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Some Open Problems in Geometric Topology of Low Dimensions

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§ 1. Three-demensional Problems

1. The 2-sphere aproximation conjecture Suppose that $f: S^2 \to S^3$ is a continuous map of the standard 2-sphere S^2 into the standard 3-sphere S^3 and denote by $\sum_f = \{x \in S^2 | f^{-1}(f(x)) \neq x\}$ the singular set of f. Suppose that dim $\sum_f = 0$. The following conjecture was set forth by the second author in 1986:

Conjecture 1. 1 For every $\varepsilon > 0$ there exists an embedding $F: S^2 \to S^3$ such that $d(f(x), F(x)) < \varepsilon$, for every $x \in S^2$.

A special case of Conjecture (1. 1) when dim $\overline{\sum_f} = 0$ (e. g. when $\sum_f$ lies in a Cantor set) was proved by M. V. Brahm [2] (see also [1]). Notice that, in general, $f(S^2)$ can be up to 3-dimensional since the map f can raise dimension. Also note that if dim $\sum_f = 1$, then approximations by embeddings need not exist. Finally, observe that an affirmative solution could possibly strengthen some results from [14] on general position properties which characterize (and detect) topological 3-manifolds (among cell-like images of 3-manifolds).

2. Controlled isotopy Let M be a closed n-dimensional topological manifold, $U = \{D_\alpha\}_{\alpha \in \Lambda}$ a covering of M consisting of closed n-cells and $h: M \to M$ a homeomorphism, is otopic to the identity $\mathrm{id}_M: M \to M$, and such that for every point $x \in M$, there exists a cell $D_\alpha \in U$ such that $\{x, h(x)\} \subset D_\alpha$.

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Problem 1.2 Does there exist an isotopy $H: M\times I\to I$ such that $H_0=h, H_1=\mathrm{id}_M$ and for every point $x\in M$ there exists a cell $D_x\in U$ such that $H(\{x\}\times I)\subset D_x$?

It was shown in [17] that the answer to Problem (1.2) is negative for the case when $M=S^1$ or $M=S^1\times S^1$. We believe that the following is true:

Conjecture 1.3 The answer to Problem (1.2) is negative for

$$M=\underbrace{S^1\times\cdots\times S^1}_{n\text{ factors}}$$

for every $n\in N$.

Conjecture 1.4 Let $\dim M=2$ and let $g\in\mathbf{Z}_+$ denote the genus of M. Then the answer to Problem 1.2 is

(i) affirmative, if $g=0$; and

(ii) negative, if $g\geq 2$.

Conjecture 1.5 Suppose that M is k-connected where $k=[\frac{n}{2}]$. Then the answer to Problem 1.2 is affirmative.

Natice that analogous problems can be considered for compact n-manifolds M with nonempty boundary (in which case one should assume that $h|_{\partial M}=\mathrm{id}_{\partial M}$ and require that $H|_{\partial M\times I}=\mathrm{id}_{\partial M\times I}$).

3. Equivalent spines It was shown in [16] and, independently, using a very different argument in [5] that there exist nonhomeomorphic compact 3-manifolds $M_1, M_2\subset S^3$ with $\partial M_i=S^1\times S^1$ such that M_1 and M_2 have the same spine K, i.e. M_i collapses onto some compact 2-polyhedron $K_i\subset\mathrm{int}\, M_i$ for $i=1,2$ and $K_1\cong K_2$.

Question 1.6 Do there exist nonhomeomorphic 3-manifolds M_1 and M_2 with boundary $\partial M_1=\partial M_2$ of genus ≥ 2 such that M_1 and M_2 have the same spinee and the fundamental group $\pi_1(M_1)\cong\pi_1(M_2)$ is not a nontrivial free product?

Note that in general one cannot get examples for ((1.6)) by simply drilling extra holes in the examples of genus one of [5] or [16] since the resulting spaces may become homeomorphic. However, these constructions yield examplex for (1.6) whose fundamental groups are nontrivial free products so the question is the open in the general case. A possible venue of attack could be an implementation of the algorithm from [5] for producing 3-manifolds with prescribed spines, except that the problem remains of determing whether different constructions are indeed nonhomeomorphic.

Also note that the case of genus zero is related to the unresolved status of the Poincaré Conjecture [18].

4. McMillan-Row problem Consider an 1-dimensional continuum (i.e. a compact and connected set) K inside a compact 3-manifold M (possibly nonorientable and with nonempty boundary).

Problem 1.7 Does K possess an open neighbourhood $N\subset M$, embeddable into $\mathbf{R}^3$?

This problem first appeared in the 1969 paper by D. R. McMillan, Jr. and H. Row [15] and is related to the study of so-called tangled embeddings of 1-dimensional continua into 3-manifolds.

5. Cell-like resolutions A compact subset K of a topological n-manifold M is cellular in M if it is the intersection of a properly nested decreasing sequence of (closed) n-cells in M. A space X is cell-like if there exists a manifold M and an emedding f from X into M such that $f(X)$ is cellular in M.

A closed map is proper if its point-inverses are compact. A map defined on an absolute neighbourhood retract (ANR) X is cell-like if its point-inverses are cell-like sets in X. A resolution of an n-dimensional ANR X is a proper, cell-like map $f: M \to X$ from a topological n-manifold M onto X. It follows by classical results (see for example [9]) that if X admits a resolution, then X must be a generalized n-manifold, i. e. an Euclidean neighbourhood retract (ENR) which is a **Z**-homology n-manifold. Given a generalized n-manifold X, the singular set of X, written $S(X)$, is by definition, the set of points $x \in X$ which have no neighbourhood in X homeomorphic to $\mathbf{R}^n$. In dimension three, the Resolution Problem, i. e. the problem of resolving generalized manifolds, still remains open (it is also entangled by the Poincaré Conjecture) for generalized 3-manifolds X whose singular sets $S(X)$ have dimension ≥ 1. For the case $\dim S(X) = 0$, we refer to [21] (see also [19]).

Question 1. 8 Suppose that there exist no fake cubes. Does there exist a nonresolvable generalized 3-manifold?

As remarked above, for such an example X, $\dim S(X) \geq 1$ is necessary. In dimensions ≥ 6 nonresolvable generalized manifolds exist [3], whereas dimensions 4 and 5 remain open:

Question 1. 9 Does there exist a nonresolvable generalized n-manifold for $n \in \{4, 5\}$?

We refer to [11] for a detailed revies of the major questions concerning the Resolution Problem (in all dimensions).

6. Group presentations

a) Takahashi manifolds In [20] a class of closed orientable 3-manifolds $M_{2n} = M(p_1, r_1; \cdots; p_{2n}, r_{2n})$ was considered, obtained by Dehn surgery along certain $(2n)$-chain links L_{2n} with surgery coefficients p_i/r_i for any $i \in \{1, \cdots, 2n\}$. The following presentation for the fundamental group of M_{2n}.

$$\pi_1(M_{2n}) \cong \langle x_1, \cdots, x_{2n} \mid x_{2i}^{r_{2i}} x_{2i+1}^{p_{2i+1}} x_{2i+2}^{-r_{2i+2}} = 1, x_{2i-1}^{-r_{2i-1}} x_{2i}^{p_{2i}} x_{2i+1}^{r_{2i+1}} = 1 \text{(indices mod } 2n) \rangle$$

corresponds to a spine of the manifold because it arises from an RR-sistem as points out to the authors by C. Hog-Angeloni. It was proved in [13] that if

$$\frac{p_{2i}}{r_{2i}} = 1 \text{ and } \frac{p_{2i+1}}{r_{2i+1}} = -1,$$

then M_{2n} is the unique prime orientable 3-manifold having a spine corresponding to the standard presentation of the Fibonacci group $F(2, 2n) = \langle x_1, \cdots, x_{2n} \mid x_i x_{i+1} x_{i+2}^{-1} = 1 \text{(indices mod} 2n) \rangle$. Moreover, this manifold is the n-fold covering of the 3-sphere S^3 branched over the figure-eight knot (see [13]).

Question 1. 10 Is $M_{2n} = M(p_1, r_1; \cdots; p_{2n}, r_{2n})$ the unique prime orientable closed

3-manifold having a spine corresponding to the presentation of $\pi_1(M_{3n})$ above? Classify the topological sructure of this class of 3-manifolds in the general case.

b) Homology 3-spheres It is well-known that surgery on the figure-eight knot yields infinitely many homology 3-spheres of Heegaard genus two (also nonfibered). The following group presentation

$$\langle x, y \mid x^{-1}yx^{-1}yxy^{-1}xy^{-1}yx^{-3} = 1, x^{-1}yx^{-1}y^3x^{-1}yx^{-4} = 1 \rangle$$

corresponds to a spine of a homology 3-sphere of Heegaard genus two. Furthermore, this manifold is proved to be a 2-fold branched covering along a knot which is not equivalent to any torus or pretzel knot.

This example might be of interest with respect to the minimum number of 3-cells in a pseudo-simplicial triangulation of the manifold. A pseudo-complex is a cell-complex in which every h-cell is abstractly isomorphic to an h-simplex. An n-dimensional pseudo-complex is said to be contracted if every n-cell has exactly $n+1$ vertices (0-cells). We know that any Heegaard genus two homology 3-sphere, which is triangulated by contracted pseudocomplexes with at most 50 3-cells, is always fibered except for perhaps the example above.

Question 1.11 Is the above-defined homology 3-sphere fibered?

Finally, we mention the problem of classification of the homeomorphism type of all Heegaard genus two homology 3-spheres (resp. 3-manifolds) which have contracted pseudo-simplicial triangulations with a fixed number n of closed 3-cells, for n at least 100. In a forthcoming paper (jointly with M. Meschiari), we shall present some results concerning these questions by using an appropriate computer algorithm.

§ 2. Four-dimensional Problems

1. Four-manifolds The first step for classifying manifolds is to determine their homotopy type and then to study the problem of when the homotopy type yields a classification up to an s-cobordism or up to a TOP homeomorphism. For 4-manifolds with free or surface fundamental groups, this program was realized in a series of papers [8]—[10]. More precisely, the closed connected smooth 4-manifolds M with free fundamental group $\pi_1(M) = *_p\mathbf{Z}$ (free product of p factors $\mathbf{Z}$) are classified, up to a homotopy or up to a TOP s-cobordism, by the isomorphism class of their intersection pairings

$$\lambda_M : H_2(M;\Lambda) \times H_2(M;\Lambda) \to \Lambda$$

over the integral group ring $\Lambda = \mathbf{Z}[\pi_1(M)]$. In particular, if $H_2(M;\mathbf{Q}) \cong 0$, then M is homotopy equivalent (TOP s-cobordant) to the connected sum $\# p(S^1 \times S^3)$.

Conjecture 2.1 Let M be as above. If $H_2(M;\mathbf{Q}) \cong 0$, then M is TOP homeomorphic to $\# p(S^1 \times S^3)$.

It was proved in [10] that a spin closed connected orientable 4-manifold M with $\pi_1(M) \cong \pi_1(F)$, F being a closed (orientable) aspherical surface, is homotopy equivalent

(TOP s-coborant) to a connected sum $(F\times S^2)\#M'$ for some simply connected 4-manifold M'.

Conjecture 2.2 The manifolds M and $(F\times S^2)\#M'$ are TOP homeomorphic.

Problem 2.3 Determine the isomorphism classes of the intersection pairings λ_M over the integral group ring $\Lambda=Z[\pi_1(M)]$, where $\pi_1(M)$ is either $*_p\mathbf{Z}$ or $\pi_1(F)$, where F is a closed surface.

Problem 2.4 Classify the distinct differentiable structures and describe the topological structure of the moduli spaces on a smooth 4-manifold with a free or a surface fundamental group.

Problem 2.5 Study the homotpy (h-cobordism, homeomorphism) type of a smooth closed 4-manifold whose fundamental group is elementary amenable, i. e. it belongs to the class of groups generated from the class of finite groups and $\mathbf{Z}$, by the operations of extension and increasing union.

Examples of groups as in Problem 2.5 are the poly-(cyclic or finite) groups, i. e. groups having finite composition series whose factors are all infinite cyclic or finite cyclic.

2. Regular genus We briefly recall the definition of regular genus for a closed connected PL n-manifold. For more details (also about other combinatorial invariants of PL manifolds) we refer to [4], [6], [7] and [12]. An $(n+1)$-colored graph is a pair (G,c), where $G=(V,E)$ is a finite multigraph, regular of degree $n+1$, and $c:E\to\triangle_n=\{0,1,\cdots,n\}$ is an edge-coloration on G with $n+1$ colors, i. e. $c(e)\neq c(f)$ for any pair of adjacent edges $e,f\in E$. The graph (G,c) is said to be contracted if the partial subgraph $G_i=(V,c^{-1}(\triangle_n\setminus\{i\}))$ is connected, for each color $i\in\triangle_n$. An n-pseudocomplex $K=K(G)$ can be uniquely associated with (G,c) so that $|G|$ becomes its dual 1-skeleton. A crystallization of a closed conected (PL) n-manifold M is a contracted $(n+1)$-colored graph (G,c) which represents M, i. e. $|K(G)|\approx_{PL}M$. It is well-known that each closed conected (PL) n-manifold admits a crystallization (see for example the quoted papers).

Given a crystallization (G,c), the minimum genus of a closed connected surface into which (G,c) regularly embeds is denoted by $g(G)$. The regular genus (or simply, the genus) of a closed (PL) n-manifold M is defined as the minimum over the genera of all crystallizations representing M. It will be denoted by $g(M)$. This topological invariant characterizes the n-sphere among closed (PL)n-manifolds in the sense that $g(M)=0$ if and only if $M\approx_{PL}S^n$. Moreover, it was proved in [4], [5] and [7] that the unique, up to a PL homeomorphism, closed connected orientable (prime) 4-manifold of genus one (resp. two) is $S^1\times S^3$ (resp. CP^2) and that the unique, up to a TOP homeomorphism, closed connected orientable (resp. nonorientable) prime 4-manifold of genus four (resp. six) is $S^2\times S^2(\mathbf{R}P^4)$.

Conjecture 2.6 The unique, up to homeomorphism, prime connected orientable 4-manifold of genus six is $\mathbf{R}P^3\times S^1$.

Finally, using the concept of regular genus, we obtain some equivalent statements of the 4-dimensional Poincaré conjecture $P(4)$ in the PL(=DIFF) category.

Conjecture 2.7 If M is a closed smooth (or PL) simply connected 4-manifold, then

$$g(M) = 2\beta_2(M) = 2\chi(M) - 4,$$

where $\beta_2(M)$ and $\chi(M)$ denote the second Betti number and the Euler characteristic of M, respectively.

Conjecture 2.8 The genus is additive with respect to the connected sum of simply-connected smooth (or PL) 4-manifolds.

Conjecture 2.9 If M is a smooth (or PL) homotopy 4-sphere, then

$$g(M \# k(S^2 \times S^2)) \geq g(M) + 4k$$

for some integer $k \geq 0$.

The Conjecture 2.7 (which is equivalent to 2.8) implies $P(4)$ while Conjecture 2.9 is equivalent to $P(4)$, as shown in [5].

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