or Seifert fibered pieces. Such a collection is minimal and unique. These decomposing tori for the θ -manifold $M(K_1, K_2, K_3)$ are three annuli, T_1, T_2 and T_3 say, which run around the knotted parts of the θ -curve $\theta(K_1, K_2, K_3)$. Tori $T_1, T_2,$ and T_3 separate $M(K_1, K_2, K_3)$ into four components: X_1, X_2, X_3 are the exteriors of the knots $K_1, K_2, K_3,$ respectively, and X_4 is a genus 2 orientable handlebody minus the three standard unknotted solid tori. Clearly, $X_1, X_2, X_3,$ and X_4 are atoroidal. Also, X_4 is not Seifert fibered since it has a genus 2 boundary component.

PROBLEM 4.2. Do there exist compact connected non-homeomorphic 3-manifolds M_1 and $M_2 \subset S^3$ such that $\partial M_1 \cong \partial M_2,$ genus $\geq 2,$ and $\pi_1(M_1) \cong \pi_1(M_2)$ is not a nontrivial free product, but M_1 and M_2 nevertheless possess the same spine?

An affirmative answer to problem above was given in [15]:

THEOREM 4.7. *Let K be any simple oriented knot in the oriented 3-sphere S^3 . Let M_1, M_2, M_3 and M_4 be the following θ -manifolds:*

- (i) $M_1 = M(K, K, K)$
- (ii) $M_2 = M(K, K, r\bar{K})$
- (iii) $M_3 = M(K, r\bar{K}, r\bar{K})$
- (iv) $M_4 = M(r\bar{K}, r\bar{K}, r\bar{K}).$

If $K = r\bar{K},$ then these manifolds are trivially all homeomorphic.

If $K \neq r\bar{K},$ then $M_4 \neq M_2 \neq M_1 \neq M_3,$ and $M_1 \cong M_4$ and $M_2 \cong M_3$ if and only if $K = \bar{K}$ (up to an orientation preserving homeomorphism). If orientation reversing homeomorphism is also permitted, then $M_1 \cong M_4 \neq M_2 \cong M_3.$

In particular, if K is the trefoil knot, then the θ -manifolds $M(K, K, K)$ and $M(K, K, \bar{K})$ are not homeomorphic since K is invertible but nonamphicheiral. However, it was proved in [15] that these 3-manifolds have the same spine. They are also irreducible, have incompressible boundaries and hence their fundamental groups are not nontrivial free products (see [15]).

THEOREM 4.8. *Let K_1, K_2, K_3 and K'_1, K'_2, K'_3 be any oriented simple knots in the oriented 3-sphere S^3 and let $M(K_1, K_2, K_3)$ and $M(K'_1, K'_2, K'_3)$ be the corresponding θ -manifolds. Suppose that*

$$h : M(K_1, K_2, K_3) \rightarrow M(K'_1, K'_2, K'_3)$$

is an orientation preserving homeomorphism.

Then either $\{K_1, K_2, K_3\} = \{K'_1, K'_2, K'_3\}$ or $\{K_1, K_2, K_3\} = \{rK'_1, rK'_2, rK'_3\}$.

Remarks:

- (1) The converse is evidently true.
- (2) Theorem 4.8 generalizes to n knots ($n > 3$).
- (3) Theorem 4.8 is also true for nonsimple knots.

Theorem 4.8 provides the answer to Problem 4.2.

Now we repeat the proof of Theorem 4.8 given in [15]. By the uniqueness of the Torus theorem we may assume that $h(T_i) = T'_j$, for some j (after an ambient isotopy if necessary), where T'_1, T'_2 and T'_3 are the decomposing tori for the θ -manifold $M(K'_1, K'_2, K'_3)$. By reindexing we get $h(T_i) = T'_i$. Then we also have $h(X_i) = X'_i$, where X'_i is the exterior of K'_i in $M(K'_1, K'_2, K'_3)$. By a theorem of Gordon and Luecke, knots are determined by their complements. Therefore, the (unoriented) meridian and longitude of a knot complement are well-defined. Thus h maps the meridian of X_i to that of X'_i , $i = 1, 2, 3$, either preserving or reversing orientation. However the oriented meridians of X_1, X_2 and X_3 represent classes of the first integral homology group of $M(K_1, K_2, K_3)$ that add up to zero and satisfy no other relation. The same holds for meridians of X'_1, X'_2 and X'_3 . Therefore, if h reverses the orientation of one meridian, then it must do so with all meridians. Since h preserves the orientation and reverses the meridian, it must reverse also the corresponding longitude. Hence, either $K_i = K'_i$ or $K_i = rK'_i$ for each $i = 1, 2, 3$. This completes the proof.

THEOREM 4.9. *Let K_1, K_2 and K_3 be oriented knots in the oriented 3-sphere S^3 . Then the θ -manifolds $M(K_1, K_2, K_3)$ and $M(K_1, K_2, r\bar{K}_3)$ have the same spine.*

Following [15], we present the proof of this theorem. Let U denote the unknot in S^3 and, as usual, let X_3 be the exterior of K_3 . Then $M(K_1, K_2, K_3)$ can be viewed as $M(K_1, K_2, U) \cup X_3$, and $M(K_1, K_2, U) \cap X_3$ is an annulus in the boundary of each part. We can collapse this to have $M(K_1, K_2, U) \cap X_3$ equal to a simple closed curve δ , a meridian of X_3 , plus the core of the annulus (in the boundary of each). Similarly, let X'_3 be the exterior of $r\bar{K}_3$. Then $M(K_1, K_2, r\bar{K}_3)$ collapses to a copy of $M(K_1, K_2, U) \cup X'_3$ with $M(K_1, K_2, U) \cap X'_3$ being the same simple closed curve δ in $\partial M(K_1, K_2, U)$. Now, there exists a homeomorphism $X_3 \rightarrow X'_3$ which is the identity on δ (it is the reflection across the plane of δ). Extend it, by identity, to a homeomorphism

$$M(K_1, K_2, U) \cup X_3 \rightarrow M(K_1, K_2, U) \cup X'_3.$$

Collapse further to get a 2-dimensional polyhedral spine. Thus the proof is complete.

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