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Peripheral Acyclicity and Homology Manifolds (*)(1)

ALBERTO CAVICCHIOLI (2) - DUŠAN REPOVŠ (3)

Abstract. – *In this paper we study various concepts of peripheral acyclicity and local homotopical properties, generalizing and extending earlier work due to R. C. Lacher, D. R. McMillan jr. and the authors. Then we obtain some applications for cell-like decompositions of manifolds and construct exotic factors (i.e. homology manifolds) of certain Euclidean spaces.*

1. – Preliminaries.

In this section we recall various concepts of peripheral acyclicity and local homotopical properties for compacta embedded into manifolds. These properties play a central role in *decomposition theory* of manifolds as shown, for example, in [16]. Indeed, some important problems of modern geometric topology of manifolds have been solved by using the techniques and results of this theory. Here we can mention e.g. the solution of the Recognition problem for higher-dimensional (TOP) manifolds, the existence of exotic factors of manifolds and a proof of the celebrated Double suspension theorem, i.e. that *the double suspension of a homology n-sphere is* (TOP) *homeomorphic to the standard* $(n + 2)$*-sphere*. As general references about these arguments

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Indirizzo degli AA.: A. CAVICCHIOLI: Dipartimento di Matematica, Università di Modena, Via Campi 213/B, 41100 Modena, Italy, e-mail: albertoc@unimo.it; D. REPOVŠ: Current address: Dipartimento di Matematica, Università di Trieste, Piazzale Europa 1, 34100 Trieste, Italy; Permanent address: Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P.O. Box 64, Ljubljana 61111, Slovenia, e-mail: dusan.repovs@uni-lj.si
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we refer, for example, to the survey papers [12], [30], [34], [40] and the book [16].

In this paper we use peripheral acyclicity to obtain some results concerning regular neighbourhoods of polyhedra embedded into (PL) n-manifolds. We also give some applications for decompositions, or partitions, of manifolds into cell-like sets. These sets behave homotopically much like points and they allow constructions of many exotic factors of manifolds.

Let X be a compact connected set in the interior of a topological n-manifold M^n. Here *manifolds* will be assumed to be connected and to have no boundary unless specified otherwise. We say that the inclusion $X \subset M$ has the *property k-uv* (R), $k \geq 0$, where R is a principal ideal domain (PID), if for each neighbourhood $U \subset M$ of X there is a neighbourhood $V \subset U$ of X such that the inclusion-induced homomorphism $H_k(V; R) \to H_k(U; R)$ is trivial. Next, X has the *property* $uv^k(R)$ (resp. $uv^\infty(R)$) if it satisfies j-*uv* (R) for any j: $0 \leq j \leq k$ (resp. for all $j \geq 0$). These local properties are topological invariants of X so they do not depend on the embedding $X \subset M$. More precisely, the *uv* conditions are related to the (reduced) Čech cohomology $\check{H}^*(X; R)$ of X as proved in [30]:

THEOREM 1.1. – (1) *If a compactum X has the properties j-uv* (R), $j \in \{k-1, k\}$, *then*

$$\check{H}^k(X; R) = 0 .$$

(2) *If* $\check{H}^j(X; R) = 0$, $j \in \{k, k+1\}$, *then X has the property k-uv* (R).

(3) *A finite-dimensional compactum X has the property* uv^∞ (R) *if and only if* $\check{H}^*(X; R) = 0$. ∎

A compactum $X \subset M$ has the *weak peripheral* (*wp*) *k-uv* (R) *property* if for each neighbourhood $U \subset M$ of X there is a neighbourhood $V \subset U$ of X such that $H_k(V \backslash X; R) \to H_k(U; R)$ is trivial. Next, X has the *wp* uv^k (R) (resp. *wp* uv^∞ (R)) *property* if it satisfies *wp j-uv* (R) for any j: $0 \leq j \leq k$ (resp. for all $j \geq 0$).

Finally, a compactum $X \subset M$ has the *strong peripheral* (*sp*) *k-uv* (R) *property* if, under the above conditions, $H_k(V \backslash X; R) \to H_k(U \backslash X; R)$ is trivial. Then X has the *sp* uv^k (R) (resp. *sp* uv^∞ (R)) *property* if it satisfies *sp j-uv* (R) for any j: $0 \leq j \leq k$ (resp. for all $j \geq 0$).

Peripheral homological properties of type *uv* were used by various authors to describe the topological structure of regular neighbourhoods of compacta (PL) embedded in 3-manifolds (see, for example [14], [31], [32], [33], [37], [39]).

If instead of homology R-modules, one uses homotopy groups, one gets the corresponding peripheral homotopical properties, denoted by *k-UV* (UV^k; UV^∞), and the *weak* (resp. *strong*) *peripheral k-UV* (UV^k; UV^∞) *property*, abbreviated as *WP* (resp. *SP*) *k-UV* (UV^k; UV^∞). In particular, property UV^∞ means that arbitrarily close neighbourhoods $V \subset U$ of X can be chosen so that V is contractible to a point in U. The local homotopical properties *WP* 1-*UV* and *SP* 1-*UV* correspond to McMillan, Jr's *WCC* (*weak cellularity criterion*) and *CC* (*cellularity criterion*), respectively (see [29], [31], [38]). Conditions of type *UV* were introduced and first analyzed by Armentrout (see [2], [3]) and were used to extend the classical notion of cellularity in the sense of [7]. A set X in a (TOP) n-manifold M^n is said to be *cellular* if there exists a properly nested decreasing sequence of closed n-cells $B_i^n \subset M$ such that $X = \bigcap_i B_i^n$. Cellularity is not an intrinsic property of X because it depends on the embedding $X \subset M$. This dependence can be avoided by using a more general concept, called cell-likeness. A space X is said to be *cell-like* if there exists a (TOP) n-manifold M^n and an embedding $f: X \to M$ such that $f(X)$ is cellular in M. As shown in [30], property UV^∞ characterizes cell-like sets:

THEOREM 1.2. – *Let X be a finite-dimensional compactum. Then X is cell-like if and only if X has property UV^∞.* ∎

Obviously, a cell-like set in M may fail to be cellular in M. For example, let $X = [0, 1]$. Then the standard embedding of X in $\mathbb{R}^n$ is a cellular arc. But there exist *wild* arcs (embeddings of X) which are not cellular in $\mathbb{R}^n$ for any $n \geq 3$ (see [1], [22]). However, every wild arc is cell-like because it is contractible. As another example, polyhedral copies of the *dunce hat* are cell-like but may be non-cellular in $\mathbb{R}^4$ as shown in [42]. It is very easy to see that the concepts of cellularity and cell-likeness agree in dimensions $n \leq 2$. A simple geometric criterion for cellularity of a compactum in a (TOP) n-manifold, $n \geq 4$, was given in [31], [38] by using a strong peripheral property of type *UV*.

THEOREM 1.3 (*The cellularity criterion*). – *Suppose that X is a cell-like compactum in the interior of a topological n-manifold, $n \geq 4$. Then X is cellular if and only if X satisfies SP 1-UV property.* ∎

In dimension three, some precaution must be taken due to the unresolved status of the Poincaré conjecture. In dimension two, a compactum X in the interior of a surface is cellular if and only if X has the property uv^∞ (R) for some PID R. Cell-like sets have been extensively studied in connection with the *decomposition theory* of manifolds (see for example [3], [16], [21], [30], [34]). If $X \subset M^n$ is a cell-like set in an n-manifold, let M^n/X denote the quotient space obtained by shrinking X to a point. Now, M/X may fail to be a genuine manifold but it is always a *homology manifold*, i.e. it possesses all the basic homology properties of manifolds. In particular, if X is a wild arc in $\mathbb{R}^n$, $n \geq 3$, then $\mathbb{R}^n/X$ is not a manifold but $(\mathbb{R}^n/X) \times \mathbb{R}$ is nevertheless (TOP) homeomorphic to $\mathbb{R}^{n+1}$ (see [1], [4], [20], [22]). Thus non-manifold (*exotic*) factors of higher-dimensional Euclidean spaces were discovered. Moreover, we have the following celebrated result, due indipendently to Edwards and Cannon (see [12], [20], [21]).

THEOREM 1.4. – *If X is a cell-like set in $\mathbb{R}^n$, then $(\mathbb{R}^n/X) \times \mathbb{R}$ is (TOP) homeomorphic to $\mathbb{R}^{n+1}$. In particular, $\mathbb{R}^n/X$ is a topological manifold if and only if X is cellular in $\mathbb{R}^n$.* ∎

Finally, we observe that the topological classification of simply-connected closed (TOP) 4-manifolds, due to FREEDMAN (see [23], [25], [28]), is based also upon a result of the previous type, where $X \subset \mathbb{R}^3$ is the *Whitehead continuum* (see for example [16], [28]).

2. – Relations between peripheral properties.

In this section we compare the concepts of peripheral acyclicity and local *UV* properties. Then we apply these results, in the next sections, to regular neighbourhoods of polyhedra PL embedded into PL manifolds and cell-like decompositions of manifolds.

First, we observe that the *sp k-uv* (R) property obviously implies the *wp k-uv* (R) property over any PID R. This implication can be partially reversed as follows:

THEOREM 2.1. – *Let X be a compactum in the interior of a (PL) n-manifold M^n, $n \geq 3$. Suppose that $X \subset M$ has the wp 1-uv property over $\mathbb{Z}_2$ (resp. $\mathbb{Z}$). Then X satisfies sp 1-uv over $\mathbb{Z}_2$ (resp. $\mathbb{Z}$).*

PROOF. – The case $n = 3$ was proved in [37] by use of Dehn's surgery. For higher dimensions, we give a different proof based on general position and existence of embedded disks. For convenience, we supress the coefficients. By hypothesis, we can express X as the intersection of compact PL n-manifolds $N_i \subset \operatorname{Int} M$ with boundary which satisfy the following properties:

(1) $N_{i+1} \subset \operatorname{Int} N_i$ for each i.

(2) The inclusion-induced homomorphism $H_1(N_i \backslash X) \to H_1(N_{i-1})$ is trivial.

We have to show that $H_1(N_i \backslash X) \to H_1(N_{i-1} \backslash X)$ is trivial, too. For this, let α be a simple closed PL curve in $N_i \backslash X$. Then there is an integer $j > i$ such that $\alpha \subset N_i \backslash N_j$. Let β_k be a simple closed PL curve whose homotopy class is a generator of $\Pi_1(\partial N_{j+1})$. Since the homomorphism

$$H_1(N_{j+1} \backslash X) \to H_1(N_j)$$

is trivial, the curve β_k bounds a PL R-orientable surface Γ_k in $N_j \subset N_i \subset N_{i-1}$. For dimension $n = 3$, we refer to [26] where this fact is extensively used. For $n \geq 5$, it is an easy consequence of general position (see [27]). For $n = 4$, we can always suppose, by general position, that β_k bounds a PL 2-dimensional pseudo-manifold (see, for example, [8], [9], [10]) with isolated singularities. A regular neighbourhood of a singular point is a cone over a disjoint union of unlinked 1-spheres. Now, we can replace it by a punctured 2-sphere inside a small 4-cell in N_j, which is a regular neighbourhood of the singular point in N_j. Since $H_1(N_i \backslash X) \to H_1(N_{i-1})$ is trivial, the curve α bounds a PL R-orientable surface Γ in N_{i-1}. By the general position theorem for piecewise-linear (PL) category (see [27], p. 97) there exists an ambient isotopy h of N_{i-1} such that

(1) h fixes the points of ∂N_{i-1}.

(2) $\tilde{\Gamma} = h(\Gamma)$ is in general position with respect to each β_k, i.e. (use $n \geq 4$)

$$\dim(\tilde{\Gamma} \cap \beta_k) \leq \dim \tilde{\Gamma} + \dim \beta_k - n \leq -1 .$$

(3) $d(h_t(x), x) < \varepsilon(x)$ for all points $x \in N_{i-1}$, where d is a metric for the topology of N_{i-1} and $\varepsilon\colon N_{i-1} \to \mathbb{R}$ is a continuous positive function.

Thus we have found an R-orientable PL surface $\tilde{\Gamma} \subset N_{i-1}$ such that $\partial \tilde{\Gamma} = \tilde{\alpha}$ is homologous with α and $\tilde{\Gamma} \cap \beta_k$ is empty for any generator $[\beta_k]$ of $\Pi_1(\partial N_{j+1})$. Now, if $\tilde{\Gamma}$ intersects N_{j+1}, it enters through ∂N_{j+1} along simple closed PL curves γ_r which are null-homotopic in ∂N_{j+1}. Let D_r^2 be a singular 2-disk in ∂N_{j+1} bounded by γ_r. If either $\dim \partial N_{j+1} = 3$ or $\dim \partial N_{j+1} \geq 5$, we can replace D_r with an embedded 2-disk $B_r^2 \subset \partial N_{j+1}$ by using Lemma 4.1 [26] (*Dehn's Lemma*) and Corollary 4.4 [24] (*Existence of embedded disks*), respectively. If $\dim \partial N_{j+1} = 4$, then D_r can be replaced with an embedded 2-disk B_r in a small collar of ∂N_{j+1} in N_{j+1}, which misses X. Finally, surgery along B_r's cuts off $\tilde{\Gamma}$ at N_{j+1}. Thus α is null-homologous in $N_i \backslash N_{j+1} \subset N_{i-1} \backslash X$ as requested. ■

EXAMPLE. – Theorem 2.1 may fail to be true in dimension two. We describe a polyhedron X in the Klein bottle which has the *wp* 1-*uv* ($\mathbb{Z}_2$) property but it does not satisfy the *sp* 1-*uv* ($\mathbb{Z}_2$) property. Furthermore, X is neither *wp* 1-*uv* ($\mathbb{Z}$) nor *wp* 1-*uv* ($\mathbb{Z}_q$), for any odd integer q. Let $X \subset M$ be the zero-section of the Klein bottle M, considered as a twisted $\mathbb{S}^1$-bundle over $\mathbb{S}^1$. Then any neighbourhood of X is homeomorphic to a Möbius band. Now, for any pair of close enough neighbourhoods $V \subset U$ of X, $V \backslash X$ is an annulus which represents the orientable double covering of V (and hence also of U). Thus the map $H_1(V \backslash X; \mathbb{Z}) \to H_1(U; \mathbb{Z})$ is the isomorphism $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$. Then X does not satisfy the *wp* 1-*uv* (R) property for $R = \mathbb{Z}$ or $R = \mathbb{Z}_q$, q odd. But X has the *wp* 1-*uv* ($\mathbb{Z}_2$) property since $H_1(V \backslash X; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \xrightarrow{\times 2} H_1(U; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ is trivial. However, X is not *sp* 1-*uv* ($\mathbb{Z}_2$) because $H_1(V \backslash X; \mathbb{Z}_2) \to H_1(U \backslash X; \mathbb{Z}_2)$ is the identity isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$.

QUESTION. – Is it possible to generalize Theorem 2.1 to the «trivial» range, i. e. to show that *wp* k-*uv* implies *sp* k-*uv* for every $k \leq [n/2]$?

THEOREM 2.2. – *Let X be a compactum in the interior of an R-orientable (TOP) n-manifold M^n, $n \geq 3$, R a PID. Suppose that X has the property uv^{n-2} (R). Then X satisfies sp uv^{n-2} (R).*

PROOF. – We shall supress the coefficients. Let $V \subset U \subset M^n$ be neighbourhoods of X such that $H_j(V) \to H_j(U)$ is trivial for any j: $0 \leq j \leq n-2$. Theorem 1.1 implies that $\check{H}^j(X) \simeq 0$ for $0 \leq j \leq n-2$. Let us consider the following commutative diagram $(1 \leq j \leq n-2)$:

$$\begin{array}{ccccccccc} \dots & & \longrightarrow & H_j(V \backslash X) & \xrightarrow{i'_*} & H_j(V) & \longrightarrow & \dots \\ & & & \downarrow{\scriptstyle j'_*} & & \downarrow{\scriptstyle j_* = 0} & & \\ \dots \longrightarrow & H_{j+1}(U, U \backslash X) & \longrightarrow & H_j(U \backslash X) & \xrightarrow[i_*]{} & H_j(U) & \longrightarrow & \dots . \end{array}$$

By the Alexander duality, we have

$$H_{j+1}(U, U \backslash X) \underset{\text{iso}}{\simeq} \check{H}^{n-j-1}(X) \underset{\text{iso}}{\simeq} 0$$

as $1 \leq n-j-1 \leq n-2$. Hence i_* is a monomorphism. Because $j_* = 0$, it follows that $j'_* = 0$. This proves the assertion. ∎

EXAMPLE. – The converse of Theorem 2.2 is false in general. Let $M^n = \mathbb{S}^k \times \mathbb{S}^{n-k}$, $n \geq 3$, $k \geq 1$, and $X = \mathbb{S}^k \vee \mathbb{S}^{n-k}$. Since $M \backslash X$ is an open n-cell, X has the *WP* (*SP*) UV^∞ and the *wp* (*sp*) uv^∞ (R) properties for any PID R. But X is clearly neither UV^k nor uv^k for any R.

For $n = 3$, Theorem 2.2 implies the following consequence about certain neighbourhoods of X in M^3.

THEOREM 2.3. – *Let X be a compactum in the interior of an orientable 3-manifold M. Then the following statements are equivalent:*

(1) *X has the property 1-uv ($\mathbb{Z}_p$,) where p is either zero or a prime number.*

(2) *There exists a neighbourhood U of X such that the inclusion-induced homomorphisms $H_1(U; \mathbb{Z}_p) \to H_1(M; \mathbb{Z}_p)$ and $H_1(U \backslash X; \mathbb{Z}_p) \to H_1(M \backslash X; \mathbb{Z}_p)$ are trivial.*

PROOF. – By Theorem 2.2, if X has the property 1-*uv* ($\mathbb{Z}_p$), then X satisfies *sp* 1-*uv* ($\mathbb{Z}_p$), hence condition (2) is verified. For the converse, let $W \subset M$ be a neighbourhood of X. We may assume that $W \subset U$ and that W is an orientable compact 3-manifold with boundary such that $X \subset \operatorname{Int} W \subset W \subset U$. For any boundary component $F \subset \partial W$ there exists a bouquet L_F of simple closed PL curves on F such that $F \backslash L_F$ is an open 2-cell. By hypothesis, each loop of L_F bounds in $M \backslash X$. Let S be the union of the surfaces bounded by L_F. Let $V \subset \operatorname{Int} W$ be a neighbourhood of X which does not intersect S. Let $\alpha \subset V$ be a 1-cycle. We may always assume that α is a piecewise-linear (PL) simple closed curve as it suffices to show that any such a curve in V bounds in W. By hypothesis, α bounds a surface Γ_α in M. Let Γ_β be a surface of S bounded by a loop β of L_F. We observe that $\alpha \cap \Gamma_\beta$ is empty. Now we can assume Γ_α and Γ_β to be in general position. Because the linking number between α and β is zero, we can adjust the intersection $\Gamma_\alpha \cap \Gamma_\beta$ to be empty. Repeating this process for any loop of L_F and for any boundary component $F \subset \partial W$, it follows that α bounds an R-orientable surface $\tilde{\Gamma}_\alpha$ in $M \setminus \bigcup\{\beta\colon \beta \in L_F$ and $F \subset \partial W\}$. Hence $\tilde{\Gamma}_\alpha$ enters ∂W through open disks and so it can be cut off at ∂W. Then we may assume that $\tilde{\Gamma}_\alpha \subset W$, i.e. $H_1(V; \mathbb{Z}_p) \to H_1(W; \mathbb{Z}_p)$ is trivial, where p is either zero or a prime number. This proves that X satisfies the property 1-*uv* $(\mathbb{Z}_p)$ as claimed. ∎

Obviously, property *k-uv* (R) (resp. *k-UV*) directly implies *wp k-uv* (R) (resp. *WP k-UV*) for any $k \geq 0$. Restricting dimension of X yields a partial converse of this implication.

PROPOSITION 2.4. – *Let R be a PID and let X be a compact set in the interior of an R-orientable n-manifold M^n, $n \geq 3$. Suppose that X has dimension $\leq n-2$. Then $X \subset M$ has the wp* 1-*uv* (R) *(resp. WP* 1-*UV) property if and only if X satisfies the property* 1-*uv* (R) *(resp.* 1-*UV).*

PROOF. – Let $V \subset U \subset M$ be neighbourhoods of X such that $H_1(V \setminus X) \to H_1(U)$ (resp. $\Pi_1(V \setminus X) \to \Pi_1(U)$) is trivial. Let α be a 1-cycle (resp. loop) in V. Since $\dim X \leq n-2$, $\Pi_1(V \setminus X) \to \Pi_1(V)$ is surjective as proved in [29]. Thus α is homologous (resp. homotopic) to a 1-cycle (resp. loop) β in $V \setminus X$. By hypothesis, β is null-homologous (resp. null-homotopic) in U, hence so is α. This completes the proof. ∎

EXAMPLES. – (1) If $\dim X = n-1$, then Proposition 2.4 is false. Let $M^n = \mathbb{S}^{n-1} \times \mathbb{S}^1$ and $X = \mathbb{S}^{n-1} \vee \mathbb{S}^1$. Then X has *wp* (*sp*) 1-*uv* and *WP* (*SP*) 1-*UV* but it does not satisfy properties 1-*uv* (R) and 1-*UV*.

(2) By definition, the *SP k-UV* property directly implies *WP k-UV* for any $k \geq 0$. The converse is in general not true. Let X be either a wild arc in $\mathbb{R}^n$, $n \geq 3$, or a wild dunce hat in $\mathbb{S}^4$. Then X is a cell-like set so it satisfies property UV^∞. Because X is non-cellular, it does not have the *SP* 1-*UV* property (see Theorem 1.3). But X has the *WP* 1-*UV* property by Proposition 2.4.

PROPOSITION 2.5. – *Let X be a compact polyhedron in the interior of a PL n-manifold M^n and suppose that X has dimension $\leq n-k-1$, $k \geq 1$. Then X satisfies WP UV^k if and only if X has the property UV^k.*

PROOF. – By hypothesis, there exists a pair of PL neighbourhoods $V \subset U \subset M$ of X such that $\Pi_j(V \setminus X) \to \Pi_j(U)$ is trivial for any $j: 0 \leq j \leq k$. We have to show that $\Pi_j(V) \to \Pi_j(U)$ is trivial, too. Let $\alpha: \mathbb{S}^j \to V$ be a PL map. We can assume that $\alpha(\mathbb{S}^j)$ and X are in general position as polyhedra, i.e.

$$\dim(\alpha(\mathbb{S}^j) \cap X) \leq \dim \alpha(\mathbb{S}^j) + \dim X - n \leq k + n - k - 1 - n \leq -1 .$$

Thus we can push $\alpha(\mathbb{S}^j)$ off X by a general position argument. This defines a PL map

$\beta: \mathbb{S}^j \to V\backslash X$ which is homotopic to α. Therefore $\Pi_j(V\backslash X) \to \Pi_j(V)$ is surjective for any $j: 0 \leq j \leq k$. Now the result follows from the following commutative diagram

$$\begin{array}{ccc} \Pi_j(V\backslash X) & \xrightarrow{\text{zero}} & \Pi_j(U) \\ {\scriptstyle\text{epi}}\downarrow & & \| \\ \Pi_j(V) & \longrightarrow & \Pi_j(U). \end{array} \quad \blacksquare$$

THEOREM 2.6. – *Let X be a compact set in the interior of a PL n-manifold M^n, $n \geq 3$. Then the following statements (for $R = \mathbb{Z}_2$ or $\mathbb{Z}$) are equivalent:*

(1) *$X \subset M$ has the wp 1-uv (R) property.*

(2) *$X \subset M$ satisfies sp 1-uv (R).*

(3) *There exists a neighbourhood $W \subset M$ such that the inclusion-induced homomorphism $H_1(W\backslash X; R) \to H_1(M\backslash X; R)$ is trivial.*

PROOF. – Since by Theorem 2.1 (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) is obvious, we only have to show that (3) $\Rightarrow$ (2). The case $n = 3$ was proved in [37] by making use of Dehn's surgery. For $n \geq 4$, we apply general position arguments. Let $U \subset M$ be a neighbourhood of X. We may assume that $U \subset W$ and that U is a compact PL n-manifold with boundary containing X in its interior. Let $\beta_k \subset \partial U$ be a simple closed curve whose homotopy class is a generator of $\Pi_1(\partial U)$. By hypothesis, $H_1(\partial U; R) \to H_1(M\backslash X; R)$ is trivial, so β_k bounds an R-orientable surface Γ_k in $M\backslash X$. Let $V = \operatorname{Int} U \backslash \bigcup_k \{\Gamma_k\}$ and let α be a simple closed PL curve in $V\backslash X$. Then α bounds an R-orientable surface Γ in $M\backslash X$. Using the same arguments as in the proof of Theorem 2.1, we can show that α bounds an R-orientable surface $\tilde{\Gamma}$ in $M\backslash\left(X \cup \bigcup_k \beta_k\right)$. Thus $\tilde{\Gamma}$ enters ∂U through null-homotopic circles and we cut it off $\tilde{\Gamma}$ at ∂U. Therefore we can assume that $\tilde{\Gamma} \subset U\backslash X$. Thus the inclusion-induced homomorphism $H_1(V\backslash X; R) \to H_1(U\backslash X; R)$ is trivial as requested. $\blacksquare$

The next corollary extends to dimension n a result proved by MCMILLAN, Jr. for $n = 3$ (see [33]).

COROLLARY 2.7. – *Let X be a compact set in the interior of a PL n-manifold M^n, $n \geq 3$, and suppose that $H_1(M\backslash X; R) \simeq 0$, where $R = \mathbb{Z}_2$ or $\mathbb{Z}$. Then $X \subset M$ has sp 1-uv (R) property.*

PROOF. – Apply Theorem 2.6 for $W = M$. $\blacksquare$

At the end of the section we describe the relationship between homological and homotopical strong peripheral properties.

PROPOSITION 2.8. – *Let X be a compactum in the interior of a TOP n-manifold.*

(1) *If X has the SP UV^k property, then it satisfies sp uv^k ($\mathbb{Z}$).*

(2) *If X has the SP UV^{k-1} and sp k-uv ($\mathbb{Z}$) properties, then it has SP UV^k, provided $k \geq 2$.*

(3) *Suppose that X has the SP 1-UV property. Then X satisfies SP UV^k if and only if X has the sp uv^k ($\mathbb{Z}$) property.*

PROOF. – (1) For a given neighbourhood U of X find open sets U_i

$$X \subset U_0 \subset U_1 \subset \ldots \subset U_{k+1} \subset U$$

such that any map $\mathbb{S}^j \to U_j \backslash X$ extends to a map $B^{j+1} \to U_{j+1} \backslash X$, $0 \leq j \leq k$. Let $V = U_0$. If K is a simplicial k-complex, then any map $K \to V \backslash X$ extends to a map $v * K \to U \backslash X$, using inductively the inclusions $U_j \backslash X \to U_{j+1} \backslash X$ (here $v * K$ represents the cone on K from the vertex v). In fact, $V \backslash X = U_0 \backslash X$, $K^0 \to U_0 \backslash X$ extends to $K^1 \to U_1 \backslash X$, $K^1 \to U_1 \backslash X$ extends to $K^2 \to U_2 \backslash X$, etc., where K^q denotes the q-skeleton of K. It follows that any singular j-cycle in $V \backslash X = U_0 \backslash X$ bounds in $U \backslash X$, $0 \leq j \leq k$. Hence X has the *sp* uv^k ($\mathbb{Z}$) property.

(2) For a given neighbourhood U of X find path-connected open neighbourhoods of X

$$V \subset U_0 \subset U_1 \subset \ldots \subset U_k \subset U$$

such that each k-cycle in $V \backslash X$ bounds in $U_0 \backslash X$ and any j-sphere in $U_j \backslash X$ is null-homotopic in $U_{j+1} \backslash X$. Let α be a map $\mathbb{S}^k \to V \backslash X$. Then α is null-homologous in $H_k(U_0 \backslash X)$. Hence $\sum_i \alpha(\tau_i) = \partial c$, where $c = \sum_j \lambda_j \sigma_j$ is a $(k+1)$-chain in $U_0 \backslash X$ and $\tau_i : \Delta^k \to \mathbb{S}^k$ are simplicial mappings. Here Δ^k represents the standard k-simplex. Let K be a geometric realization of $\{\sigma_j\}$. We can extend α to a map $\beta : |K| \to U_0 \backslash X$. Define $K' = K \cup \mathcal{C}(K^{k-1})$, where $\mathcal{C}(K^{k-1})$ is the cone over K^{k-1}. Using the inclusions $U_j \backslash X \to U_{j+1} \backslash X$, β extends to a map $\bar{\alpha} : K' \to U \backslash X$, where K' is $(k-1)$-connected, $\sum_i \tau_i \sim 0$ in K and whence ~ 0 in K'. Now the Hurewicz theorem implies that $\mathbb{S}^k$ is null-homotopic in $|K'|$. Therefore, $\bar{\alpha}|_{\mathbb{S}^k} = \alpha$ is null-homotopic in $U \backslash X$.

(3) is a direct consequence of (2). ∎

3. – Embeddings of polyhedra into manifolds.

Throughout this section we shall study regular neighbourhoods of compact polyhedra embedded into PL n-manifolds, extending earlier results of the authors (see [14], [37], [39]). Our investigations were motivated by the following question which includes a problem settled by R. C. LACHER in [29].

QUESTION. – *Let X be a compact polyhedron. Suppose $f: X \to M$ is an embedding of X into the interior of a compact PL n-manifold M which is homotopic (isotopic) to the inclusion $X \subset M$. If $X \subset M$ has the weak (strong) peripheral UV^k (uv^k) property, does $f(X) \subset M$ have the same property?*

In general, the answer is negative. Let $M^n = \mathbb{S}^k \times \mathbb{S}^{n-k}$, $2k = n+1$, $n \geq 3$, and let $X = \mathbb{S}^k \vee \mathbb{S}^{n-k}$. It is well-known that $X \subset M$ has the *WP* (*SP*) UV^∞ property and hence it has *wp* (*sp*) uv^∞ (R) over any PID R. Let $\varphi: X \to \mathbb{R}^n$ and $\psi: \mathbb{R}^n \to M$ be PL embeddings. We show that $X' = (\psi \circ \varphi)(X) \subset M$ does not satisfy *wp* ($k-1$)-*uv* (R), nor *sp* ($k-1$)-*uv* (R). We supress the coefficients. Setting $U = M$ we would have V such that $\theta: H_{k-1}(V \backslash X') \to H_{k-1}(M)$ is trivial. But $H_{k-1}(M) \simeq R$ as $2k = n+1$ and $H_j(V \backslash X') \simeq H_j(\mathbb{R}^n \backslash \varphi(X)) \simeq H_{j+1}(\mathbb{R}^n, \mathbb{R}^n \backslash \varphi(X)) \simeq H^{n-j-1}(\varphi(X)) \simeq H^{n-j-1}(X) \simeq$ $\simeq H^{n-j-1}(\mathbb{S}^k) \oplus H^{n-j-1}(\mathbb{S}^{n-k}) \simeq R$ for any $j \in \{n-k-1, k-1, n-1\}$ and $H_j(V \backslash X') \simeq 0$ otherwise. This implies that θ is an isomorphism. If $n = 3$, $k = 2$, then X' does not have *WP* (*SP*) 1-*UV* property, nor *wp* (*sp*) 1-*uv* (R) as $\Pi_1(M) \simeq$ $\simeq \Pi_1(V \backslash X') \simeq \mathbb{Z}$ and $H_1(V \backslash X'; R) \simeq R$.

However, under certain conditions on X, we can answer in the affirmative Lacher's question stated below.

THEOREM 3.1. – *Let $f: X \to M$ be a PL embedding of a polyhedron X in the interior of a compact connected PL n-manifold M^n, $n \geq 3$. Suppose that f is homotopic to the inclusion $X \subset M$ and that $X \subset M$ has the wp uv^{n-2} (R) property for $R = \mathbb{Z}_2$ or $\mathbb{Z}$. Then $f(X) \subset M$ satisfies wp uv^{n-2} (R).*

In order to prove Theorem 3.1 we need some algebraic lemmas. For convenience, we use homology with $\mathbb{Z}_2$ coefficients. One can easily modify the argument to obtain the result in the other case.

LEMMA 3.2. – *Let X be a k-polyhedron and let $f: X \to M$ be a PL embedding of X into a closed connected PL n-manifold M^n, $k \leq n-1$, $n \geq 3$. If N is a regular neighbourhood of $f(X)$ in M, then we have that ($0 \leq p \leq n$)*

$$b_p(M \backslash f(X)) = b_p(M) - b_{n-p}(X) + \dim \operatorname{Ker} f_{*n-p} + \dim \operatorname{Ker} f_{*n-p-1}$$

and

$$b_p(\partial N) = b_{p+1}(M) + b_p(M) - b_{p+1}(X) - b_{n-p}(X) + \dim \operatorname{Ker} f_{*p} + \dim \operatorname{Ker} f_{*p+1} + \dim \operatorname{Ker} f_{*n-p} + \dim \operatorname{Ker} f_{*n-p-1}$$

where $f_*: H_*(X; \mathbb{Z}_2) \to H_*(M; \mathbb{Z}_2)$ and $b_p(\cdot)$ is the p-th Betti number (mod 2).

PROOF. – We supress the $\mathbb{Z}_2$ coefficients. By the Alexander-Poincaré duality, it follows that

$$H_p(M, M\backslash f(X)) \simeq H^{n-p}(f(X)) \simeq H^{n-p}(X) \simeq H_{n-p}(X).$$

Let $i: f(X) \to M$ be the inclusion and let $\tilde{f}: X \to f(X)$ be the restriction of f.

The commutative diagram

$$\begin{array}{ccc} f(X) & \xrightarrow{i} & M \\ {\scriptstyle \tilde{f}}\uparrow & & \uparrow{\scriptstyle f} \\ X & = & X \end{array}$$

implies

$$\begin{array}{ccc} H^p(f(X)) & \xleftarrow{i^{*p}} & H^p(M) \\ {\scriptstyle \tilde{f}^{*p}}\downarrow{\scriptstyle \text{iso}} & & \downarrow{\scriptstyle f^{*p}} \\ H^p(X) & = & H^p(X) \end{array}$$

hence $\operatorname{rk} i^{*p} = \operatorname{rk} f^{*p} = \operatorname{rk} f_{*p} = b_p(X) - \dim \operatorname{Ker} f_{*p}$.

We consider the diagram

$$\begin{array}{ccccccc} H^{n-p}(M) & \xrightarrow{i^{*n-p}} & H^{n-p}(f(X)) & & & & \\ {\scriptstyle \text{iso}}\uparrow & & \uparrow{\scriptstyle \text{iso}} & & & & \\ H_p(M) & \xrightarrow[j_{*p}]{} & H_p(M, M\backslash f(X)) & \xrightarrow[\partial_p]{} & H_{p-1}(M\backslash f(X)) & \xrightarrow[\alpha_{*p-1}]{} & H_{p-1}(M). \end{array}$$

Then we have that $\dim \operatorname{Ker} \partial_p = \operatorname{rk} j_{*p} = \operatorname{rk} i^{*n-p} = b_{n-p}(X) - \dim \operatorname{Ker} f_{*n-p}$.

On the other hand,

$$\begin{aligned} \dim \operatorname{Ker} \partial_p &= b_{n-p}(X) - \operatorname{rk} \partial_p = b_{n-p}(X) - \dim \operatorname{Ker} \alpha_{*p-1} \\ &= b_{n-p}(X) - b_{p-1}(M\backslash f(X)) + \operatorname{rk} \alpha_{*p-1} \\ &= b_{n-p}(X) - b_{p-1}(M\backslash f(X)) + \dim \operatorname{Ker} j_{*p-1} \\ &= b_{n-p}(X) - b_{p-1}(M\backslash f(X)) + b_{p-1}(M) - \operatorname{rk} j_{*p-1} \\ &= b_{n-p}(X) - b_{p-1}(M\backslash f(X)) + b_{p-1}(M) - \dim \operatorname{Ker} \partial_{p-1} \\ &= b_{n-p}(X) - b_{p-1}(M\backslash f(X)) + b_{p-1}(M) - b_{n-p+1}(X) + \dim \operatorname{Ker} f_{*n-p+1}. \end{aligned}$$

Hence we obtain that

$$b_{p-1}(M\backslash f(X)) = b_{p-1}(M) - b_{n-p+1}(X) + \dim \operatorname{Ker} f_{*n-p+1} + \dim \operatorname{Ker} f_{*n-p},$$

which is the first part of the assertion. The second formula follows by considering the Mayer-Vietoris sequence of the pair $(N, M\backslash \mathrm{Int}\, N)$. ∎

COROLLARY 3.3. – *Let X be a polyhedron and let $f_1, f_2 : X \to M$ be PL embeddings of X into the interior of a compact connected PL n-manifold M^n, $n \geq 3$. If* $\mathrm{rk}\, f_{1*} = \mathrm{rk}\, f_{2*}$ (mod 2) *and $N_i \subset \mathrm{Int}\, M$ is a regular neighbourhood of $f_i(X)$ in M, then we have that*

$$b_p(\partial N_1) = b_p(\partial N_2) \quad \text{and} \quad b_p(M\backslash \mathrm{Int}\, N_1) = b_p(M\backslash \mathrm{Int}\, N_2),$$

where $b_p(\cdot)$ is the p-th Betti number (mod 2). ∎

COROLLARY 3.4. – *Let the maps $f_1, f_2 : X \to M^n$ be homotopic and suppose that N_i is a regular neighbourhood of $f_i(X)$ in M^n, $n \geq 3$. Then we have that*

$$H_*(\partial N_1; R) \simeq H_*(\partial N_2; R) \quad \textit{for } R = \mathbb{Z}_2 \textit{ or } \mathbb{Z}. \qquad ∎$$

LEMMA 3.5. – *Let X be a polyhedron in the interior of a compact connected PL n-manifold M^n, $n \geq 3$. Then the following statements are equivalent (for $R = \mathbb{Z}_2$ or $\mathbb{Z}$):*

(1) *$X \subset M$ has the wp uv^{n-2} (R) property;*

(2) *For any regular neighbourhood $N \subset M$ of X, ∂N is a collection of R-homology $(n-1)$-spheres.*

PROOF. – (1) ⇒ (2). We use homology with $\mathbb{Z}_2$ coefficients. By hypothesis, there exists a regular neighbourhood $N^* \subset \mathrm{Int}\, N$ of X in M such that $H_j(N^*\backslash X) \to H_j(N)$ is trivial for any j: $0 \leq j \leq n-2$. Hence $H_j(\partial N^*) \to H_j(N^*)$ is zero, too, since $N^*\backslash X \underset{\text{top}}{\simeq} \partial N^* \times [0, 1)$ is homotopic to ∂N^*.

Consider the following commutative diagrams over $\mathbb{Z}_2$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_n(N^*, \partial N^*) & \longrightarrow & H_{n-1}(\partial N^*) & \longrightarrow & H_{n-1}(N^*) \\
 & & \downarrow \text{iso} & & \downarrow \text{iso} & & \downarrow \text{iso} \\
0 & \longleftarrow & H_0(N^*) & \longleftarrow & H_0(\partial N^*) & \longleftarrow & H_1(N^*, \partial N^*)
\end{array}$$

$$\begin{array}{ccccccc}
H_{n-1}(N^*) & \longrightarrow & H_{n-1}(N^*, \partial N^*) & \longrightarrow & H_{n-2}(\partial N^*) & \longrightarrow & 0 \\
\downarrow \text{iso} & & \downarrow \text{iso} & & & & \\
H_1(N^*, \partial N^*) & \longleftarrow & H_1(N^*) & \longleftarrow & 0 & &
\end{array}$$

where the vertical isomorphisms follow from the Poincaré duality plus the Universal coefficient theorem. Due to the exactness, the alternating sum of the ranks is zero, hence $\mathrm{rk}\, H_{n-2}(\partial N^*) = 0$. Obviously, ∂N^* is orientable because otherwise there exists $TH_{n-2}(\partial N^*; \mathbb{Z}) \simeq \mathbb{Z}_2$ which contradicts $b_{n-2}(\partial N^*; \mathbb{Z}_2) = 0$. Hence $H_{n-2}(\partial N^*; \mathbb{Z}) = 0$, i.e. $FH_1(\partial N^*; \mathbb{Z}) = 0$. By induction, let $\mathrm{rk}\, H_j(\partial N^*) = 0$, for any j: $1 < q \leq j \leq n-2$. We have to show that $\mathrm{rk}\, H_{q-1}(\partial N^*) = 0$.

Consider the exact homology sequence over $\mathbb{Z}_2$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_q(N^*) & \longrightarrow & H_q(N^*, \partial N^*) & \longrightarrow & H_{q-1}(\partial N^*) & \longrightarrow & 0 \\ & & \downarrow \text{iso} & & \downarrow \text{iso} & & & & \\ 0 & \longleftarrow & H_{n-q}(N^*, \partial N^*) & \longleftarrow & H_{n-q}(N^*) & \underset{i_* = 0}{\longleftarrow} & H_{n-q}(\partial N^*). & & \end{array}$$

The map i_* is null because the inequalities $1 < q \leq j \leq n-2$ imply that $n-2 \geq n-q > 1$. Since the alternating sum of the ranks is zero, we have $\mathrm{rk}\, H_{q-1}(\partial N^*) = 0$. This implies that $H_j(\partial N^*) \simeq 0$ for any j: $1 \leq j \leq n-2$. Moreover, $H_{n-1}(\partial N^*; \mathbb{Z}) \simeq \mathbb{Z}$ since ∂N^* is an orientable $(n-1)$-manifold. Thus we have proved that ∂N^* is a R-homology $(n-1)$-sphere as requested.

(2) $\Rightarrow$ (1). We have to show that $H_j(N^* \backslash X) \to H_j(N)$ is trivial for any $0 \leq j \leq n-2$. Since $N^* \backslash X \underset{\text{top}}{\simeq} \partial N^* \times [0, 1)$ is homotopic to ∂N^* and ∂N^* is a R-homology $(n-1)$-sphere, the result follows. ∎

PROOF OF THEOREM 3.1. - Let N be a regular neighbourhood of X in M and let N^* be one of the regular neighbourhoods of $f(X)$ in M. By Lemma 3.5, ∂N is a collection of R-homology $(n-1)$-spheres for $R = \mathbb{Z}_2$ or $\mathbb{Z}$. By Corollary 3.4, ∂N^* is also a collection of R-homology $(n-1)$-spheres. Thus $f(X)$ has the weak peripheral uv^{n-2} (R) property. ∎

COROLLARY 3.6. - *Suppose $X \subset M^n$ is a simply connected polyhedron in the interior of a compact connected PL n-manifold M^n, $n \geq 3$. If* $\dim X \leq n-3$ *and X has the wp uv^{n-2} $(\mathbb{Z})$ property, then X satisfies WP UV^{n-2}.*

PROOF. - Let us consider the exact homotopy sequence of the pair $(N, \partial N)$, where N is a regular neighbourhood of X in $\mathrm{Int}\, M$. Let $f: \mathbb{S}^j \to N$, $j \leq 2$, be a PL map. We can suppose that $f(\mathbb{S}^j)$ and X are in general position, i.e. (use $\dim X \leq n-3$)

$$\dim (f(\mathbb{S}^j) \cap X) \leq j + n - 3 - n \leq -1 .$$

Then $f: \mathbb{S}^j \to N$ is replaced by a homotopic map in $N \backslash X \sim \partial N$ and hence $\Pi_j(\partial N)$ covers $\Pi_j(N)$, $j \leq 2$. Thus the exact sequence mentioned above implies that $\Pi_1(\partial N) \simeq \Pi_1(N)$. Since $\Pi_1(N) \simeq \Pi_1(X) \simeq 0$, it follows that $\Pi_1(\partial N) \simeq 0$. But ∂N is a collection of $\mathbb{Z}$-homology $(n-1)$-spheres, hence by the Hurewicz theorem, ∂N consists of homotopy $(n-1)$-spheres. This implies that X has the *WP UV^{n-2}* property. ∎

4. – Generalized manifolds.

In this section we apply the previous results on peripheral properties to study cell-like decompositions of manifolds. In particular, we consider the problem of determining whether the elements of a given decomposition of a manifold are cell-like subsets of that manifold. First we recall some definitions and recent results about decomposition theory of manifolds. A *decomposition* G of a topological n-manifold M is a partition of M into compact connected subsets. The *decomposition space* M/G is the quotient space obtained by shrinking each element of G to a point. Let $\pi: M \to M/G$ denote the quotient map, H_G the set of nondegenerate ($\neq$ point) elements of G and N_G their union. If G consists of a compactum $X \subset M$ plus the individual points of $M \setminus X$, then M/G is usually called M *modulo* X and denoted by M/X. A decomposition G is said to be *upper semicontinuous* if π is a closed map. A *cell-like decomposition* of a topological manifold is an upper semicontinuous decomposition whose elements are cell-like sets. Topologists are much interested in cell-like decompositions of manifolds because the corresponding decomposition spaces (which may fail to be genuine manifolds) possess all the basic algebraic topology properties of manifolds (for example, they satisfy the Poincaré duality). These spaces are examples of generalized manifolds in the sense of [12] (if $n \geq 4$, one must assume, in addition, that $\dim M/G < \infty$).

A locally compact separable metric space E is said to be a *generalized n-manifold* if it satisfies the following properties:

(1) E is an Euclidean neighbourhood retract (ENR), i.e. for some integer m, E embeds in $\mathbb{R}^m$ as a retract of an open subset of $\mathbb{R}^m$.

(2) E is a $\mathbb{Z}$-homology n-manifold, i.e.

$$H_*(E, E \setminus \{e\}; \mathbb{Z}) \underset{\text{iso}}{\approx} H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$$

for any point $e \in E$.

A *generalized n-manifold with boundary* is an ENR E such that E is a $\mathbb{Z}$-homology n-manifold with boundary and ∂E is a generalized $(n-1)$-manifold. Any generalized n-manifold (with boundary), $n \leq 2$, is a genuine manifold. In dimension $n \geq 3$, a generalized n-manifold E may fail to be locally Euclidean at some (or perhaps all) points. These points, called *singularities* of E, form the *singular set* of E, written $S(E)$. If $S(E) \neq E$, then the complement $M(E) = E \setminus S(E)$ is an open n-manifold, called the *manifold set* of E. If a generalized 3-manifold E admits a piecewise-linear (PL) structure, then E is a PL 3-manifold (see, for example, [10] and [13]).

A *resolution* of a generalized n-manifold E is a pair (M, G) consisting of a TOP n-manifold M and a cell-like decomposition G of M such that M/G is TOP homeomorphic to E. In this case, we say that E is a *resolvable generalized n-manifold.* There is a single integer obstruction $I(E) \in H_0(E; \mathbb{Z})$ to the resolution of a generalized n-manifold E, which can be described as follows (for more details see [35] and [36]). Let $U \subset E$

be an open set with a proper degree one map $f: U \to \mathbb{R}^n$. Make f transverse to zero and define the *local index* $I(E)$ to be the cardinality of $f^{-1}(0)$. In [35], [36] Quinn proved some characteristic properties of this obstruction, collected in the next theorem.

THEOREM 4.1. – *Let E^n and F^m be connected generalized manifolds of dimensions n and m respectively. Then the local obstruction index satisfies the following properties*:

(1) $I(E) \equiv 1 \mod 8$.

(2) *If E is a topological manifold, then* $I(E) = 1$.

(3) *If U is an open subset of E, then* $I(U) = I(E)$.

(4) $I(E \times F) = I(E)I(F)$.

(5) *For $n \geqslant 5$, there exists a resolution of E if and only if* $I(E) = 1$. ■

Recently, it was announced in [6] that there exist examples of non-resolvable generalized n-manifolds, $n \geqslant 6$, having arbitrary local obstruction index and homotopy equivalent to any simply connected closed n-manifold.

One of the central problems of modern geometric topology is how to detect TOP manifolds among more general TOP spaces (see for example [12], *The recognition problem*). For dimension $n \geqslant 5$, fundamental results, due to Cannon, Edwards and Quinn (see [11], [12], [20], [21], [35], [36]), solved the recognition problem among resolvable generalized manifolds. More precisely, we have the following recognition theorem.

THEOREM 4.2. – *A locally compact separable metric space E is a TOP n-manifold, $n \geqslant 5$, if and only if E is a resolvable generalized n-manifold (i.e. the local obstruction index $I(E)$ equals one) which satisfies the disjoint disks property (DDP): any two maps of a 2-cell to E can be ε-approximated by maps having disjoint images in E for any $\varepsilon > 0$.* ■

Two important consequences of Theorem 4.2 are the Double suspension theorem [11] and the fact that any resolvable generalized n-manifold E is a (possibly exotic) factor of the topological $(n+2)$-manifold $E \times \mathbb{R}^2$ for any $n \geqslant 0$ (see [17]).

The DDP is clearly inappropriate for dimensions three and four (see for example [24]). However there exist other general position properties which effectively work for detecting TOP 3-manifolds among resolvable generalized ones (see [18], [19]). Essentially nothing is known in dimension four except for the fact that a generalized 4-manifold E has a resolution if and only if $E \times \mathbb{R}$ has one (see [35], [36]).

Now we prove some results which represent partial converses to the fact that finite-dimensional cell-like decompositions of manifolds yield generalized manifolds. This allows us to construct further exotic factors of manifolds and, in particular, of Euclidean spaces.

THEOREM 4.3. – *Let X be a k-dimensional connected compactum in the interior of a compact PL n-manifold M^n, $2k+1 \leq n$, $k \geq 1$. Suppose that X satisfies the WP 1-UV property. If M/X is a generalized n-manifold, then X is a cell-like set in M. In particular, for $n \geq 4$, $(M^n/X) \times \mathbb{R}$ is a topological $(n+1)$-manifold.*

PROOF. – Since M/X is a generalized n-manifold, X has a compact connected manifold neighbourhood N^n in M^n such that ∂N is a $\mathbb{Z}$-homology $(n-1)$-sphere. We prove that X satisfies the uv^∞ property over $\mathbb{Z}$. We supress the coefficients. For any PL 1-cycle $z \in H_1(N)$, let $f\colon \mathbb{S}^1 \to N$ be a PL loop which is homologous to z in $H_1(N)$. Since $k \leq n-2$, a general position argument (see the proof of Proposition 2.4) deforms f to a PL map $\mathbb{S}^1 \to N\backslash X \underset{\text{top}}{\simeq} \partial N \times [0, 1)$ and hence to a map $\mathbb{S}^1 \to \partial N$ (up to homotopy). Then this map is null-homologous in ∂N because ∂N is a $\mathbb{Z}$-homology $(n-1)$-sphere. This implies that f is also null-homologous, i.e. $H_1(N) \simeq 0$. Obviously, $H_j(N) \simeq H_j(X) \simeq 0$ for any $j > k$ as N contracts onto X and $\dim X = k$. For any j, $0 < j \leq k < 2k \leq n-1$, we have $H_j(\partial N) \simeq 0$ and $H_j(N, \partial N) \simeq H^{n-j}(N) \simeq 0$, as $n - j > k$. Thus the exact homology sequence of the pair $(N, \partial N)$ yields that $H_j(N) \simeq 0$ for any j, $0 < j \leq k$. Therefore N is acyclic, i.e. X satisfies uv^∞ property. Now, we have to prove that X has the property UV^∞. By the Hurewicz theorem it suffices to show that X satisfies the 1-*UV* property. But this follows from Proposition 2.4 since X has the *WP* 1-*UV* property and since $k \leq n-2$. Thus X is a cell-like set by Theorem 1.2, i.e. M/X is a resolvable generalized n-manifold. For any $n \geq 4$, $(M/X) \times \mathbb{R}$ is a resolvable generalized $(n+1)$-manifold which satisfies the DDP (see Corollary 4 A [16], p. 289 and [17]). Now the map $\pi \times id : M^n \times \mathbb{R} \to (M^n/X) \times \mathbb{R}$, $n \geq 4$, can be approximated by TOP homeomorphisms, i.e. $M \times \mathbb{R} \underset{\text{top}}{\simeq} (M/X) \times \mathbb{R}$ (see [20]). Thus the proof is completed. ■

REMARK. Under the hypothesis of Theorem 4.3, for $n = 3$, we can only conclude that $(M^3/X) \times \mathbb{R}^2$ is (TOP) homeomorphic to $M^3 \times \mathbb{R}^2$, due to the unresolved status of the Poincaré conjecture. Namely, let M^3 be a fake 3-sphere and $X \subset M$ its spine. Then X is obviously cell-like but $(M/X) \times \mathbb{R} \underset{\text{top}}{\simeq} \mathbb{S}^3 \times \mathbb{R}$ is clearly not homeomorphic to $M \times \mathbb{R}$. On the other hand, if the Poincaré conjecture is true, every cell-like subset X of a 3-manifold M has a neighbourhood in M embeddable in $\mathbb{R}^3$ (see [33]) and so we can apply Theorem 1.3 to conclude that $(M/X) \times \mathbb{R} \underset{\text{top}}{\simeq} M \times \mathbb{R}$.

LEMMA 4.4. – *Let X be a compact, locally connected, simply connected set in the interior of a (TOP) n-manifold M. Then $X \subset M$ satisfies the WP 1-UV property.*

PROOF. – Choose a neighbourhood U of X in M. There exists an integer $\theta > 0$ such that θ-close maps of $\mathbb{S}^1$ into U are homotopic. Let $V = U \cap N_\lambda(X)$ where

$$\lambda = \min\left\{\frac{\theta}{4}, \frac{\eta}{4}\right\}.$$

Here $N_\lambda(X)$ denotes the λ-neighbourhood of X, i.e.

$$N_\lambda(X) = \{x \in M : d(x, X) < \lambda\}.$$

Furthermore, the real number η shows X as locally connected for $\theta/4$, i.e. for any pair $x, x' \in X$ such that $d(x, x') < \eta$ there exists a path σ in X from x to x' with $\operatorname{diam}\sigma < \theta/4$. Given a loop $f: \mathbb{S}^1 \to V \backslash X$, pick $a_i \in \mathbb{S}^1$, cyclically ordered, such that

$$\operatorname{diam} f([a_i, a_{i+1}]) < \lambda$$

for any i. Thus, in particular, we have $d(f(a_i), f(a_{i+1})) < \lambda$. Choose points $z_i \in X$ such that $d(z_i, f(a_i)) < \lambda$. This is possible by the definition of V. Then for any i, we obtain

$$d(z_i, z_{i+1}) \leq d(z_i, f(a_i)) + d(f(a_i), f(a_{i+1})) + d(f(a_{i+1}), z_{i+1}) < 3\lambda < \eta$$

so we can join z_i and z_{i+1} by a path $\gamma_i : [0, 1] \to X$. Now it follows that

$$d(\gamma_i, f|_{[a_i, a_{i+1}]}) \leq \operatorname{diam} \gamma_i([0, 1]) + d(z_i, f(a_i)) + \operatorname{diam} f([a_i, a_{i+1}]) < 3\lambda < \theta.$$

So the loops $\gamma = \gamma_1 \gamma_2 \dots \gamma_n$ and f are θ-close and hence homotopic in U, i.e. $[f] = [\gamma] \in \Pi_1(U)$. This means that $[f] = [\gamma] \simeq 0$ since $[\gamma] \in \Pi_1(X) \simeq 0$ by hypothesis. ■

REMARKS. – (1) If instead of $\Pi_1(X) \simeq 0$, we assume $H_1(X) \simeq 0$ we get the *wp* 1-*uv* property with appropriate coefficients.

(2) For any k-cell D^k, let $f: D^k \to M^n$ be a wild embedding of D^k into M^n. Then $f(D^k)$ satisfies the *WP* 1-*UV* property by Lemma 4.4. Since $f(D^k)$ is contractible, it has UV^∞ property, i.e. $f(D^k)$ is cell-like. But $f(D^k)$ does not satisfy *SP* 1-*UV* as it is non-cellular (see Theorem 1.3).

Theorem 4.3 and Lemma 4.4 directly imply the following result:

COROLLARY 4.5. – *Let X be a k-dimensional locally connected compactum in the interior of a PL n-manifold M^n, $2k+1 \leq n$. If X is simply connected and M/X is a generalized n-manifold (possibly with boundary), then X is a cell-like set in M.* ■

We say that a locally connected compactum X in the interior of a TOP manifold M is a *pseudo-spine* of M if $M \backslash X$ is TOP homeomorphic to $\partial M \times [0, 1)$. Because M/X is homotopy equivalent to the cone over ∂M, the quotient space M/X is a generalized n-manifold with boundary. Thus the previous results directly imply the following corollary:

COROLLARY 4.6. – *Let M^n be a compact connected PL n-manifold with boundary ∂M a $\mathbb{Z}$-homology $(n-1)$-sphere, $n \geq 4$. Suppose that M admits a simply connected (or WP 1-UV) pseudo-spine X of dimension k, $2k+1 \leq n$. Then the generalized*

n-manifold M/X with boundary is a cartesian factor of the TOP $(n+1)$-manifold $(M/X) \times \mathbb{R} \underset{\text{top}}{\simeq} M \times \mathbb{R}$. ∎

To complete the paper we present a result about certain exotic factors of the Euclidean 4-space $\mathbb{R}^4$.

THEOREM 4.7. – *Let G be an upper semicontinuous decomposition of $\mathbb{R}^3$ with the following properties:*

(1) *H_G consists of the components of some compact set.*

(2) *There exists $n \in \mathbb{N}$ such that $(\mathbb{R}^3/G) \times \mathbb{R}^n$ is contractible and simply connected at infinity.*

(3) *$\mathbb{R}^3/G$ is a generalized 3-manifold.*

Then each component in the closure of N_G is cell-like. In particular, $(\mathbb{R}^3/G) \times \mathbb{R}$ is topologically homeomorphic to $\mathbb{R}^4$.

PROOF. – Consider the topological product $E = (\mathbb{R}^3/G) \times \mathbb{R}^n$. Then E is a non-compact generalized $(n+3)$-manifold. We may assume that $n \geqslant 2$ so by Daverman's theorem (see [17]) E has the DDP. Because H_G consists of the components of some compact set, $S(\mathbb{R}^3/G) \neq \mathbb{R}^3/G$, hence the local obstruction index $I(\mathbb{R}^3/G)$ equals one. By Theorem 4.1, we have

$$I(E) = I(\mathbb{R}^3/G)I(\mathbb{R}^n) = 1\,,$$

hence E admits a resolution as $\dim E \geqslant 5$. Consequently E is a non-compact topological $(n+3)$-manifold by Theorem 4.2. Also, E is contractible and simply connected at infinity so Siebenmann's theorem implies that E is TOP homeomorphic to $\mathbb{R}^{n+3}$ (see [41]). It now follows by a result of MCMILLAN Jr. (see [33]) that each component in the closure of N_G is cell-like. Finally, we apply Proposition 2 of [16], p. 206, to conclude that $(\mathbb{R}^3/G) \times \mathbb{R}$ is TOP homeomorphic to $\mathbb{R}^4$. ∎

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