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TOPOLOGICAL PROPERTIES OF CYCLICALLY PRESENTED GROUPS

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ABSTRACT

We introduce a family of cyclic presentations of groups depending on a finite set of integers. This family contains many classes of cyclic presentations of groups, previously considered by several authors. We prove that, under certain conditions on the parameters, the groups defined by our presentations cannot be fundamental groups of closed connected hyperbolic 3–dimensional orbifolds (in particular, manifolds) of finite volume. We also study the split extensions and the natural HNN extensions of these groups, and determine conditions on the parameters for which they are groups of 3–orbifolds and high–dimensional knots, respectively.

Keywords: Manifold, orbifold, branched covering, knot, homology, cyclically presented group, split extension, natural HNN extension, high–dimensional knot group, hyperbolic structure, isometry group.

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1. Cyclically Presented Groups

Let F_n be the free group on n free (ordered) generators $x_1, \dots, x_n$. Let θ denote the automorphism of F_n defined by setting $\theta(x_i) = x_{i+1}$, where the subscripts are reduced modulo n. For any cyclically reduced word w in F_n, let us consider the normal closure $N_n(w)$ in F_n of the set $\{w, \theta(w), \dots, \theta^{n-1}(w)\}$, and let $G_n(w)$ denote the factor group $F_n/N_n(w)$. Of course, $G_n(w)$ admits a balanced presentation with n (ordered) generators, also denoted by $x_1, \dots, x_n$, and n relators obtained from the single word w by all cyclic permutations of the generators. Following [23], we say that a group G has a *cyclic presentation* (or, equivalently, G is a *cyclically presented group*) if G is isomorphic to $G_n(w)$ for some n and w. There is a *polynomial* $f_w(t)$ associated with $G_n(w)$, whose i–th coefficient is the exponent sum of the generator x_i in the defining word w. This polynomial is very useful to determine when the abelianized group $A_n(w)$ of $G_n(w)$ is finite or not. It was proved in [24] (see also [23]) that $A_n(w)$ is infinite if and only if $f_w(t)$ vanishes on an n–th root of unity. If $A_n(w)$ is finite, then its order is the module of the product of all complex numbers $f_w(\xi)$' s, where ξ runs through the set of (primitive) n–th roots of unity. Furthermore, $A_n(w)$ is trivial if and only if $f_w(t)$ is a unit in the ring $\mathbb{Z}[t]/(t^n - 1)$. The automorphism θ of F_n naturally induces an automorphism ρ of order n on $G_n(w)$. Let $H_n(w)$ denote the *split extension group* of $G_n(w)$ by the cyclic group $\mathbb{Z}_n$ (generated by ρ). Then $H_n(w)$ has a finite presentation with two generators, x_1 and ρ say, and two relations of the form $\rho^n = 1$ and $v(\rho, x_1) = 1$. This relation is obtained from the defining word w substituting any generator x_i with the formula $x_i = \rho^{i-1}x_1\rho^{-(i-1)}$ (indices mod n). It is known that $G_n(w)$ is trivial if and only if $H_n(w)$ is cyclic of order n, and that $G_n(w)$ is finite if and only if $H_n(w)$ is finite (see the references above). Cyclically presented groups are very interesting from a geometric point of view since there are many connections with the theory of closed connected 3–manifolds. It is an open problem to determine which cyclic presentations of groups correspond to spines of closed 3–manifolds, and, in particular, for which classes of knots the cyclic branched coverings give rise to such presentations (for knot theory we refer for example to [3] and [32]). Another question is to establish when a cyclically presented group cannot be the fundamental group of a closed hyperbolic 3–orbifold (in particular, 3–manifold) of finite volume. In the present paper we discuss the above questions for a new family of cyclic presentations of groups, which depends on a finite set of positive integers. This family contains many classes of cyclic presentations which have appeared in recent years. So, our results generalize and extend earlier work due to Maclachlan, Szczepański, Vesnin and others. We also study the split extensions and the natural HNN extensions of the cyclically presented groups defined by presentations of our family, and determine conditions on the parameters for which they are 3–orbifold groups and high–dimensional knot groups, respectively.

2. Some Series of Cyclic Presentations

We introduce the following family of cyclic presentations of groups depending on positive integers:

$$G_n(h,k;p,q;r,s;\ell) = G_n\Big(\big(\prod_{j=0}^{r-1} x_{1+jp}\big)^{\ell} \big(\prod_{j=0}^{s-1} x_{1+h+jq}\big)^{-k}\Big)$$

$$=< x_1,\ldots,x_n : \quad (x_i x_{i+p} \cdots x_{i+p(r-1)})^{\ell} = (x_{i+h} \cdots x_{i+h+q(s-1)})^{k} \quad (i=1,\ldots,n) \quad >$$

where the subscripts are reduced modulo n, and $r \geq 2$.

This family contains many classes of cyclic presentations of groups, previously considered by several authors.

(1) The groups

$$G_n(2,1;1,1;2,1;1) = G_n(x_1 x_2 x_3^{-1})$$

are called the *Fibonacci groups* $F(2,n)$ introduced in [15] (see also [16]). The group $F(2,2m)$, $m \geq 2$, is the fundamental group of the m–fold cyclic covering of the 3–sphere branched over the figure–eight knot, as proved in [21] (a surgery description of these manifolds can be found in [14]). For any $m \geq 4$, the group $F(2,2m)$ is hyperbolic, i.e., it can be regarded as a discontinuous subgroup of PSL$(2;\mathbb{C})$, the group of orientation–preserving isometries of the hyperbolic 3–space $\mathbb{H}^3$ (see [20] for the proof). The Fibonacci groups with odd number of generators ($n \geq 3$) cannot be fundamental groups of closed hyperbolic 3–orbifolds (in particular, 3–manifolds) of finite volume (see [26]). Recall that a *hyperbolic 3–orbifold* is a quotient space of the form $\mathbb{H}^3/\Gamma$, where Γ is a discrete group of isometries of $\mathbb{H}^3$ (in particular, if Γ is torsion–free, we obtain the notion of *hyperbolic 3–manifold*). For the theory of geometric orbifolds see for example [18] and [42].

(2) The groups

$$G_n(1,1;2,2;r,r-1;1) = G_n\Big(\big(\prod_{j=0}^{r-1} x_{2j+1}\big)\big(\prod_{j=0}^{r-2} x_{2j+2}\big)^{-1}\Big)$$

are called the *generalized Sieradski groups* $S(r,n)$, $r \geq 2$, $n \geq 2$, introduced and geometrically studied in [7]. For $r = 2$, we get the so-called *Sieradski groups* $S(n) = G_n(x_1 x_3 x_2^{-1})$, first considered in [35]. The group $S(r,n)$ is the fundamental group of the n–fold cyclic covering of the 3–sphere branched over the torus knot of type $(2r-1,2)$, as shown in [7] (see also [8] for cyclic branched coverings of torus knots). This manifold is homeomorphic to the Brieskorn manifold $M(2,2r-1,n)$ in the sense of [28]. Thus the groups $S(r,n)$ are non–hyperbolic.

(3) The groups

$$G_n(r,1;1,1;r,1;1) = G_n(x_1x_2\cdots x_rx_{1+r}^{-1})$$

were introduced in [25] (and denoted by $F(r,n)$, $r \geq 2$, $n \geq 3$) as a natural generalization of the Fibonacci groups $F(2,n)$ (and in fact they are called by the same name in the current literature). The groups $F(n-1,n)$, $n \geq 3$, are the fundamental groups of closed connected orientable 3–manifolds (see [29]). These manifolds are known to be homeomorphic to Seifert fibered spaces (see [6]), and so the groups $F(n-1,n)$ are non-hyperbolic. If r is even ($r \geq 4$) and n is odd and coprime with $r+1$, then $F(r,n)$ cannot be the fundamental group of a closed hyperbolic 3–orbifold of finite volume (see [36]).

(4) The groups

$$G_n(r+k-1,1;1,1;r,1;1) = G_n(x_1x_2\cdots x_rx_{r+k}^{-1})$$

and

$$G_n(r,1;1,1;r,k;1) = G_n((x_1x_2\cdots x_r)(x_{1+r}\cdots x_{r+k})^{-1})$$

were defined and algebraically studied in [4] (and denoted by $F(r,n,k)$ and $H(r,n,k)$, respectively) for any $r \geq 2$, $n \geq 3$, and $k \geq 1$. Obviously, they represent further generalizations of the Fibonacci groups $F(r,n)$. If r is even and n is odd and coprime with $r+2k-1$, then $F(r,n,k)$ cannot be the fundamental group of a hyperbolic 3–orbifold of finite volume (see [39] for the proof). Analogous results have also been obtained in that paper for two extremal cases with $n = r+2k-1$, i.e., $k=1$ and $r=2$. For any $k \geq 2$, the groups $H(k,2k-1,k-1)$ were proved to be isomorphic to generalized Sieradski groups $S(k,2k-1)$, so they are non-hyperbolic (see [39]).

(5) The groups

$$G_n(s,c;1,1;r,1;1) = G_n(x_1x_2\cdots x_rx_{1+s}^{-c})$$

were considered in [24] (and denoted by $F(r,s,c,n)$). Of course, they are further generalizations of the groups $F(r,n)$ and $F(r,n,k)$. In [24] the authors determined necessary and sufficient conditions under which the abelianized group of $F(r,s,c,n)$ is finite. These conditions will be extended for more general groups of Fibonacci type in Section 5, in order to study connections between cyclic presentations and higher dimensional knot groups. The family $F(r,s,c,n)$ of cyclically presented groups contains the *generalized Neuwirth groups*

$$\Gamma_n^k = G_n(x_1x_2\cdots x_{n-1}x_n^{-k}) = F(n-1,n-1,k,n)$$

which were discussed and geometrically studied in [40]. The groups Γ_n^k are fundamental groups of closed connected orientable 3–manifolds $M_n(k)$ which are homeomorphic to Seifert fibered spaces (see [40]). Moreover, the Seifert invariants of these fibered spaces are completely determined in the quoted paper. As a consequence, the groups $\Gamma_n^k = F(n-1,n-1,k,n)$ are non-hyperbolic.

(6) The groups

$$G_n(k-1,1;q,q;r,s;1) = G_n((\prod_{j=0}^{r-1} x_{1+jq})(\prod_{j=0}^{s-1} x_{k+jq})^{-1})$$

were defined in [31] (and denoted by $P(r,n,k,s,q)$). These groups contain for example the generalized Sieradski groups, which are non-hyperbolic. In [31] the author studied the asphericity and the atoricity for this class of groups and some other classes of symmetrically presented groups.

(7) The groups

$$G_n(k,1;h,1;2,1;1) = G_n(x_1 x_{1+h} x_{1+k}^{-1})$$

were introduced in [9] (and denoted by $G_n(h,k)$). They are natural generalizations of the Gilbert–Howie groups $H(n,h)$ defined in [17] as

$$H(n,h) = G_n(x_1 x_{1+h} x_2^{-1}) = G_n(h,1).$$

Obviously, we have $G_n(2,1) = S(n)$ and $G_n(1,2) = F(2,n)$. Some algebraic and topological properties of the groups $G_n(h,k)$ will appear in a forthcoming paper of Bardakov and Vesnin (as announced by the second author during a talk held at the University of Modena in March 2001). In particular, they will prove that if n is odd and coprime with $h-2k$, and $h-k$ is even, then $G_n(h,k)$ cannot be the fundamental group of a closed connected hyperbolic 3–orbifold of finite volume.

(8) The groups

$$G_n(n-1,k;1,1;n,1;\ell) = G_n((x_1 x_2 \cdots x_n)^\ell x_n^{-k})$$

were defined and geometrically studied in [34] (and denoted by $\Gamma(k,\ldots,k;\ell)$ for any $\ell \geq 1$ and $k \geq 2$). The authors proved that the groups $\Gamma(k,\ldots,k;\ell)$ (and some further generalizations of them) are fundamental groups of closed connected orientable 3–manifolds $M_n(k;\ell)$. The class of manifolds of type $M_n(k;\ell)$ contains the generalized Neuwirth manifolds discussed in (5) which are homeomorphic to $M_n(k+1;1)$. As proved in [34], the manifold $M_n(k;\ell)$ is homeomorphic to the Seifert fibered 3-manifold $\Sigma(k;\ell)$ defined by the Seifert invariants

$$\Sigma(k;\ell) = (O\,o\,O : -1 \quad \underbrace{(k,1)\cdots(k,1)}_{n \quad \text{times}} \quad (\ell,\ell-1)).$$

In particular, these groups are non-hyperbolic. We consider again the fibered manifolds $M_n(k;\ell)$ in Section 4, and prove that they are homeomorphic to n-fold cyclic coverings of lens spaces (in particular, the 3–sphere whenever a certain condition on the parameters is satisfied) branched over genus one 1-bridge knots. The proof is based on algebraic and geometric arguments and involves the split extension group of $\Gamma(k,\ldots,k;\ell)$.

3. Non-hyperbolic Groups

In this section we shall apply some techniques developed in [26], [36], and [39] to the groups arising from our family of cyclic presentations. This allows us to prove that, under certain conditions on the parameters, these groups are non-hyperbolic. More precisely, we obtain the following result:

Theorem 3.1. *Suppose that the number n ($n \geq 3$) of generators is odd and coprime with $2h + q(s-1) - p(r-1)$, and that one of the following conditions is satisfied:*

i) *either r or ℓ is even, and k and s are odd; or*
ii) *either s or k is even, and ℓ and r are odd.*

Then the cyclically presented group

$$G_n(h,k;p,q;r,s;\ell) = G_n((\prod_{j=0}^{r-1} x_{1+jp})^{\ell} \ (\prod_{j=0}^{s-1} x_{1+h+jq})^{-k})$$

cannot be the fundamental group of a closed connected hyperbolic 3–orbifold (in particular, 3–manifold) of finite volume.

Proof. Suppose conversely that $G_n = G_n(h,k;p,q;r,s;\ell)$ is an hyperbolic group, and let $\mathrm{Isom}(\mathbb{H}^3)$ denote the group of isometries of the hyperbolic 3–space $\mathbb{H}^3$. Then there is a faithful representation γ from G_n into $\mathrm{Isom}(\mathbb{H}^3)$ such that the image $\Gamma_n = \gamma(G_n)$ is a discrete group of finite covolume. Of course, Γ_n admits an automorphism α of order n which cyclically permutes the generators, also denoted by $x_1, \dots, x_n$. By the Mostow rigidity theorem (see for example [2] and [42]) there exists an isometry $t \in \mathrm{Isom}(\mathbb{H}^3)$ such that $\alpha(g) = tgt^{-1}$ for any element $g \in \Gamma_n$. Let H_n be the split extension group of Γ_n by the cyclic group generated by the isometry t. Then H_n is isomorphic to the fundamental group of a closed hyperbolic 3–orbifold of finite volume. Since α has order n, it follows that t^n commutes with all elements of Γ_n, i.e., t^n belongs to the center of Γ_n. This implies that $t^n = 1$ because every non-elementary Kleinian group has trivial center (see for example [1]). Thus we obtain $t^n = 1$, hence t is of order n_1, where n_1 divides n. Now we substitute the relation

$$x_{i+1} = t^i x_1 t^{-i}$$

into the initial relation of Γ_n:

$$(x_1 x_{1+p} \cdots x_{1+p(r-1)})^{\ell} = (x_{1+h} \cdots x_{1+h+q(s-1)})^{k}.$$

Then we get the relation

$$((x_1\, t^p)^r\, t^{-pr})^{\ell} = (t^h\, (x_1\, t^q)^s\, t^{-h-qs})^k.$$

Of course, H_n has the finite presentation

$$H_n =< x_1, t: \quad t^{n_1} = 1, \quad ((x_1\, t^p)^r\, t^{-pr})^{\ell} = (t^h\, (x_1\, t^q)^s\, t^{-h-qs})^k > .$$

Let us consider the group generated by the finite products of the squares of the elements in H_n, i.e.,

$$H_n^{(2)} :=< \eta^2 \, : \, \eta \in H_n > .$$

Since n is odd, it follows that n_1 is odd, hence $t \in H_n^{(2)}$. If one of the conditions of the statement is satisfied, we get $x_1 \in H_n^{(2)}$ by using the second relation of H_n. Thus we have $H_n = H_n^{(2)}$, hence H_n is a subgroup of the group of orientation–preserving isometries of $\mathbb{H}^3$. Therefore we set $H_n < \mathrm{PSL}(2;\mathbb{C})$. Let us denote by PA the image in $\mathrm{PSL}(2;\mathbb{C})$ of a matrix $A \in \mathrm{SL}(2;\mathbb{C})$ under the 2–fold covering $P : \mathrm{SL}(2;\mathbb{C}) \to \mathrm{PSL}(2;\mathbb{C}) = \mathrm{SL}(2;\mathbb{C})/\{\pm I_2\}$. Since t is of order n_1, we can assume without loss of generality that

$$t = P\begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix}$$

where φ is a (primitive) root of unity (in $\mathbb{C}$) of order $2n_1$. Furthermore, we set

$$x_1 = P\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

where $xw - yz = 1$, and $yz \neq 0$ because Γ_n has finite covolume. Substituting the matrices in the second relation of H_n, we get

$$\left(\begin{pmatrix} x\varphi^p & y\varphi^{-p} \\ z\varphi^p & w\varphi^{-p} \end{pmatrix}^r \begin{pmatrix} \varphi^{-pr} & 0 \\ 0 & \varphi^{pr} \end{pmatrix}\right)^\ell$$

$$= \left(\begin{pmatrix} \varphi^h & 0 \\ 0 & \varphi^{-h} \end{pmatrix} \begin{pmatrix} x\varphi^q & y\varphi^{-q} \\ z\varphi^q & w\varphi^{-q} \end{pmatrix}^s \begin{pmatrix} \varphi^{-h-qs} & 0 \\ 0 & \varphi^{h+qs} \end{pmatrix}\right)^k .$$

It was proved in [39] by induction on j that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^j = \begin{pmatrix} S_j & bR_j \\ cR_j & T_j \end{pmatrix}$$

where

$$\begin{pmatrix} S_{j+1} \\ T_{j+1} \\ R_{j+1} \end{pmatrix} = \begin{pmatrix} a & 0 & bc \\ 0 & d & bc \\ 1 & 0 & d \end{pmatrix} \begin{pmatrix} S_j \\ T_j \\ R_j \end{pmatrix}$$

with

$$\begin{pmatrix} S_1 \\ T_1 \\ R_1 \end{pmatrix} = \begin{pmatrix} a \\ d \\ 1 \end{pmatrix} .$$

Applying this formula to our case, we get

$$\left(\begin{pmatrix} S_r & y\varphi^{-p}R_r \\ z\varphi^p R_r & T_r \end{pmatrix} \begin{pmatrix} \varphi^{-pr} & 0 \\ 0 & \varphi^{pr} \end{pmatrix}\right)^\ell$$

$$= \left(\begin{pmatrix} \varphi^h & 0 \\ 0 & \varphi^{-h} \end{pmatrix} \begin{pmatrix} S_s & y\varphi^{-q}R_s \\ z\varphi^q R_s & T_s \end{pmatrix} \begin{pmatrix} \varphi^{-h-qs} & 0 \\ 0 & \varphi^{h+qs} \end{pmatrix}\right)^k$$

hence

$$\begin{pmatrix} \varphi^{-pr}S_r & y\varphi^{pr-p}R_r \\ z\varphi^{p-pr}R_r & \varphi^{pr}T_r \end{pmatrix}^\ell = \begin{pmatrix} \varphi^{-qs}S_s & y\varphi^{2h+q(s-1)}R_s \\ z\varphi^{-2h-q(s-1)}R_s & \varphi^{qs}T_s \end{pmatrix}^k .$$

Finally, we have

$$\begin{pmatrix} S_\ell & y\varphi^{pr-p}\overline{R}_\ell \\ z\varphi^{p-pr}\overline{R}_\ell & T_\ell \end{pmatrix} = \begin{pmatrix} S_k & y\varphi^{2h+q(s-1)}\overline{R}_k \\ z\varphi^{-2h-q(s-1)}\overline{R}_k & T_k \end{pmatrix}$$

where $\overline{R}_\ell = R_r R_\ell$ and $\overline{R}_k = R_s R_k$. Therefore, we obtain

$$\begin{cases} y\varphi^{pr-p}\overline{R}_\ell = y\varphi^{2h+q(s-1)}\overline{R}_k \\ z\varphi^{p-pr}\overline{R}_\ell = z\varphi^{-2h-q(s-1)}\overline{R}_k \end{cases}$$

Since $yz \neq 0$, we get

$$\varphi^{2h+q(s-1)-p(r-1)} = \varphi^{-2h-q(s-1)+p(r-1)}$$

and so

$$\varphi^{2(2h+q(s-1)-p(r-1))} = 1.$$

Now φ is a primitive root of unity of order $2n_1$ and n (and hence n_1) is coprime with $2h+q(s-1)-p(r-1)$. This gives a contradiction, so the proof is completed.□

As one can easily verify, Theorem 3.1 includes all results on non-hyperbolicity of groups of Fibonacci type with an odd number of generators proved in [26], [36], and [39], and summarized in Section 2. Moreover, we obtain further consequences concerning some groups discussed in Section 2.

Corollary 3.2. *Suppose that $r+k$ (≥ 3) is odd, and n (≥ 3) is odd and coprime with $r+k$. Then the Campbell–Robertson group*

$$H(r,n,k) = G_n((x_1x_2\cdots x_r)(x_{1+r}\cdots x_{r+k})^{-1})$$

cannot be the fundamental group of a hyperbolic 3–orbifold of finite volume.

In particular, for $k=1$, we get the main theorem of [36] on non-hyperbolicity of the Fibonacci groups $F(r,n)$ with an odd number of generators.

Corollary 3.3. *Suppose that $c+r$ (≥ 3) is odd, and n (≥ 3) is odd and coprime with $2s-r+1$. Then the Johnson–Odoni group*

$$F(r,s,c,n) = G_n(x_1x_2\cdots x_r x_{1+s}^{-c})$$

cannot be the fundamental group of a hyperbolic 3–orbifold of finite volume.

In particular, for $c=1$ and $s=r+k-1$, we get the main theorem of [39] concerning with the non-hyperbolicity of the generalized Fibonacci groups $F(r,n,k)$ with an odd number of generators.

Corollary 3.4. *Suppose that $r+s$ (≥ 3) is odd, and n (≥ 3) is odd and coprime with $2(k-1)+q(s-r)$. Then the Prishchepov group*

$$P(r,n,k,s,q) = G_n((\prod_{j=0}^{r-1} x_{1+jq})(\prod_{j=0}^{s-1} x_{k+jq})^{-1})$$

cannot be the fundamental group of a hyperbolic 3–orbifold of finite volume.

Corollary 3.5. *Let n (≥ 3) be odd and coprime with $2k-h$. Then the group $G_n(h,k) = G_n(x_1 x_{1+h} x_{1+k}^{-1})$ cannot be the fundamental group of a hyperbolic 3–orbifold of finite volume.*

4. Split Extensions and Orbifold Groups

In this section we show that the split extensions of some groups discussed in Section 2 are very interesting from a topological point of view. In fact, they are fundamental groups of some 3–dimensional orbifolds. First, we consider the group $\Gamma(k,\dots,k;\ell)$, introduced in [34], (see (8), Section 2), and denote it by $\Gamma_n(k;\ell)$, i.e.,

$$\begin{aligned}\Gamma_n(k;\ell) &= G_n((x_1 x_2 \cdots x_n)^{\ell} x_n^{-k}) \\ &= \langle x_1,\dots,x_n : (x_i x_{i+1} \cdots x_{i+n-1})^{\ell} = x_{i+n-1}^k,\ i=1,\dots,n\rangle\end{aligned}$$

where $n \geq 2$, $k \geq 2$, and $\ell \geq 1$. In the case $\ell = 1$, the group $\Gamma_n(k+1;\ell)$ coincides with Γ_n^k of [40]. Let us denote by $H_n(k;\ell)$ the split extension group of $\Gamma_n(k;\ell)$ by $\mathbb{Z}_n = \langle t : t^n = 1\rangle$, where t is the automorphism given by $t(x_i) = x_{i+1}$ (subscripts mod n). From Section 3, $H_n(k;\ell)$ has a finite presentation with generators t and x and relations $t^n = 1$ and $v(t,x) = 1$, where

$$v(t,x) = \left((xt)^n t^{-n}\right)^{\ell} \left(t^{n-1}(xt)t^{-(n-1)-1}\right)^{-k}$$

as $p = q = s = 1$, $r = n$, and $h = n-1$. Therefore, $H_n(k;\ell)$ admits the following presentation

$$H_n(k;\ell) = \langle x,t \,:\, t^n = 1,\ (xt)^{n\ell} = t^{-1}x^k t\rangle.$$

Setting $y = xt$, we get the presentation

$$H_n(k;\ell) = \langle x,y \,:\, (x^{-1}y)^n = 1,\ y^{n\ell} = x^k\rangle.$$

Suppose now that $n\ell - k = \pm 1$. Observe that the group $\langle x,y : y^{n\ell} = x^k\rangle$ is the group of the torus knot $T(n\ell,k)$, and $x^{-1}y$ represents its meridian. Denote by $\mathcal{O}(T(n\ell,k),n)$ the orbifold whose underlying topological space is the 3–sphere $\mathbb{S}^3$ and whose singular set is the torus knot $T(n\ell,k)$ with singularity index n. Thus we have proved the following result.

Theorem 4.1. *If $n\ell - k = \pm 1$, then the split extension group $H_n(k;\ell)$ of $\Gamma_n(k;\ell) = G_n((x_1 x_2 \cdots x_n)^{\ell} x_n^{-k})$ is the fundamental group of the 3–dimensional fibered orbifold $\mathcal{O}(T(n\ell,k),n)$.*

As proved in [34], $\Gamma_n(k;\ell)$ is the fundamental group of the orientable Seifert fibered 3–manifold $M_n(k;\ell)$ (see (8), Section 2). The following result describes the manifolds $M_n(k;\ell)$ as cyclic branched coverings of lens spaces.

Theorem 4.2. *The Seifert fibered manifold $M_n(k;\ell)$ is an n–fold cyclic branched covering of the lens space $L(n\ell - k, 1)$ (in particular, $\mathbb{S}^1 \times \mathbb{S}^2$ when $n\ell = k$). If $n\ell = k \pm 1$, then $M_n(k;\ell)$ is the n–fold cyclic covering of the 3–sphere branched over the torus knot $T(n\ell,k)$, i.e., $M_n(k;\ell)$ is the Brieskorn manifold $M(n\ell,k,n)$ in the sense of* [28].

As a consequence of Theorem 4.2, the commutator group of the centrally extended triangle group [28] with parameters $(k \pm 1, k, n)$ admits a cyclic presentation. To prove Theorem 4.2, we briefly recall the combinatorial construction of $M_n(k;\ell)$, given in [34], by pairwise identification of oppositely oriented boundary faces of a polyhedral 3–ball B^3. Let $P_n(k;\ell)$ be the cellular decomposition of ∂B^3 shown in Figure 1 (case $n = \ell = 3$ and $k = 4$), where left and right boundaries are assumed to be identified. It consists of $2n$ faces labeled by A_i and $\overline{A}_i$, which are $(k+n\ell-2)$–gons. The labeling of the edges of $P_n(k;\ell)$ and their orientations are shown in Figure 1. Identifying A_i with $\overline{A}_i$ for any $i = 1, \ldots, n$ yields the manifold $M_n(k;\ell)$ (for details see [34]).

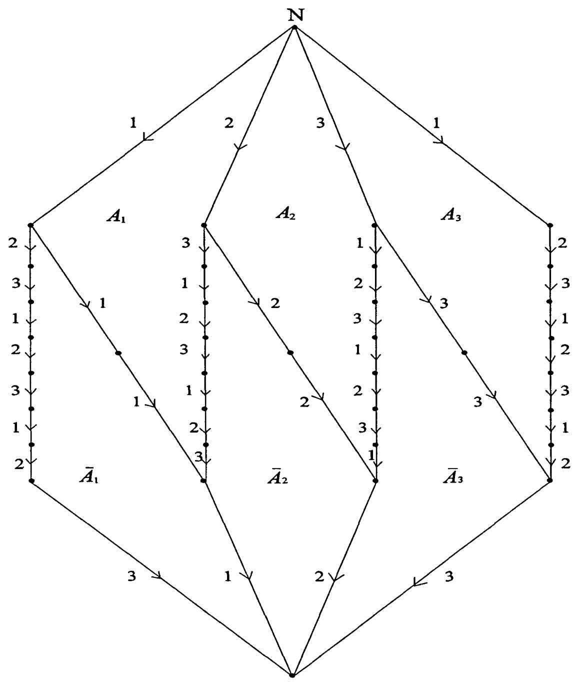
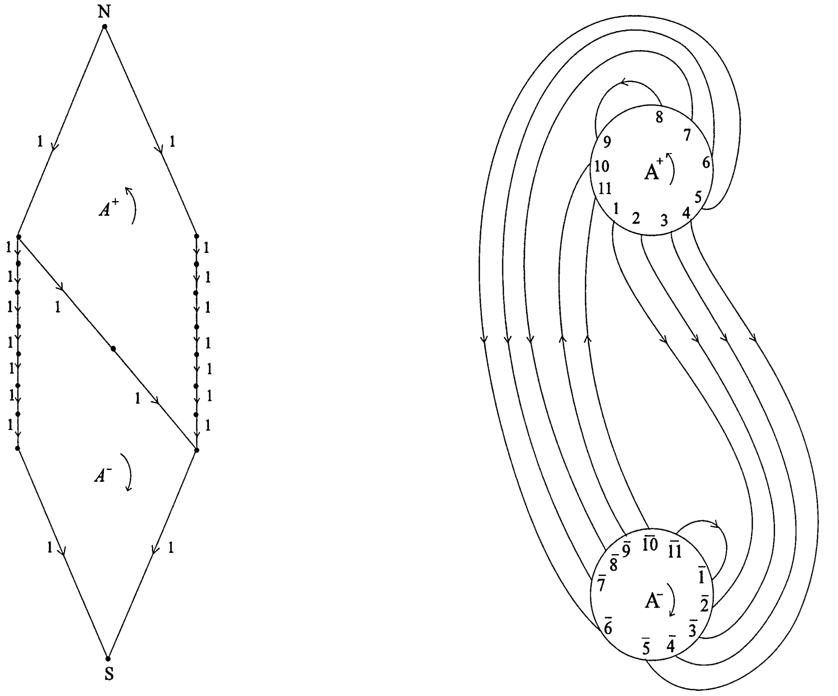


Figure 1. The polyhedron $P_n(k;\ell) = P_3(4;3)$

Rotation by an angle $2\pi/n$ about the north–south axis NS defines an action of the cyclic group $\mathbb{Z}_n$ on $M_n(k;\ell)$. The quotient space $M_n(k;\ell)/\mathbb{Z}_n$ is obtained by taking a fundamental domain $\Pi(k;\ell)$ for the action and making identifications (see Figure 2 (a)). A Heegaard diagram for this quotient space is pictured in Figure 2 (b). From the diagram one can easily see that the quotient is topologically the lens space $L(n\ell - k, 1)$. Note that $\Gamma_1(k;\ell) =< x : x^{n\ell-k} = 1 >\cong \mathbb{Z}_{|n\ell-k|}$.

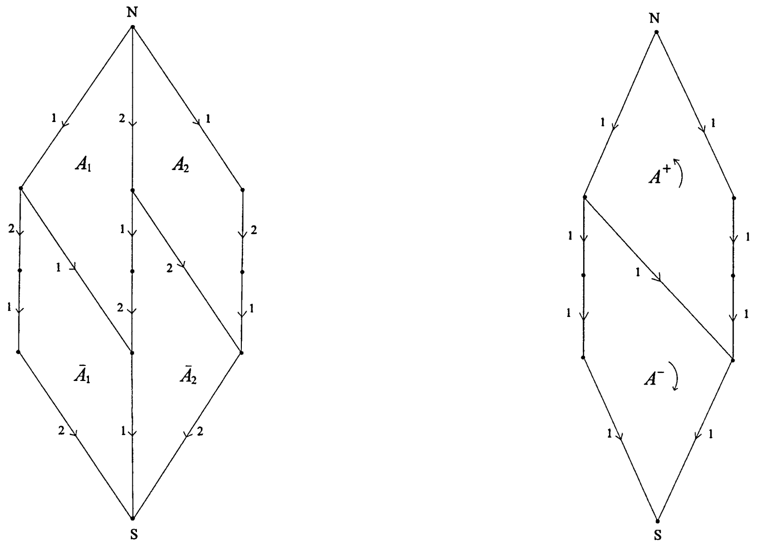


(a) The fundamental domain $\Pi(k;\ell)$ for the action of $\mathbb{Z}_n$ on $M_n(k;\ell)$ (case $k = 4$ and $\ell = 3$)

(b) A Heegaard diagram for the quotient $M_n(k;\ell)/\mathbb{Z}_n$ (case $k = 4$ and $\ell = 3$)

Figure 2

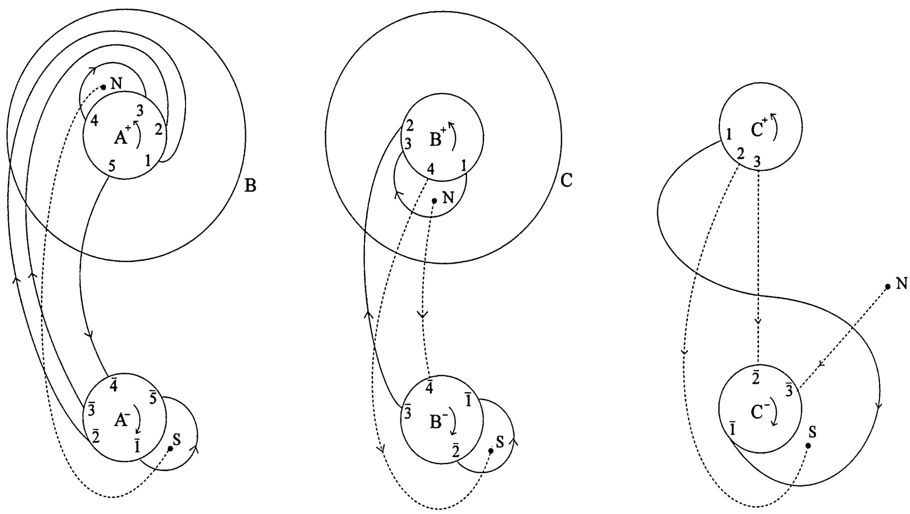
If $n\ell - k = \pm 1$, then $M_n(k;\ell)/\mathbb{Z}_n$ is homeomorphic to the 3–sphere (see Figures 3 and 4). The axis of the rotation is drawn in Figure 4 (a) as a dotted curve. It lies below the diagram, inside the ball whose boundary is being identified along the disc pair (A^+, A^-). Now we apply the method used in [22] for the figure–eight knot to our case, and modify Figure 4 (a) to Figure 5 (a) by simplifications along closed curves and cancellations of handles. Figure 4 (b) is obtained from Figure 4 (a) by a simplification along the closed simple curve B (also called the *Whitehead–Zieschang reduction*). Figure 4 (c) is obtained from Figure 4 (b) by a simplification along the closed simple curve C. It is also a Heegaard diagram for the quotient space, that is the 3–sphere. Figure 5 (a) is obtained from Figure 4 (c) by a cancellation of handles. By using Reidemeister moves, it is easy to see that the knot pictured in Figure 5 (a) is the torus knot $T(n\ell, k) = T(4, 3)$. This completes the proof of Theorem 4.2. $\square$



(a) The polyhedron $P_n(k;\ell) = P_2(3;2)$

(b) The fundamental domain $\Pi(k;\ell) = \Pi(3;2)$

Figure 3



(a) (b) (c)

Figure 4. A Heegaard diagram for the quotient $M_n(k;\ell)/\mathbb{Z}_n \cong \mathbb{S}^3$ (case $k = 3$ and $\ell = 2$)

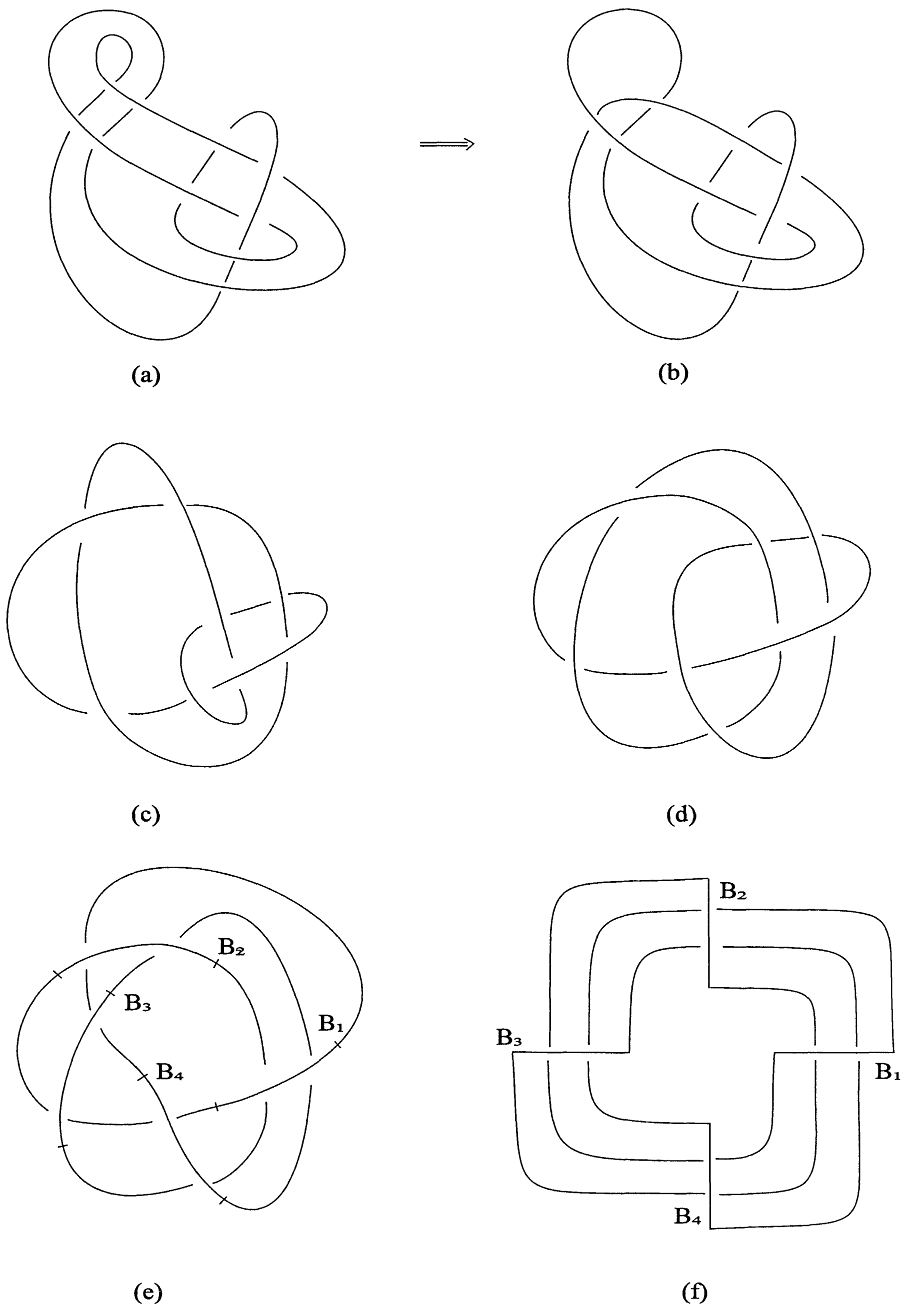


Figure 5. A sequence of Reidemeister moves yielding the torus knot $T(n\ell, k) = T(4,3)$

Now we consider the Prishchepov group

$$P(r,n,h+1,s,p) = G_n(h,1;p,p;r,s;1) = G_n((\prod_{j=0}^{r-1} x_{1+jp})\ (\prod_{j=0}^{s-1} x_{1+h+jp})^{-1})$$

where $n \geq 2$, $r > s \geq 1$, and p, $h \geq 1$. Recall that this class of cyclic presentations contains in particular the generalized Sieradski groups. Suppose that $h \equiv \pm 1$ (mod n), and $p(r-s) \equiv 2h$ (mod n). Let $H_n = H_n(r,n,h+1,s,p)$ be the split extension group of $P(r,n,h+1,s,p)$ by $\mathbb{Z}_n =< t : t^n = 1 >$, where t is the automorphism given by $t(x_i) = x_{i+1}$ (subscripts mod n). By Section 3, H_n has a finite presentation with generators t and x, and relations $t^n = 1$ and

$$(xt^p)^r t^{-pr} = t^h (xt^p)^s t^{-h-ps}$$

as $k = \ell = 1$ and $p = q$. The algebraic conditions on the parameters imply that H_n admits the following presentation

$$\begin{aligned} H_n &=< x,t \ :\ t^n = 1, \quad (xt^p)^r = t^h (xt^p)^s t^{-h} t^{p(r-s)} > \\ &=< x,t \ :\ t^n = 1, \quad (xt^p)^r = t^h (xt^p)^s t^h > \\ &=< x,t \ :\ t^n = 1, \quad (xt^p)^{r+s} = ((xt^p)^s t^{\pm 1})^2 > . \end{aligned}$$

We set $y = xt^p$ and $z = (xt^p)^s t^{\pm 1}$. These relations are invertible, i.e., we can express x and t in terms of y and z. In fact, we have $t^{\pm 1} = y^{-s} z$ and $x = y(z^{-1} y^s)^{\pm p}$. So the group H_n admits the finite presentation

$$H_n =< y,z \ :\ (y^{-s} z)^n = 1, \quad y^{r+s} = z^2 > .$$

Suppose moreover that $r+s$ is odd. Recall that the group of the torus knot $T(\alpha,\beta)$ has the finite presentation $< y,z : y^\alpha = z^\beta >$. Since α and β are coprime, there exist integers ξ and η such that $\alpha\xi - \beta\eta = 1$. We can always choose the path $\mathbf{m} = y^{-\eta} z^\xi$ as a meridian of $T(\alpha,\beta)$. In our case, we have $\alpha = r+s$, $\beta = 2$, $\xi = 1$, and $\eta = (r+s-1)/2$, so we get $\mathbf{m} = y^{-(r+s-1)/2} z$. Let us denote by $\mathcal{O}(T(r+s,2),n)$ the 3–dimensional orbifold whose underlying space is the 3–sphere and whose singular set is the torus knot $T(r+s,2)$ with singularity index n. We have proved the following result.

Theorem 4.3. *If $h \equiv \pm 1$ (mod n), $p(r-s) \equiv 2h$ (mod n), and $r+s$ is odd, then the split extension of the Prishchepov group $P(r,n,h+1,s,p)$ is the fundamental group of the fibered 3–orbifold $\mathcal{O}(T(r+s,2),n)$.* □

If $h \equiv \pm 1$ (mod n), $p(r-s) \equiv 2h$ (mod n), $\gcd(r,s) = 1$ and $r+s$ is odd, then the Prishchepov group $P(r,n,h+1,s,p)$ ($n \geq 2$, $r > s \geq 1$, and h, $p \geq 1$) is the fundamental group of a closed connected orientable 3–manifold which we denote by $M_n = M(r,n,h+1,s,p)$. To construct M_n we use the face identification procedure. Let us consider a tessellation $Q_n = Q(r,n,h+1,s,p)$ on the boundary of a 3–ball, which consists of n $(r+s)$–gons R_i in the northern hemisphere and n $(r+s)$–gons $\overline{R}_i$ in the southern hemisphere. The edge labels and their orientations are obtained as depicted in Figure 6 for the tessellation $Q(8,4,2,3,2)$. For each i the boundary cycle of the $(r+s)$–gons R_i and $\overline{R}_i$ is given by the sequence

$$x_i x_{i+p} \cdots x_{i+p(r-1)} \big(x_{1+h} \cdots x_{1+h+p(s-1)}\big)^{-1}$$

where the subscripts are taken modulo n. Now we identify the faces R_i with $\overline{R}_i$ so that the corresponding oriented edges carrying the same label match well. The resulting complex is a closed orientable 3–manifold $M_n = M(r,n,h+1,s,p)$. Rotation by an angle $2\pi/n$ about the north–south axis NS defines an action of $\mathbb{Z}_n$ on the manifold M_n. The quotient space $M_n/\mathbb{Z}_n$ is homeomorphic to the lens space $L(r-s,s)$. One can construct a Heegaard diagram of it from a fundamental domain $\Pi(r,n,h+1,s,p)$ of the action (see Figure 7). The algebraic conditions on the parameters permit to make the edge identifications coherently, and to construct an admissible Heegaard diagram for the quotient space.

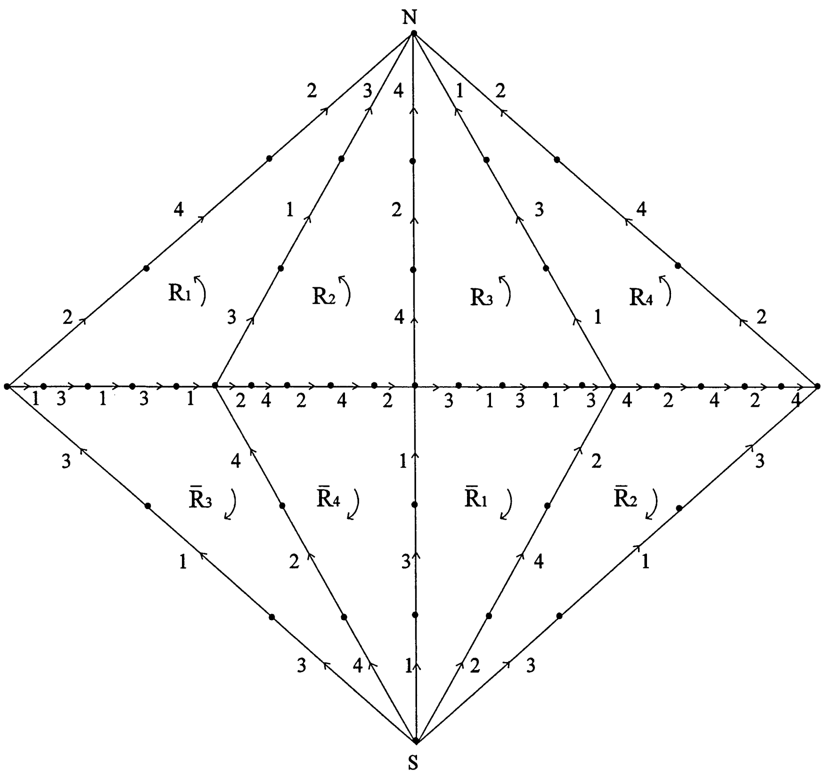
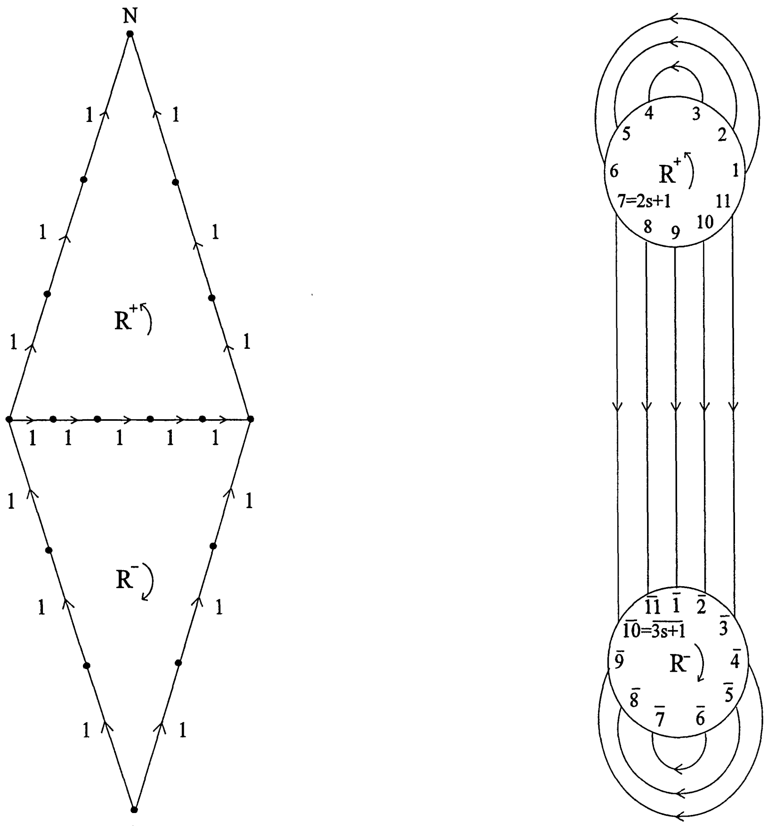


Figure 6. The polyhedron $Q(r,n,h+1,s,p) = Q(8,4,2,3,2)$

If further $s = r-1$, then the quotient space is homeomorphic to the 3–sphere as shown in Figure 8 for the tessellation $Q(4,4,2,3,2)$. A Heegaard diagram for this quotient space is pictured in Figure 9 (a). The axis of the rotation is drawn, as usual, as a dotted curve. Figure 9 (b) is obtained from Figure 9 (a) by a simplification along the closed curve V. Figure 10 (a) is obtained from Figure 9 (b) by a cancellation of handles. It is easy to see by Reidemeister moves that the knot pictured in Figure 10 (a) is the torus knot $T(7,2) = T(2r-1,2)$. So we have the following result.

Theorem 4.4. *If $h \equiv \pm 1$ (mod n), $p(r-s) \equiv 2h$ (mod n), $gcd(r,s) = 1$ and $r+s$ is odd, then the Prishchepov group $P(r,n,h+1,s,p)$ ($1 < s \leq r-1$, p, $h \geq 1$, and $n \geq 2$) is the fundamental group of a closed connected orientable 3-manifold $M_n = M(r,n,h+1,s,p)$ which is an n-fold cyclic covering of the lens space $L(r-s,s)$. If further $s = r-1$, then M_n is the n-fold covering of the 3-sphere branched over the torus knot $T(2r-1,2)$, i.e., M_n is the Brieskorn manifold $M(2,2r-1,n)$.* □



(a) The fundamental domain $\Pi(8,4,2,3,2)$

(b) A Heegaard diagram for the quotient $M(r,n,h+1,s,p)/\mathbb{Z}_n \cong L(r-s,s)$ (case $r=8$, $h=1$, $s=3$ and $p=2$)

Figure 7

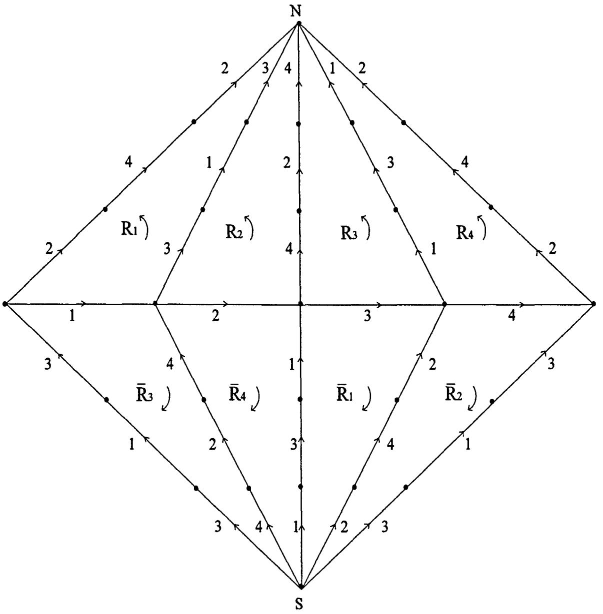


Figure 8. The polyhedron $Q(4,4,2,3,2)$

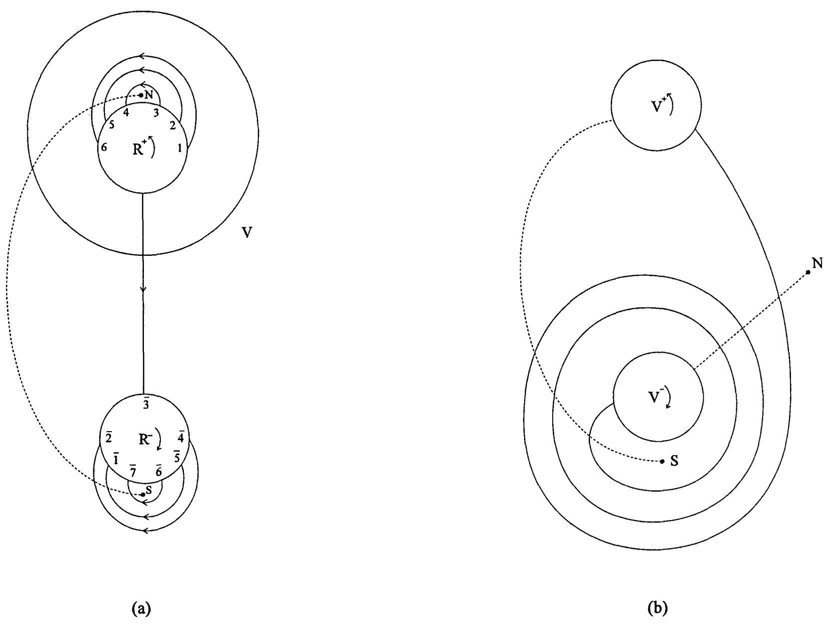


Figure 9. A Heegaard diagram for the quotient $M(r,n,h+1,s,p)/\mathbb{Z}_n \cong \mathbb{S}^3$
(case $r=4$, $h=1$, $s=3$ and $p=2$)

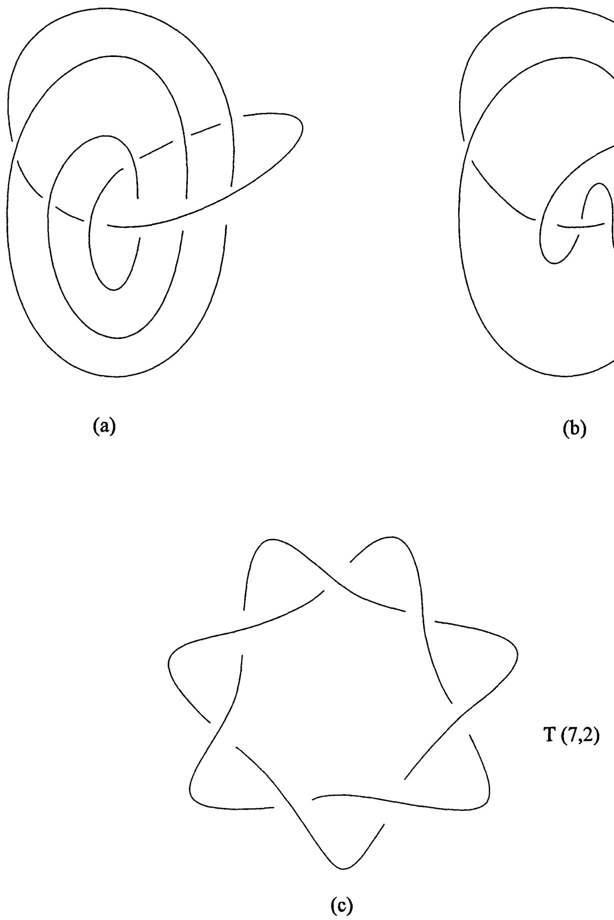


Figure 10. A sequence of Reidemeister moves yielding the torus knot $T(2r-1,2)=T(7,2)$

As a consequence, we obtain isomorphisms between Prishchepov groups for different values of h and p, and fixed n and $s=r-1$ (satisfying conditions above). Finally, it follows that the commutator group of the centrally extended triangle group [28] with parameters $(2,2r-1,n)$ admits a cyclic presentation.

5. HNN Extensions and Knot Groups

One of the first question in knot theory is which abstract group is an *m–knot group*, i.e., the fundamental group of the complement of an m–dimensional knot in the $(m+2)$–sphere. For $m \geq 3$, Kervaire gave necessary and sufficient conditions to solve the above question (see for example [32], Section 11D). More precisely, a group G is a 3–knot group (and so, an m–knot group for any $m \geq 3$) if and only if G is finitely presentable, $H_1(G;\mathbb{Z}) \cong \mathbb{Z}$, $H_2(G;\mathbb{Z}) \cong 0$, and G has weight 1 (that is, G is the normal closure of some single element). It is well–known that these conditions are necessary but not sufficient in dimension $n \leq 2$. Recently, Szczepański and Vesnin [41] have obtained conditions for which the natural HNN extension of a cyclically presented group is the group of an high–dimensional knot. We briefly recall their result which is based on Kervaire's characterization. Let $G_n(w)$ be a cyclically presented group with generators $x_1, \ldots, x_n$ and relations $w = \theta(w) = \cdots = \theta^{n-1}(w) = 1$, where θ is the automorphism of $G_n(w)$ given by $\theta(x_i) = x_{i+1}$ (subscripts mod n). Following [41], we say that the defining word w is *admissable* if $|f_w(1)| = 1$, where $f_w(t)$ is the polynomial associated to $G_n(w)$. Let $\mathcal{G}_n(w)$ denote the *natural* HNN *extension* of $G_n(w)$, that is,

$$\mathcal{G}_n(w) =< G_n(w), t: \quad t^{-1}gt = \theta(g),\ g \in G_n(w) > .$$

Suppose that the abelianized group of $G_n(w)$ is finite, or, equivalently, that $f_w(t)$ has no roots in common with $t^n - 1$ (see [24]). Under this assumption, it was proved in [41] that $\mathcal{G}_n(w)$ is an m–knot group, $m \geq 3$, if and only if w is admissable. In particular, the natural HNN extension of the Fibonacci group $F(r,n)$ is a 3–knot group if and only if $r = 2$ (see [37] for the proof). A similar result was also proved in [38] for the class of *fractional Fibonacci groups* $F^{k/\ell}(2,n) = G_n((x_1^{\ell} x_2^{k} x_3^{-\ell}))$. More precisely, the natural HNN extensions of these groups are high–dimensional knot groups if and only if $k = 1$. The present section is devoted to construct a new class of high–dimensional knot groups arising from the cyclic presentations defined in Section 2. To prove our results we apply the main theorem of [41], cited above, and some algebraic methods developed by Johnson and Odoni in [24] and [30].

Let us consider the cyclically presented groups

$$G_n(h,k;p,p;r,s;\ell) = G_n((\prod_{j=0}^{r-1} x_{1+jp})^{\ell}\ (\prod_{j=0}^{s-1} x_{1+h+jp})^{-k}),$$

where $n \geq 2$, $r \geq 2$, $s \geq 1$, $h,k,p,\ell \in \mathbb{Z}$ and $\ell \neq 0$. The polynomial $f_w(t)$ associated to the word

$$w = (\prod_{j=0}^{r-1} x_{1+jp})^{\ell}\ (\prod_{j=0}^{s-1} x_{1+h+jp})^{-k}$$

or, equivalently, to

$$\theta^{n-1}(w) = (\prod_{j=0}^{r-1} x_{jp})^{\ell}\ (\prod_{j=0}^{s-1} x_{h+jp})^{-k}$$

is

$$f_w(t) = \ell(\sum_{j=0}^{r-1} t^{jp}) - kt^h(\sum_{j=0}^{s-1} t^{jp})$$

and hence

$$f_w(t) = \ell \frac{t^{pr}-1}{t^p-1} - kt^h \frac{t^{ps}-1}{t^p-1}$$

for $t^p - 1 \neq 0$.

We seek necessary and sufficient conditions for the parameters under which $f_w(t)$ vanishes on an n-th root of unity. First we consider five cases.

case (i): $ks = r\ell$. Then $f_w(1) = 0$.

case (ii): $k = 0$, $\gcd(pr, n) = a > 1$, and a does not divide p. Then $f_w(t)$ vanishes on a primitive a-th root of unity.

case (iii): $k = \pm\ell$ and there exists $m \in \mathbb{Z}$, $m > 1$, such that m divides n but does not divide ps, $p(s-r) \equiv 0 \pmod{m}$, and either $h \equiv 0 \pmod{m}$ and $k = \ell$ or $h \equiv m/2 \pmod{m}$ and $k = -\ell$, with m even in the second case. Then $f_w(t)$ vanishes on a primitive m-th root of unity.

case (iv): $k = (-1)^u 2\ell$, $h = ps(1+3u)$, $r \equiv 3s \pmod{6s}$, and $n \equiv 0 \pmod{6ps}$. If λ is a primitive $(6ps)$-th root of unity, then

$$\begin{aligned} f_w(\lambda) &= \ell\frac{\lambda^{3ps}-1}{\lambda^p-1} - (-1)^u 2\ell\lambda^{ps}\lambda^{3psu}\frac{\lambda^{ps}-1}{\lambda^p-1} \\ &= \ell\frac{-2}{\lambda^p-1} - (-1)^u 2\ell\lambda^{ps}(-1)^u\frac{\lambda^{ps}-1}{\lambda^p-1} \\ &= \frac{-2\ell(1-\lambda^{ps}+\lambda^{2ps})}{\lambda^p-1} = 0. \end{aligned}$$

In fact, $\lambda^{6ps} = 1$ implies $(\lambda^{3ps}+1)(\lambda^{3ps}-1) = (\lambda^{3ps}+1)(-2) = 0$. But we have $(\lambda^{3ps}+1) = (\lambda^{ps}+1)(\lambda^{2ps}-\lambda^{ps}+1)$, and $\lambda^{ps} \neq -1$.

case (v): $k = \pm\ell$ and there exists $m \in \mathbb{Z}$, $m > 1$, such that m divides n but does not divide ps, $p(r+s) \equiv 0 \pmod{m}$, and either $h + ps \equiv 0 \pmod{m}$ and $k = -\ell$ or $h + ps \equiv m/2 \pmod{m}$ and $k = \ell$, with m even in the second case. Then $f_w(t)$ vanishes on a primitive m-th root of unity. In fact, for the first case we have

$$f_w(\lambda) = \ell\frac{\lambda^h-1}{\lambda^p-1} + \ell\lambda^h\frac{\lambda^{-h}-1}{\lambda^p-1} = \frac{\ell(\lambda^h-1+1-\lambda^h)}{\lambda^p-1} = 0,$$

while in the second case

$$\begin{aligned} f_w(\lambda) &= \ell\frac{\lambda^h\lambda^{-m/2}-1}{\lambda^p-1} - \ell\lambda^h\frac{\lambda^{-h}\lambda^{m/2}-1}{\lambda^p-1} \\ &= \ell\frac{-\lambda^h-1}{\lambda^p-1} - \ell\frac{-1-\lambda^h}{\lambda^p-1} \\ &= \frac{\ell(-\lambda^h-1+1+\lambda^h)}{\lambda^p-1} = 0. \end{aligned}$$

Theorem 5.1. *The abelianized group of $G_n(h, k; p, p; r, s; \ell)$ is infinite in the above five cases, and finite otherwise.*

Theorem 5.1 together with the main result of [41] gives a new class of examples of high–dimensional knot groups.

Theorem 5.2 *Suppose that conditions (i)–(v) are not satisfied. Then the natural HNN extension of the group $G_n(h,k;p,p;r,s;\ell)$ is a m–knot group, $m \geq 3$, if and only if $r\ell - ks = \pm 1$.*

Proof of Theorem 5.1. We have to prove only the second assertion so we assume that $ks \neq r\ell$ (case (i)). If $k = 0$ and $\gcd(pr, n) = 1$, then $f_w(t)$ does not vanish on an n-th root of unity. In fact, suppose conversely that λ is a primitive n-th root of unity with $f_w(\lambda) = 0$. It follows that n does not divide p; otherwise, we have

$$f_w(\lambda) = \ell(\sum_{j=0}^{r-1} \lambda^{jp}) = \ell r \neq 0.$$

Since

$$f_w(\lambda) = \ell \frac{\lambda^{pr} - 1}{\lambda^p - 1} = 0,$$

we obtain $\lambda^{pr} = 1$, and so $\lambda = 1$ because $\gcd(pr, n) = 1$. This gives a contradiction. If $k = 0$, $\gcd(pr, n) = a > 1$ and a divides p, then $f_w(t)$ does not vanish on an n-th root of unity. In fact, if $\lambda^n = 1$ and $f_w(\lambda) = 0$, then n does not divide p, as above. But $f_w(\lambda) = 0$ implies that $\lambda^{pr} = 1$, hence n divides pr. This gives $a = n$, where a divides p, that is, a contradiction. Thus we may also assume that $k \neq 0$ (case (ii)). Now let λ be a primitive m-th root of unity, $m > 1$, such that m divides n but does not divide ps, and $f_w(\lambda) = 0$. Then we claim that the parameters satisfy the conditions in cases (iii), (iv) or (v). From $f_w(\lambda) = 0$, we obtain

$$\ell(\lambda^{pr} - 1) = k\lambda^h(\lambda^{ps} - 1) \tag{5.1}$$

hence

$$\frac{k}{\ell}\lambda^h - 1 = \lambda^{pr}(\frac{k}{\ell}\lambda^{h+p(s-r)} - 1). \tag{5.2}$$

Since $|\lambda| = 1$, we get

$$|\frac{k}{\ell}\lambda^h - 1| = |\frac{k}{\ell}\lambda^{h+p(s-r)} - 1|.$$

Since $\frac{k}{\ell} \neq 0$, we have

$$\lambda^{h+p(s-r)} = \lambda^{\pm h}.$$

If $\lambda^{h+p(s-r)} = \lambda^h$, then $\lambda^{p(s-r)} = 1$ so $p(s-r) \equiv 0 \pmod{m}$. It follows from (5.2) that

$$\frac{k}{\ell}\lambda^h - 1 = \lambda^{ps}(\frac{k}{\ell}\lambda^h - 1)$$

hence

$$(\frac{k}{\ell}\lambda^h - 1)(\lambda^{ps} - 1) = 0.$$

But $\lambda^{ps} \neq 1$ because m does not divide ps, so we get $\frac{k}{\ell}\lambda^h = 1$. This implies that $\frac{k}{\ell} = \pm 1 = \lambda^h$, which gives case (iii). There remains the possibility that

$$\lambda^{h+p(s-r)} = \lambda^{-h}$$

which we have to reduce to cases (iii), (iv) or (v). Thus we have $\lambda^{pr} = \lambda^{2h+ps}$ and

$$\frac{\lambda^{2h+ps} - 1}{\lambda^{ps} - 1} = \frac{k}{\ell}\lambda^h \tag{5.3}$$

using (5.1) with $\frac{k}{\ell} \notin \{0, \frac{r}{s}\}$, and $m = \text{ord}(\lambda) > 1$, $m \backslash n$, m does not divide ps. Suppose now $\frac{k}{\ell} \neq \pm 1$, and let $\text{ord}(\lambda^{ps}) = e \backslash m$ and $\text{ord}(\lambda^{2h+ps}) = d \backslash m$. Using a nice argument from [24], we take norms from $\mathbb{Q}(\lambda)$ to $\mathbb{Q}$ in (5.3), so we get

$$(\frac{k}{\ell})^{\phi(m)}(-1)^{h\phi(m)} = \frac{(\Phi_d(1))^{\frac{\phi(m)}{\phi(d)}}}{(\Phi_e(1))^{\frac{\phi(m)}{\phi(e)}}} \tag{5.4}$$

where Φ_d is the d-th cyclotomic polynomial and ϕ is the Euler totient function. Now let c be a prime which divides $\frac{k}{\ell}$, $c^t \backslash \frac{k}{\ell}$ say. Then c divides $\Phi_d(1)$ by (5.4) which implies that $d = c^\eta$, $\eta \geq 1$, and $\Phi_d(1) = c$. Comparing powers of c in both sides of (5.4) we have

$$t\phi(m) \leq \frac{\phi(m)}{\phi(d)}.$$

This forces $t = 1$ and $\phi(d) = 1$. Since $d = c^\eta$, it follows that $c = 2$ and $\eta = 1$, hence $k = \pm 2\ell$ and $\lambda^{2h+ps} = -1$. Substituting into (5.3) yields

$$2 = \pm 2\lambda^h(1 - \lambda^{ps}). \tag{5.5}$$

Squaring this gives $4 = 4\lambda^{2h}(1 - \lambda^{ps})^2$ that is $\lambda^{2ps} - \lambda^{ps} + 1 = 0$, so we obtain $e = \text{ord}(\lambda^{ps}) = 6$, and $m = \text{ord}(\lambda) = 6ps$. It now follows from (5.5) that $h \equiv ps \pmod{3ps}$, $h = ps(1 + 3u)$ say, and that

$$\text{sgn}(\frac{k}{\ell}) = \lambda^{ps}\lambda^{3psu}(1 - \lambda^{ps}) = \lambda^{ps}(1 - \lambda^{ps})(-1)^u$$

that is

$$(-1)^{u+1}\,\text{sgn}(\frac{k}{\ell}) = \lambda^{ps}(\lambda^{ps} - 1) = \lambda^{2ps} - \lambda^{ps} = -1,$$

hence $\text{sgn}(\frac{k}{\ell}) = (-1)^u$. So $\lambda^{p(r-s)} = \lambda^{2h}$ and $2h = 2ps + 6psu$ imply $\lambda^{p(r-s)} = \lambda^{2ps}$, hence $p(r - s) \equiv 2ps \pmod{6ps}$, that is, $r \equiv 3s \pmod{6s}$. This yields case (iv).

Suppose now $\lambda^{pr} = \lambda^{2h+ps}$ and $\frac{k}{\ell} = \pm 1$. Substituting into (5.1) yields

$$|\lambda^{2h+ps} - 1| = |\lambda^{ps} - 1|$$

which implies $\lambda^{2h+ps} = \lambda^{\pm ps}$.

If $\lambda^{2h+ps} = \lambda^{ps}$, then $\lambda^{2h} = 1$ so $2h \equiv 0 \pmod m$, and $\lambda^h = \pm 1$. From $2h \equiv 0 \pmod m$ we get $\lambda^{pr} = \lambda^{ps}$, so $p(r - s) \equiv 0 \pmod m$. This gives again case (iii).

If $\lambda^{2h+ps} = \lambda^{-ps}$, then $\lambda^{2h+2ps} = 1$, so $2h + 2ps \equiv 0 \pmod m$, and $\lambda^{h+ps} = \pm 1$. Hence $h + ps$ is congruent to either 0 or $m/2 \pmod m$, with m even in the second case. Of course, if $\lambda^{pr} = \lambda^{2h+ps} = \lambda^{-ps}$ and $\lambda^{h+ps} = \pm 1$, then $f_w(\lambda) = 0$ if and only if $k = -(\pm 1)\ell$. Finally, from $\lambda^{pr} = \lambda^{-ps}$, we get $\lambda^{p(r+s)} = 1$, hence $p(r+s) \equiv 0 \pmod m$. This gives case (v). □

Rewriting conditions (i)–(v) for the classes of cyclic presentations listed in Section 2 produces a lot of corollaries. For example, we immediately re-obtain that the Fibonacci groups $F(r, n)$ have a finite abelianization, as it is well-known [23]. Using Zeeman's twist spun construction of knots [43], our results imply the following consequences.

Corollary 5.3. *The abelianization of the generalized Sieradski group $S(r,n)$, $r \geq 2$, is infinite if and only if $n \equiv 0 \pmod{4r-2}$. If $4r-2$ does not divide n, then the natural HNN extension of $S(r,n)$ is a fibred 2–knot group with the Brieskorn manifold $M(2, 2r-1, n)$ as closed fibre.*

For $r = 2$, the first part of the statement was proved in [24].

Corollary 5.4. *The abelianization of the Campbell–Robertson group $F(r,n,k)$, where $r \geq 2$, $n \geq 3$ and $k \geq 1$, is infinite if and only if there exists $m \in \mathbb{Z}$, $m > 1$, $m \backslash n$ such that either $k \equiv 0 \pmod m$ and $r \equiv 1 \pmod m$ or $k \equiv 1 + m/2 \pmod m$ and $r \equiv -1 \pmod m$, with m even in the second case. If the abelianization is finite, then the natural HNN extension of $F(r,n,k)$ is a 3–knot group if and only if $r = 2$.*

For $k = 1$, the last part of the statement was proved in [37].

Corollary 5.5. *The abelianization of the Campbell–Robertson group $H(r,n,k)$, where $r \geq 2$, $n \geq 3$, and $k \geq 1$, is infinite if and only if $k = r$. In the case $k \neq r$, the natural HNN extension of $H(r,n,k)$ is a 3–knot group if and only if $r = k \pm 1$.*

For the Johnson–Odoni group $F(r,s,c,n)$, conditions (i)–(v) become the following sentences (also denoted by the same labels):

case (i): $c = r$

case (ii): $c = 0$ and $\gcd(r,n) = a > 1$

case (iii): $c = \pm 1$, and there exists $m \in \mathbb{Z}$, $m > 1$, $m \backslash n$, with $r \equiv 1 \pmod m$ and either $s \equiv 0 \pmod m$ and $c = 1$, or $s \equiv m/2 \pmod m$ and $c = -1$, with m even in the second case

case (iv): $c = (-1)^u 2$, $s = 1 + 3u$, $r \equiv 3 \pmod 6$ and $n \equiv 0 \pmod 6$

case (v): $c = \pm 1$, and there exists $m \in \mathbb{Z}$, $m > 1$, $m \backslash n$, with $r \equiv -1 \pmod m$ and either $s \equiv -1 \pmod m$ and $c = -1$, or $s \equiv -1 + m/2 \pmod m$ and $c = 1$, with m even in the second case.

We remark that cases (i)–(iv) are exactly those presented in [24], while case (v) was not considered in that paper. So the first part of the next corollary completes Proposition 4.2 of [24].

Corollary 5.6. *The abelianization of the Johnson–Odoni group $F(r,s,c,n)$, where $r \geq 2$, $n \geq 2$, $s, c \in \mathbb{Z}$, is infinite in the above five cases, and finite otherwise. In the finite case, the natural HNN extension of $F(r,s,c,n)$ is a 3–knot group if and only if $r = c \pm 1$.*

For the Prischepov group $P(r,n,k,s,q)$, conditions (i)–(v) reduce to only three cases (denoted by the same corresponding labels):

case (i): $s = r$

case (iii): there exists $m \in \mathbb{Z}$, $m > 1$, $m \backslash n$, m does not divide qs, with $qs \equiv qr \pmod m$, and $k \equiv 1 \pmod m$

case (v): there exists $m \in \mathbb{Z}$, $m > 1$, $m \backslash n$, m does not divide qs, with $qs \equiv -qr \pmod{m}$, and $k + qs \equiv 1 + m/2 \pmod{m}$, m even.

Corollary 5.7 *The abelianization of the Prischepov group $P(r, n, k, s, q)$ is infinite in the above three cases, and finite otherwise. In the finite case, the natural HNN extension of $P(r, n, k, s, q)$ is a 3–knot group (resp. 2–knot group if the hypothesis of Theorem 4.4 are satisfied) if and only if $s = r \pm 1$.*

In particular, if $s = r - 1$, then $P(r, n, k, s, q)$ has infinite abelianization if and only if $n \equiv 0 \pmod{q(2r-1)}$ and $2(k-1) \equiv q \pmod{q(2r-1)}$. If further $k = q = 2$, then these conditions reduce to that in the statement of Corollary 5.3.

Now let us consider the group $G_n(h, k) = G_n(x_1 x_{1+h} x_{1+k}^{-1})$ which can also be regarded as $G_n(k, 1; h, h; 2, 1; 1)$. From Theorems 5.1 and 5.2 we have

Corollary 5.8. *The abelianization of the group $G_n(h, k)$ is infinite if and only if $n \equiv 0 \pmod{6}$, $k + h \equiv 3 \pmod{6}$, and $3h \equiv 0 \pmod{6}$. In the finite case, the natural HNN extension of $G_n(h, k)$ is a 3–knot group.*

For $k = 1$, $G_n(h, k)$ coincides with the Gilbert–Howie group $H(n, h)$. So Corollary 5.8 immediately implies that $H(n, h)$ has infinite abelianization if and only if $n \equiv 0 \pmod{6}$ and $h \equiv 2 \pmod{6}$. This is a well-known result, proved by Odoni in [30].

Finally, the group $\Gamma_n(k; \ell) = G_n(n-1, k; 1, 1; n, 1; \ell)$ has finite abelianization if and only if $k \neq n\ell$ and $k \neq 0$. In fact, the abelianized group is isomorphic to $\mathbb{Z}_k^{n-2} \oplus \mathbb{Z}_{k|k-n\ell|}$, as one can directly verify by standard computations on the circulant matrix having the generator exponents as entries. So we have

Corollary 5.9. *If $k \neq n\ell$ and $k \neq 0$, then the natural HNN extension of $\Gamma_n(k; \ell)$ is a fibred 2–knot group (with the Brieskorn manifold $M(n\ell, k, n)$ as closed fibre) if and only if $n\ell = k \pm 1$.*

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