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Open problems on graphs arising from geometric topology

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Abstract

We have collected several open problems on graphs which arise in geometric topology, in particular in the following areas: (1) basic embeddability of compacta into the plane $\mathbb{R}^2$; (2) approximability of maps by embeddings; (3) uncountable collections of continua in $\mathbb{R}^2$ and their span; and (4) representations of closed PL manifolds by colored graphs. These problems should be of interest to both topologists and combinatorists. © 1998 Elsevier Science B.V.

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1. Introduction

A well-known Moore's theorem on triods states that there are no uncountable collections of pairwise disjoint (homeomorphic) copies of the letter T in the plane [36]. This theorem was subsequently widely generalized [1,4,8,38,41,42,47,57]. Proofs of these generalizations include a reduction to their graph analogues. These analogues are interesting elementary theorems concerning the narrowness properties of trees in the plane. Among them is the following generalization of Konstantinov's Fatties Theorem (a short proof is sketched in Section 2):

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Theorem 1.1 [42]. *Suppose that $K \subset \mathbb{R}^2$ is a tree and $\vec{a} \in \mathbb{R}^2$ is a vector such that $K \cap (K+\vec{a}) = \emptyset$. Then two fatties (i.e., round disks of diameters $|\vec{a}|$) cannot exchange their positions, by moving their centers on K continuously and not intersecting each other.*

In the solution of Hilbert's 13th problem [3,24], the notion of a *basic embedding* appeared for the first time. In particular, an embedding $K \subset \mathbb{R}^2$ is said to be *basic*, if for every continuous function $f : K \to \mathbb{R}$ there exist continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x)+h(y)$, for every pair $(x, y) \in K$ (we assume that $\mathbb{R}^2$ is endowed with a fixed Cartesian coordinate system). This is equivalent to a certain elementary geometric property [51] (see Section 3). Thus the description of graphs, basically embeddable in $\mathbb{R}^2$, can be reduced to an elementary problem, concerning a certain linear property of trees in the plane. It turns out that finite graphs, basically embeddable in the plane, have a simple description in the spirit of Kuratowski's description of planar graphs [27] (a short proof of PL-embeddings of graphs is sketched in Section 3).

Theorem 1.2 [49]. *A finite graph K is basically embeddable in $\mathbb{R}^2$ if and only if it does not contain any of the following three graphs: a circle S, a pentod P or a cross C with branched ends (see Fig. 1) (or, equivalently, if it is contained in V_n, for some n, see Fig. 2).*

A classical problem in topology is to describe compacta, embeddable in the plane. Such a description for graphs and Peanian continua was given by Kuratowski and Claytor [15,27] (see also [50,56]). Using the inverse limit techniques, embeddability of compacta can be reduced, roughly speaking, to *embeddability of maps* of graphs (see the definition in Section 4) [46,48]. This concept generalizes:

(1) ordinary embeddability of graphs in $\mathbb{R}^2$;
(2) embeddability of graphs in $\mathbb{R}^2$ with two vertices 'close together' or with a given cycle bounding a face [53]; and
(3) isotopness of different embeddings of a given graph in $\mathbb{R}^2$.

Examples show that this notion is rather geometric and interesting in itself. For the general case there are apparently no geometric descriptions (in the spirit of Kuratowski) of maps, embeddable in $\mathbb{R}^2$. But for this problem, an analogue of van Kampen's obstruc-

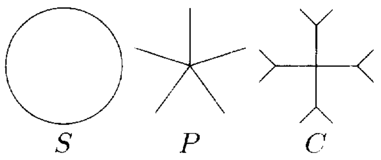


Fig. 1.

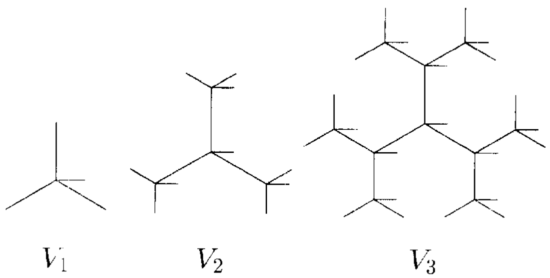


Fig. 2.

tion $\vartheta(f) \in H^2_S(\widetilde{K}, \widetilde{K}^f)$ and a difference element $\omega(f) \in H^1_S(\widetilde{K}^f)$ (cf. [40]) can be introduced (see Section 4).

Theorem 1.3. *If a PL-map $f: K \to \mathbb{R}^2$ is approximable by embeddings then:*

(a) $\vartheta(f) = 0$; *and*

(b) *there exists an embedding $\varphi: K \to \mathbb{R}^2$ such that $\omega(\varphi)|_{\widetilde{K}^f} = \omega(f)$.*

We also discuss the combinatorial approach to the topology of triangulated manifolds based on the theory of colored graphs. Moreover, we present some open problems which are of interest to both topologists and combinatorists.

2. Generalizations of Moore's theorem

Let us denote by $|x, y|$ the distance between points $x, y \in \mathbb{R}^2$. An *arc* is a path without self-intersections. A *triod* is the union of three piecewise-rectilinear arcs a_0, b_0 and c_0, intersecting only at the point 0. For a subset $Z \subset \mathbb{R}^2$, we denote $|x, Z| = \sup\{|x, y| \mid y \in Z\}$. A triod is called *$\varepsilon$-triod* if $|a, b_0 \cup c_0| > \varepsilon$, $|b, c_0 \cup a_0| > \varepsilon$ and $|c, a_0 \cup b_0| > \varepsilon$. A *tree* is the union of (intersecting) rectilinear arcs in $\mathbb{R}^2$, which is connected and does not contain any cycles.

Definition 2.1 [42]. A tree $K \subset \mathbb{R}^2$ is said to be *ε-disjoinable* if there is a vector $\vec{a} \in \mathbb{R}^2$ such that $K \cap (K + \vec{a}) = \emptyset$ and $|\vec{a}| < \varepsilon$.

This property is a simplified graph analogue of the existence of an uncountable collection of disjoint copies of a continuum K in $\mathbb{R}^2$. The graph analogue of Moore's theorem on triods is that no ε-triod is ε-disjoinable (cf. [36]). Our further considerations are related to the well-known Konstantinov Fatties Theorem and Mountain Climbers Theorem.

Theorem 2.2 [2, 1.1.2].

(a) *Suppose that two byciclists, connected by a rope of length ε, can ride along two roads from A to B and from A' to B', respectively. Then two ε-fatties (round disks of diameters ε) cannot ride along the first road from A to B and along the second road from B' to A', respectively, not intersecting each other.*

(b) ([19,23,55], see also [5]) *Two mountain climbers begin at the sea level, at opposite ends of a (two-dimensional) chain of mountains, lying above the sea level and having finitely many summits. Then they can find routes along which to travel, always maintaining equal altitudes, until they eventually meet.*

Definition 2.3 [17]. A tree $K \subset \mathbb{R}^2$ is said to be *symmetrically ε-spanned* if and only if two ε-fatties cannot exchange their positions, moving continuously their centers on K and not intersecting each other.

Evidently, no ε-triod is symmetrically ε-spanned (cf. [17]). However, for each $\varepsilon > 0$, there is a tree in $\mathbb{R}^2$ which is not symmetrically 1-spanned and does not contain any ε-triod (see Fig. 3) [20].

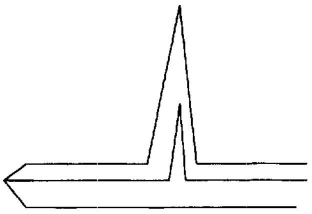


Fig. 3

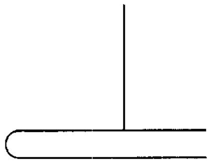


Fig. 4.

Now let us sketch a *new* proof of Theorem 1.1, which may be of help in attacking unsolved problems below. Let $\varepsilon = |\vec{a}|$. For $x, y \in K$ such that $|x, y| > \varepsilon$, dispose two men (not fatties!) at x and $y + \vec{a}$, and move them to y and $x + \vec{a}$ along the arcs $l \subset K$ and $l + \vec{a} \subset K + \vec{a}$, joining x to y and $y - \vec{a}$ to $x - \vec{a}$, respectively. If the resulting rotation of the vector from the first man to the second one is in the clockwise direction, let $x < y$. It is easy to show that then for each $x, y \in K$ such that $|x, y| > \varepsilon$, either $x < y$ or $y < x$. The relation '$<$' is well-defined and continuous. Therefore ε-fatties cannot exchange their positions. Thus every ε-disjoinable tree is symmetrically ε-spanned. A stronger result is possible. First, we need a definition.

Definition 2.4 [30]. A tree $K \subset \mathbb{R}^2$ is said to be *ε-spanned* if two ε-fatties cannot move continuously their centers on K and not intersecting each other, so that their traces would be the same.

Evidently, any ε-spanned tree is symmetrically ε-spanned [17]. It is easy to show that every subtree of an ε-disjoinable tree K has a $<$-minimal point (i.e., a point $u \in K$ such that $u < x$ whenever $x \in K$ and $|x, u| > \varepsilon$). It follows that any ε-disjoinable tree is ε-spanned [42]. Note that in general, the relation $<$ is not transitive (see Fig. 5) [42].

The genuine graph analogue of the existence of an uncountable collection of disjoint copies of a continuum K in $\mathbb{R}^2$ is defined as follows. A tree $K \subset \mathbb{R}^2$ is said to be *genuinely ε-disjoinable* if there is a continuous (or piecewise linear) map $f : K \to \mathbb{R}^2$ such that $K \cap f(K) \neq \emptyset$ and $|x, f(x)| < \varepsilon$ for each $x \in K$. The proof of Theorem 1.1 remains valid if by ε-disjoinability we understand this weaker property. The method above can be applied to solve affirmatively the following famous problem from continua theory:

Problem 2.5 [16, 430].

(a) Is any genuinely ε-disjoinable K ε-spanned?

(b) Is any symmetrically ε-spanned tree ε-spanned?

To state the graph analogue of another generalization of Moore's theorem, let us introduce:

Definition 2.6 [8]. A tree $K \subset \mathbb{R}^2$ is said to be *ε-narrow* if there is an arc $l \subset K$ such that $|x, l| < \varepsilon$, for each $x \in K$. A tree $K \subset \mathbb{R}^2$ is said to be *hereditarily ε-narrow*, if every one of its subtrees is ε-narrow.

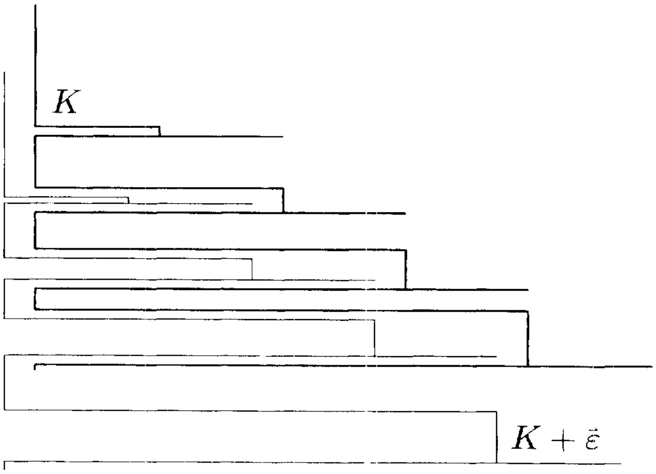


Fig. 5.

Evidently, no ε-triod is ε-narrow [9]. However, for each $\varepsilon > 0$, there is an ε-narrow tree containing a 1-triod (and hence not hereditarily ε-narrow) (see Fig. 4) [9].

Problem 2.7.

(a) [9] Show that a tree, containing no ε-triods, is 10ε-narrow.

(b) [31,34] Prove that any ε-spanned tree is 10ε-narrow.

In the proof of Problems 2.7 Konstantinov's Theorem could perhaps be useful. If we change ε to 10ε (or to $M\varepsilon$, for some fixed M) in the conclusions of Problems 2.7 (or in any other problem from our list), we obtain weaker assertions. But to prove them it still suffices to prove their continuous analogues, which involve no ε at all. So it is as much interesting as to prove nonweakened assertions (which could be false).

The graph analogue of [8] is that any ε-disjoinable tree is ε-narrow (and hence any ε-disjoinable tree is hereditarily ε-narrow). Now, let us introduce a property, sufficient for genuine ε-disjoinability.

Definition 2.8 [6]. A tree $K \subset \mathbb{R}^2$ is said to be *ε-chainable* if for some n, K is the union of its closed subsets $C_1, \dots, C_n$ such that $\operatorname{diam} C_i < \varepsilon$ for $i = 1, 2, \dots, n$, and C_i intersects only C_{i-1} and C_{i+1} for $i = 2, 3, \dots, n-1$.

Evidently, any ε-chainable tree is symmetrically ε-spanned [17] and ε-spanned [30]. It is not difficult to show that any ε-chainable tree is also genuinely ε-disjoinable [43]. However, for each $\varepsilon > 0$ there is an ε-disjoinable but not 1-chainable tree (see Fig. 5) [38,42].

Problem 2.9. Prove that every (genuinely) ε-spanned thee $K \subset \mathbb{R}^2$ of diameter 1 is 10ε-chainable.

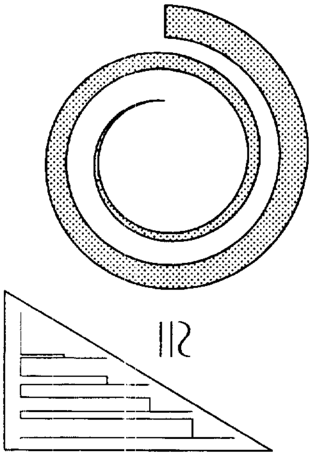
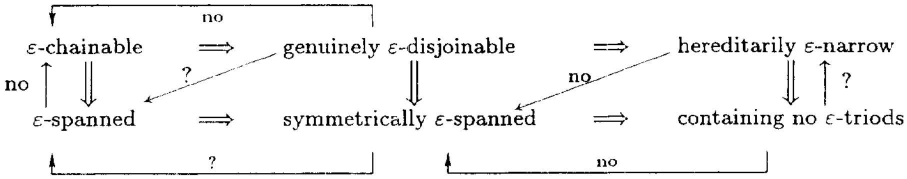


Fig. 6.

In searching a counterexample perhaps Fig. 6 could be useful. For an affirmative proof [21] may be useful.

We have the following connections between the properties introduced above:



3. Basic embeddings

Fix a Cartesian coordinate system in $\mathbb{R}^2$. Denote by $[ab]$ the rectilinear arc, joining points $a, b \in \mathbb{R}^2$. A sequence $\{a_1, \ldots, a_n\} \subset \mathbb{R}^2$ is said to be an *array*, if for each $i = 2, \ldots, n-1$, $a_{i-1} \neq a_i \neq a_{i-1}$ and $[a_{i-1}a_i]$, $[a_i a_{i+1}]$ are orthogonal to each other and parallel to the coordinate axes.

Definition 3.1. An embedding $K \subset \mathbb{R}^2$ is said to be *basic*, and it is denoted by $K \subset_b \mathbb{R}^2$, if for some integer n there are no arrays of n points in K.

Evidently, the set $K = \{(0,0), (0,1), (1,0), (1,1)\}$ contains arrays of arbitrary great length. Clearly, it is not basically embedded in $\mathbb{R}^2$, by the definition from Section 1. This example illustrates the equivalence of the two definitions, proved in [51] for compact subsets K.

We outline the idea of proof of Theorems 1.2 in Lemmas 3.2–3.9 below. Their proofs are easy and are left to the reader. These ideas are useful in attacking open Problems 3.10–3.15. For simplicity, we restrict ourselves to PL-embeddings.

It will be convenient to use another equivalent definition of basic embedding. Denote by p and q the projections of $\mathbb{R}^2$ onto the coordinate axes. Then for a point $z \in \mathbb{R}^2$, $p^{-1}(p(z))$ and $q^{-1}(q(z))$ are lines, going through z and parallel to the coordinate axes. For $Z \subset \mathbb{R}^2$ let us define:

$$E(Z) = \{ z \in \mathbb{R}^2 \mid |Z \cap p^{-1}(p(z))| \geqslant 2 \quad \text{and} \quad |Z \cap q^{-1}(q(z))| \geqslant 2 \}.$$

It is easy to show that any embedding $K \subset \mathbb{R}^2$ is basic if and only if $E^n(K) = \emptyset$, for some n.

Lemma 3.2 [51]. *The circle S from Fig. 1 is not basically embeddable in $\mathbb{R}^2$.*

Lemma 3.3.
(a) *If $P \subset_b \mathbb{R}^2$ then a cross with center d is basically embeddable into the half-plane $[0, +\infty) \times \mathbb{R}$ so that $d = (0, 0)$.*
(b) *If a cross with center d is basically embeddable into the half-plane $[0, +\infty) \times \mathbb{R}$ so that $d = (0, 0)$, then a triod with center d is basically embeddable into the quarter-plane $[0, +\infty) \times [0, +\infty)$ so that $d = (0, 0)$.*
(c) *A triod with center d cannot be basically embedded into $[0, +\infty) \times [0, +\infty)$ so that $d = (0, 0)$.*
(d) *The pentod P is not basically embeddable in $\mathbb{R}^2$.*

Lemma 3.4. *If the cross with the 'center' d is basically embedded in $\mathbb{R}^2$, then one of its branches contains a rectilinear arc with the end d, parallel to one of coordinate axes.*

Basic nonembeddability of C is reduced to that of P (see Fig. 1) using Lemma 3.4 and the following device.

Definition 3.5. Suppose that $K \subset \mathbb{R}^2$ and $\{0\} \times [0, 1] \subset K$. Define a *compression* $p : \mathbb{R}^2 \to \mathbb{R}^2$ by:

$$p(x, y) = \left\{ \begin{array}{ll} (x, y), & x \leqslant 0, \\ (0, y), & 0 \leqslant x \leqslant 1, \\ (x - 1, y), & x \geqslant 1. \end{array} \right\} \quad \text{for every pair } (x, y) \in \mathbb{R}^2.$$

Lemma 3.6. *Under the hypotheses of Definition 3.5, if $K \subset_b \mathbb{R}^2$, then the restriction of p on $K \setminus (\{0\} \times [0, 1])$ is a 1–1 map and $p(K) \subset_b \mathbb{R}^2$.*

This lemma can also be proved using the definition of basic embedding from Section 1. Note that the reverse implication of Lemma 3.6 is also true (we do not need this in the proof of Theorem 1.2).

Lemma 3.7. *The graph C of Fig. 1 is not basically embeddable in $\mathbb{R}^2$.*

Lemma 3.8. *If a graph K does not contain any of S, P, C, then K is contained in the graph V_n for some n (see Fig. 2).*

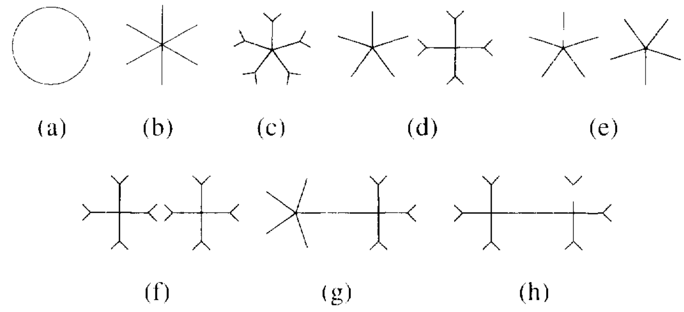


Fig. 7.

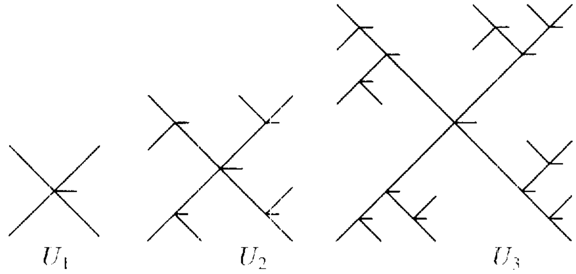


Fig. 8.

Lemma 3.9. *V_n is basically embeddable in $\mathbb{R}^2$ for every integer n.*

In [49] the description of compacta, basically embeddable in $\mathbb{R}^2$, was established for *pathwise connected* compacta and conjectured for connected compacta.

Problem 3.10. Prove this conjecture.

The definition of a basic embedding into $\mathbb{R}^2$ could obviously be generalized to embeddings into $X \times Y$, for arbitrary spaces X, Y. Some problems from the preliminary version of this paper have already been solved. Now they appear as Theorems 3.11 and 3.12. Let T_i be an i-od.

Theorem 3.11 [28]. *A finite graph K is basically embeddable into $T_3 \times T_2$ if and only if either of the following equivalent conditions hold:*
(a) *K does not contain any of graphs of Figs.* 7(a)–(h);
(b) *K is contained in U_n for some n (Fig.* 8).

Call a vertex of a graph K *horrid* (respectively *awful*) if its degree is greater than 4 (respectively, its degree equals 4 and it has no hanging edges). The *defect* of graph K is the sum

$$\delta(K) = (\deg A_1 - 2) + \cdots + (\deg A_n - 2),$$

where $A_1, \ldots, A_n$ are all horrid and awful vertices of K.

Theorem 3.12 [29]. *A finite (not necessarily connected) graph K is basically embeddable into $T_m \times T_n$ if and only if it is a tree and either $\delta(K) < m + n - 2$ or K has a horrid vertex with hanging edge and $\delta(K) \leqslant m + n - 2$.*

Problem 3.13. Find a description of graphs, basically embeddable into
(a) $\mathbb{R} \times S$;
(b) $T_3 \times S$;
(c) $S \times S$;
(d) (a graph) $\times$ (a graph).

Problem 3.14. Show that for every triple of finite graphs K, X, Y there is an algorithm for checking whether K is basically embeddable in $X \times Y$;

Problem 3.15 (cf. [44]). Prove that for every pair of finite graphs X, Y there is a finite number of 'prohibited' subgraphs for basic embeddings in $X \times Y$.

Note that a minor of a basically embeddable graph is not necessarily basically embeddable.

4. Approximability of maps by embeddings

Definition 4.1. A map $f : K \to M$ between graphs K and M is said to be *embeddable* in $\mathbb{R}^2$ if there is an embedding $\varphi : M \to \mathbb{R}^2$ such that the composition $\varphi \circ f$ is approximable by embeddings (i.e., for each $\varepsilon > 0$ there exists an embedding $\psi : K \to \mathbb{R}^2$ which is ε-close to $\varphi \circ f$).

For a graph K, let

$$\widetilde{K} = \bigcup \{\sigma \times \tau \mid \sigma, \tau \text{ are cells (edges or vertices) of } K,\ \sigma \cap \tau = \emptyset\}$$

be the deleted product of K (it is a 2-dimensional polyhedron). For a map $f : K \to \mathbb{R}^2$ let

$$\widetilde{K}^f = \bigcup \{\sigma \times \tau \mid \sigma, \tau \text{ are cells of } K,\ f(\sigma) \cap f(\tau) = \emptyset\}.$$

We omit $\mathbb{Z}$-coefficients from our homology and cohomology groups. Let us construct a generalization of the van Kampen obstruction $\vartheta(f) \in H^2_{\mathbb{Z}}(\widetilde{K}, \widetilde{K}^f)$ and a difference element $\omega(f) \in H^1_{\mathbb{Z}}(\widetilde{K}^f)$, for an arbitrary PL-map $f : K \to \mathbb{R}^2$ (not necessarily an embedding).

Definition 4.2. Take a general position map $g : K \to \mathbb{R}^2$, sufficiently close to the map f. Fix an orientation of $\mathbb{R}^2$ and for any two disjoint oriented edges σ and τ of K, count an intersection where the orientation of $g(\sigma)$ followed by that of $g(\tau)$ agrees with that of $\mathbb{R}^2$ as $+1$, and -1 otherwise. Then $\vartheta(f)$ is the class of the cocycle $\vartheta_g(f)(\sigma, \tau)$ which counts the intersections of $g(\sigma)$ and $g(\tau)$ algebraically in this fashion. If f maps all K to a point, then $\vartheta(f)$ is the van Kampen obstruction to embeddability of K in $\mathbb{R}^2$.

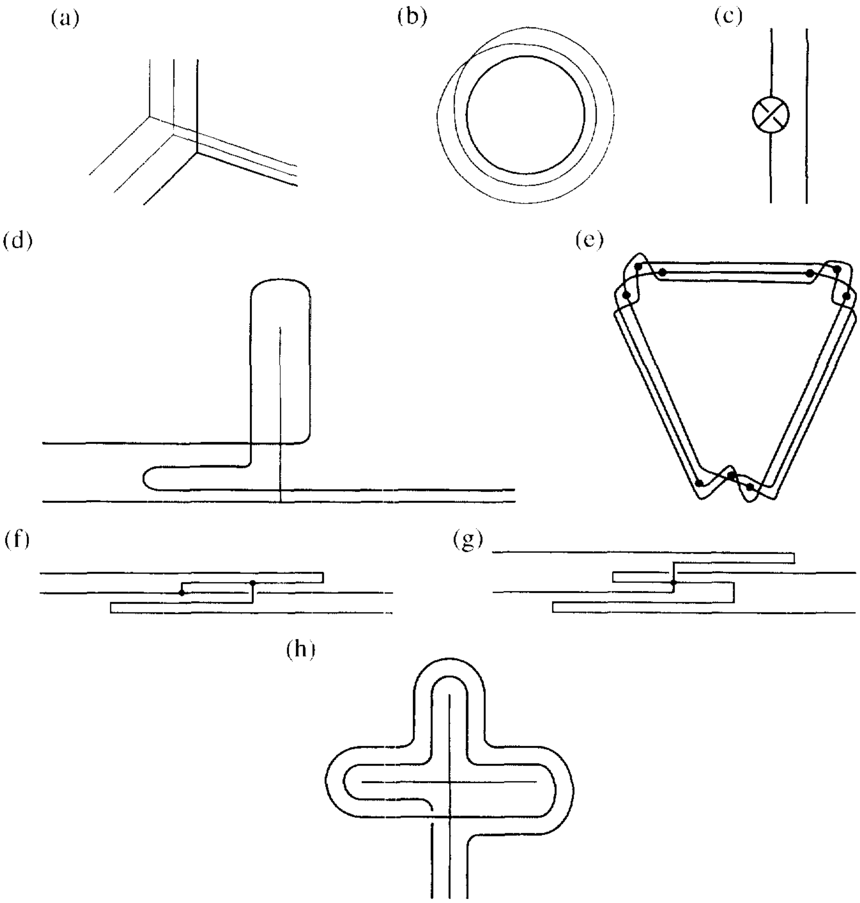


Fig. 9.

Definition 4.3. Approximate f by a PL-map, sufficiently close to f. Denote this new map also by f. Take a point $x \in S^1$. Fix an orientation of $\mathbb{R}^2$ and for each oriented edge σ and vertex a of K (where $f(a) \notin f(\sigma)$) count the passage of the oriented path $\tilde{f}:\{a\} \times \sigma \to S^1$ through x when the path goes through x in the clockwise direction, as $+1$, and -1 otherwise. Choosing x in general position, we may assume that $x \notin \tilde{f}(\{a\} \times \partial\sigma)$ and that the path goes through x transversally. Then $\omega(f)$ is the class of the cocycle $\omega_x(f)(\{a\},\sigma)$, which counts the intersection of $\tilde{f}(a \times \sigma)$ and x algebraically in this fashion.

Evidently, these definitions are correct and Theorem 1.3 is true. Theorem 1.3 can be applied to show that maps in Fig. 9 are not approximable by embeddings [40,48].

Example 4.4 [40].
(a) Map f in Fig. 9(e) is not approximable by embeddings; however, $\vartheta(f) = 0$.

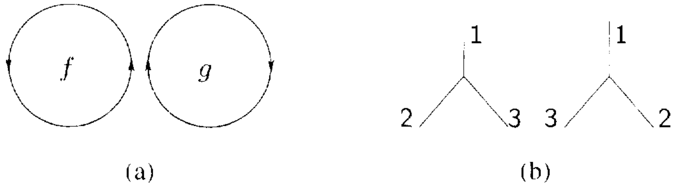


Fig. 10.

(b) Maps f in Figs. 9(f) and 9(g) are not approximable by embeddings; however there is an embedding φ such that $\omega(\varphi)|_{\widetilde{K}_f} = \omega(f)$.

Problem 4.5. Show that the reverse implication in Theorem 1.3(a) is true provided that:
(a) K is arbitrary and f is monotone;
(b) K is an arc; or
(c) K is a tree.

Evidently, if two embeddings $f, g : K \to \mathbb{R}^2$ are isotopic, then $\omega(f) = \omega(g)$. From this it follows that embeddings in Figs. 10(a) and 10(b) are not isotopic. It is not difficult to show that embeddings $f, g : K \to \mathbb{R}^2$ of a tree K are isotopic if and only if for each triod $\gamma \in K$, embeddings $f|_\gamma$ and $g|_\gamma$ do not constitute the pair, shown in Fig. 10(a).

Theorem 4.6 [35]. *Embeddings $f, g : K \to \mathbb{R}^2$ of a graph K are isotopic if and only if for each triod or simple closed curve $\gamma \subset K$, embeddings $f|_\gamma$ and $g|_\gamma$ do not constitute a pair, shown in Figs.* 10(a) *or* 10(b).

Therefore embeddings $f, g : K \to \mathbb{R}^2$ are isotopic if and only if $w(f) = w(g)$. In [45], a beautiful description of $H_2(\widetilde{K})$, related to Kuratowski's criterion, was established.

Theorem 4.7 [45]. *$H_2(\widetilde{K})$ is generated by*

$$\big\{[\gamma_1 \times \gamma_2] \in H_2(\widetilde{K}) \mid \gamma_1, \gamma_2 \subset K \textit{ are disjoint simple closed curves}\big\}$$
$$\cup \big\{[\widetilde{\gamma}] \in H_2(\widetilde{K}) \mid \gamma \subset K \textit{ is homeomorphic to either } K_5 \textit{ or } K_{3,3}\big\}.$$

Furthermore, $\dim H_2(\widetilde{K})$ *is either equal to the maximal number of independent tori $\gamma_1 \times \gamma_2$ contained in it, or else one more than this number. The first alternative occurs if and only if K is planar.*

We conjecture an analogous description of $H_1(\widetilde{K})$ and $H_1^S(\widetilde{K})$ exists, related to the MacLane–Adkisson criterion (perhaps one should first try the $\mathbb{Z}_2$-coefficients). For a proof the method of [52] could perhaps be useful.

Problem 4.8. (a) $H_2^S(\widetilde{K})$ is generated by

$$\big\{[\gamma_1 \times \gamma_2 \cup \gamma_2 \times \gamma_1] \in H_2^S(\widetilde{K}) \mid \gamma_1, \gamma_2 \subset K \text{ are disjoint simple closed curves}\big\}$$
$$\cup \big\{[\widetilde{\gamma}] \in H_2^S(\widetilde{K}) \mid \gamma \subset K \text{ is homeomorphic to either } K_5 \text{ of } K_{3,3}\big\}.$$

(b) $H_1(\widetilde{K})$ is generated by

$$\{[\tilde{\gamma}] \in H_1(\widetilde{K}) \mid \gamma \subset K \text{ is homeomorphic to either } S^1 \text{ or } T\}$$
$$\cup \{[\gamma \times a_\gamma \cup a_\gamma \times \gamma] \in H_1(\widetilde{K}) \mid \gamma \in K \text{ is homeomorphic to } S^1,\ a_\gamma \subset K \setminus \gamma\}.$$

Find $\dim H_1(\widetilde{K})$.

(c) $H_1^S(\widetilde{K})$ is generated by

$$\{[\tilde{\gamma}] \in H_1^S(\widetilde{K}) \mid \gamma \subset K \text{ is homeomorphic to either } S^1 \text{ or } T\}$$
$$\cup \{[\gamma \times a_\gamma \cup a_\gamma \times \gamma] \in H_1^S(\widetilde{K}) \mid \gamma \subset K \text{ is homeomorphic to } S^1,\ a_\gamma \in K \setminus \gamma\}.$$

Find $\dim H_1^S(\widetilde{K})$.

5. PL manifolds and colored graphs

In this section we investigate the connection between colored graphs and triangulated (PL) manifolds. For general references on graph-theoretical representations of PL manifolds we refer to [7,10–14,18,22,32,54]. We shall list some open problems from this area.

An $(n+1)$*-colored graph* is a pair (G, c), where $G = (V(G), E(G))$ is a multiple graph, regular of degree $n+1$ (possibly with multiple edges but without loops), together with a proper edge-coloring $c: E(G) \to \Delta_n = \{0, 1, \ldots, n\}$, by means of $n+1$ colors (hence $c(e) \neq c(f)$, for any pair e, f of adjacent edges of G).

To every colored graph (G, c) is associated an n-dimensional polyhedron $|G|$ as follows:

(1) for each vertex $v \in V(G)$, consider an n-simplex $\sigma^n(v)$ and label its $n+1$ vertices by Δ_n; and

(2) if v and w are joined in G by an i-colored edge, $i \in \Delta_n$, then identify the $(n-1)$-faces of $\sigma^n(v)$ and $\sigma^n(w)$ opposite to the vertex labelled by i, so that equally labelled vertices coincide.

We say that an $(n+1)$-colored graph (G, c) *represents* a closed (i.e., compact, connected and having no boundary) PL n-manifold M^n if $|G|$ is PL homeomorphic to M.

Theorem 5.1 (Existence). *Let M^n be a closed PL n-manifold. Then there exists an $(n+1)$-colored graph G which represents M, i.e., $|G| \approx_{\mathrm{PL}} M$.*

Proof. Suppose K is a (simplicial) triangulation of M. Then the first barycentric subdivision K' of K is vertex-colorable, in the sense that each n-simplex of K' contains $n+1$ differently colored vertices. In fact, we color by $i \in \Delta_n$ all the vertices of K' which are barycenters of the i-simplices of K. Let now G be the 1-skeleton of the dual cellular subdivision of K'. Obviously, G is a multiple graph and is regular of degree $n+1$. Furthermore, each edge of G inherits the color of the vertex of K' which does not belong to the dual $(n-1)$-simplex. The graph G with the edges colored in this way represents M, i.e., $|G| \approx_{\mathrm{PL}} M$. $\square$

This result can be extended to PL manifolds with nonempty boundary and to PL generalized (homology) manifolds by suitable modifications of the definition of the colored graph (see [12]).

By means of Theorem 5.1, we can analyze PL manifolds using colored graphs which represent them and reduce the study of PL manifolds to problems from graph theory. Unfortunately, there are many different colored graphs representing the same manifold. However, these graphs are related by a finite sequence of *moves*, defined as follows:

Given an $(n+1)$-colored graph G, an *m-residue* of G is a connected component of a subgraph generated by any m specified colors. A subgraph ϑ of G, formed by two vertices v and w, joined by k edges, $1 \leqslant k \leqslant n$, with colors $c_1, c_2, \ldots, c_k$ is called a *dipole of type k* if v and w belong to distinct $(n+1-k)$-residues, generated by $\Delta_n \setminus \{c_1, c_2, \ldots, c_k\}$. *Cancelling* ϑ means the following:

(1) delete v, w and the k edges joining them;
(2) to paste together the pairs of dangling edges (the ones which had an end-point in the deleted vertices) of the same color.

Adding ϑ means the inverse process.

Theorem 5.2 (equivalence). *Two colored graphs represent PL homeomorphic manifolds if and only if one can be transformed into the other by a finite sequence of cancelling and/or adding dipoles.*

It follows that every topological property of a closed PL manifold can be deduced from the properties of the colored graph which represents it. For example, it can be proved that the orientability of a manifold is equivalent to the bipartiteness of the representing graphs. Furthermore, connected sums of PL n-manifolds M_1 and M_2 correspond to *connected sums* of representing graphs G_1 and G_2, respectively. Indeed, one can match arbitrarily the colors of G_1 and G_2, take off a vertex from either graph and paste together the dangling edges with colors corresponding in the matching.

5.1. Characterizations

An immediate characterization of colored graphs representing manifolds is provided by the following:

Theorem 5.3. *Let (G, c) be an $(n+1)$-colored graph. Then G represents a closed PL n-manifold if and only if every n-residue (i.e., a component of a partial subgraph obtained by deleting one color at a time) represents a standard $(n-1)$-sphere.*

In dimension three it is very easy to find an arithmetical condition for recognizing colored graphs which represent 3-manifolds. Namely, one can show that a 4-colored graph G represents a closed 3-manifold if and only if the relation $q_0 + q_3 = q_2$ holds, where q_i denotes the number of i-residues of G. For this purpose recall that the Euler characteristic of the manifold on one hand vanishes and on the other hand it equals $q_3 - q_2 + q_0$.

Problem 5.4. Find algebraic characterizations of colored graphs representing closed PL 4-manifolds with special fundamental groups (e.g., finite, free products, surface groups, poly-(finite or cyclic), amenable group, etc.).

Problem 5.5. Find algebraic characterizations of colored graphs representing PL (singularly) fibered manifolds (e.g., good torus fibrations, 4-manifolds with torus actions, circle bundles, surface bundles, Seifert manifolds, etc.).

Problem 5.6. Find an algebraic characterization of colored graphs representing the standard n-sphere S^n, $n \geqslant 3$.

Problem 5.7. Heegaard diagrams and branched coverings can be described by graph-theoretic tools. Find a simple algorithm, determining a colored graph which represents the 3-manifold obtained by surgery along an oriented link (with coefficients) in the oriented 3-sphere S^3.

5.2. Homotopy

It is well known (see, e.g., [18]) how to deduce a presentation of the fundamental group $\Pi_1(M)$ of a closed PL n-manifold M from a colored graph G representing M. We shall sketch this construction. Assume that each n-subgraph of G (i.e., each partial graph obtained by deleting a color at a time) is connected (eventually, cancelling suitable dipoles). Choose two colors α and β in the color set Δ_n and denote by $x_1, x_2, \ldots, x_p$ the connected components, except one, of the $(n-1)$-subgraph of G obtained by deleting α- and β-colored edges. It is easy to see that the connected components of the complementary 2-subgraph are simple cycles, whose edges are alternatively colored by α and β. If $n = 2$, let y_1 be the only connected component. If $n \geqslant 3$, denote by $y_1, y_2, \ldots, y_q$ all components, except one; fix an orientation and a starting point for each of them. Compose the word r_j from the cycle y_j, by the following rule: follow the chosen direction, starting from the chosen vertex and list consecutively every x_i you meet with exponent $+1$ or -1 according to whether α or β is the color of the edge which leads you into x_i.

Theorem 5.8. *With the above notation, the fundamental group of the closed PL n-manifold M admits the following finite group presentation*:

$$\Pi_1(M) \cong \langle x_1, x_2, \ldots, x_p : r_1, r_2, \ldots, r_q \rangle.$$

Problem 5.9. Deduce combinatorial descriptions of the homotopy groups $\Pi_i(M)$, $i \geqslant 2$, from a colored graph representing M.

Problem 5.10. Under which combinatorial conditions on colored graphs representing M certain homotopy groups (e.g., $\Pi_1(M)$ or $\Pi_2(M)$) vanish?

5.3. Homology

A homology theory for colored graphs was developed in [14] as follows:

Let G be an $(n+1)$-colored graph. Two vertices v and w of G are said to be Γ-*connected*, $\Gamma \subset \Delta_n$, if they are joined by a finite sequence of edges with colors in Γ. Let us define:

$$S_k(G) = \{(\Gamma, V) \mid \#\Gamma = k \wedge V \text{ is a } \Gamma\text{-connected component of } G\}.$$

We say that the free R-module (R a principal ideal domain) $C_k(G)$ on the set $S_k(G)$ is the module of k-*chains* of G. For simplicity, we shall suppress the R-coefficients. We set $C_*(G) = \{C_k(G)\}_{k\in\mathbb{Z}}$ and $C_k(G) = 0$ if $S_k(G) = \emptyset$.

Next, we define the boundary operator on $C_*(G)$. Let

$$\sigma_A : A \to A \quad (A \subset \mathbb{Z} \text{ a finite set})$$

be the cyclic permutation defined by

$$\sigma_A(m) = \begin{cases} \min\{z:\ z \in A \wedge z > m\} & \text{if } m < \max A, \\ \min A & \text{if } m = \max A. \end{cases}$$

Let $\operatorname{ord}(A, m)$ be an integer such that $\sigma_A^{\operatorname{ord}(A,m)}(\min A) = m$ and define $\partial : C_*(G) \to C_*(G)$ by setting

$$\partial(\Gamma, V) = \sum_{i\in\Gamma} (-1)^{\operatorname{ord}(\Gamma,i)} \sum \{(\Gamma \setminus \{i\}, W)\}$$

where the second summation is taken over all pairs $(\Gamma \setminus \{i\}, W)$ such that

(1) W is a $(\Gamma \setminus \{i\})$-connected component of G; and

(2) $W \subset V$.

Since $\partial C_{p+1}(G) \subset C_p(G)$ and $\partial\circ\partial = 0$, the pair $(C_*(G), \partial)$ is a chain complex. Thus the homology $H_*(C_*(G))$ and the cohomology $H^*(C_*(G))$ of G are defined in the usual way, denoted by $H_*(G)$ and $H^*(G)$, respectively. Obviously, a (colored) isomorphism between colored graphs induces an isomorphism between the respective (co)homology groups.

Theorem 5.11. *Let G be an $(n+1)$-colored graph. Then we have $H_p(G) \cong H^{n-p}(|G|)$ and $H^p(G) \cong H_{n-p}(|G|)$. Furthermore, if G represents an orientable closed homology n-manifold, then $H_p(G) \cong H^{n-p}(G)$.*

We refer to [14] for details and more information on the combinatorial analogs to exact homology sequences, products, duality, etc.

Problem 5.12. Find combinatorial conditions for recognizing colored graphs which represent homology n-spheres, $n \geqslant 3$.

5.4. Combinatorial invariants

Various numerical invariants can be associated to PL manifolds via representations by colored graphs (see, e.g., [7,10,11,18,22] for various concepts of complexity, regular genus, etc.). Here, we shall only discuss one which was first introduced in [7]. Let M^n be a closed PL n-manifold and G a colored graph representing M. Denote by $\mathcal{G}(M)$ the set of colored graphs which represent M and define $\beta_k(G) = \operatorname{rank} C_{n-k}(G)$ and $c_k(M) = \min\{\beta_k(G) \mid G \in \mathcal{G}(M)\}$. We say that $c_k(M)$ is the *k-complexity* of M.

The *reduced complexities* of M are:

$$\tilde{c}_k(M) = c_k(M) - \binom{n+1}{k+1}, \quad 0 \leqslant k \leqslant n-1,$$

$$\tilde{c}_n(M) = c_n(M) - 2.$$

The following was proved in [14]:

Theorem 5.13.

(1) *Let M and M' be closed PL n-manifolds. Then the reduced complexities are subadditive, i.e., $\tilde{c}_k(M \# M') \leqslant \tilde{c}_k(M) + \tilde{c}_k(M')$.*

(2) *$\tilde{c}_0(M) = 0$ and $\tilde{c}_n(M) = 0$ if and only if $M \approx_{\mathrm{PL}} S^n$ (n-sphere).*

(3) *For every closed surface M, $\tilde{c}_1(M) = \frac{3}{2}\tilde{c}_2(M) = 6 - 3\chi(M)$, where $\chi(M)$ is the Euler–Poincaré characteristic of M. Furthermore, the reduced complexities are additive in dimension two.*

(4) *If M is a closed 3-manifold, then $\tilde{c}_1(M) = \frac{1}{2}\tilde{c}_2(M) = \tilde{c}_3(M)$.*

(5) *If M is a closed PL 4-manifold, then $\tilde{c}_1(M) = 6 - 3\chi(M) + \frac{1}{2}\tilde{c}_4(M)$, $\tilde{c}_2(M) = 4 - 2\chi(M) + 2\tilde{c}_4(M)$, and $\tilde{c}_3(M) = \frac{5}{2}\tilde{c}_4(M)$.*

Conjecture 5.14. Let M and M' be two closed orientable PL 4-manifolds. Then

$$\tilde{c}_4(M \# M') = \tilde{c}_4(M) + \tilde{c}_4(M').$$

Conjecture 5.15. If M is a simply-connected closed PL 4-manifold, then $\tilde{c}_4(M) = 6\chi(M) - 12$.

Remark. Conjecture 5.14 implies the PL (DIFF) 4-dimensional Poincaré conjecture, i.e., that every homotopy 4-sphere is PL (DIFF) homeomorphic to the standard 4-sphere S^4. Indeed, let M be a homotopy 4-sphere. Then there exists a nonnegative integer k such that $M \# k(S^2 \times S^2)$ is diffeomorphic to $k(S^2 \times S^2)$, by a well-known theorem of Wall. Here $k(S^2 \times S^2)$ represents the connected sum of k copies of $S^2 \times S^2$. Conjecture 5.14 implies that

$$\tilde{c}_4(M \# k(S^2 \times S^2)) = \tilde{c}_4(M) + \tilde{c}_4(k(S^2 \times S^2)) = \tilde{c}_4(k(S^2 \times S^2)),$$

hence $\tilde{c}_4(M) = 0$ and so by Theorem 5.13(2), $M \approx_{\mathrm{PL}} S^4$.

Problem 5.16. Complexity gives rise to an interesting family of colored graphs: $G \in \mathcal{G}(M^n)$ is said to be *minimal* if $c_n(M^n) = \beta_n(G)$. Study the properties of these graphs and find combinatorial characterizations of them.

As it was pointed out by Kühnel in [26] our equations for $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$, and $\tilde{c}_4$ in the case of a closed PL 4-manifold are similar to the Dehn–Sommerville equations. Indeed, the expression $2 - \chi(M)$ occurs in a natural way in Theorem 5.13(5) as well as in [25], in particular in the following conjecture:

Conjecture 5.17* (Kühnel). For every triangulation of a closed PL $(2k)$-manifold M^{2k} with n vertices, the following inequality holds:

$$\binom{n-k-2}{k+1} \geqslant (-1)^k \binom{2k+1}{k+1} (\chi(M) - 2),$$

with equality if and only if the triangulation is $(k+1)$-neighbourly (see [25] for the definition).

Problem 5.18 (Kühnel). Find the relationship between the two concepts above.

5.5. Homeomorphism type

From now on we shall consider 3-dimensional closed manifolds. It is well known that the Heegaard genus one 3-manifolds are topologically classified. In particular, a colored graph $G(p,q)$ representing the lens space $L(p,q)$ is easily constructed as follows:

Take two cycles of length $2p$ and let $v_1, v_2, \ldots, v_{2p}$ and $w_1, w_2, \ldots, w_{2p}$ be their vertices ordered cyclically and with indices in $\mathbb{Z}_p$. Color the edges alternatively with 0 and 1 so that the indices correspond. Then put edges of color 2 between v_i and w_i and those of color 3 between v_i and w_{i+2q}.

One can extend this construction in a natural way to represent all Heegaard genus two 3-manifolds by 4-colored graphs, depending upon 6-tuples of positive integers:

Theorem 5.19. *Let M be a closed 3-manifold of Heegaard genus two. Then there exists a 4-colored graph, depending upon a 6-tuple of positive integers $(n_1, h_1; n_2, h_2; n_3, h_3)$, which represents M.*

Proof (*Sketch*). Take three cycles C_0, C_1, and C_2 of length $2n_1$, $2n_2$, and $2n_3$, respectively, set cyclically in the plane and with indices in $\mathbb{Z}_3$. Color the edges of the cycles alternatively with 0 and 1. Connect, in a standard fashion, C_i with C_{i+1} with a set of $n_i + n_{i+1} - n_{i-2}$ parallel edges of color 2 (indices mod 3) in order to obtain a planar 3-colored graph (which of course represents the standard 2-sphere S^2). Repeat the same construction with sets of parallel edges colored by 3 after the clockwise rotations described by three positive integers h_1, h_2 and h_3.

Deleting all 2-colored edges obviously yields a planar graph. □

Problem 5.20. What relations exist between two 6-tuples generating graphs which represent the same manifold? Classify the homeomorphism type of the Heegaard genus two 3-manifolds in terms of 6-tuples representing them.

Problem 5.21. What arithmetic conditions on these 6-tuples of integers yield homology 3-spheres?

There is a simple combinatorial proof of the classical Viro theorem to the effect that every genus two orientable closed 3-manifold is a 2-fold covering of the 3-sphere S^3 branched over a 3-bridge link, by using 6-tuples representing graphs. Namely, one constructs a suitable colored involution acting on the graph.

Problem 5.22. Write a finite presentation of the fundamental group of a Heegaard genus two 3-manifold in terms of a 6-tuple of integers giving a colored graph which represents it.

In a forthcoming paper we shall classify the topological structure of all genus two homology 3-spheres, represented by graphs arising from 6-tuples as above, up to 50 vertices. Moreover, we shall present a computer program which generates a catalogue (with possible repetitions) of the Heegaard genus two 3-manifolds together with a presentation of their fundamental groups.

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