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SPECTRAL SEQUENCES IN K-THEORY FOR A TWISTED QUADRATIC EXTENSION

By

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Abstract. We describe spectral sequences of K-groups which can be constructed for a twisted quadratic extension of antistructures. These spectral sequences are based on Tate cohomology groups of the groups $K_i(R)$, $i \in \{0,1\}$. The existence of such spectral sequences follows from an earlier work by Muranov if one uses the methods of construction of spectral sequences originally due to Hambleton and Kharshiladze. We describe the first differentials in these spectral sequences and then give some examples related to surgery.

1. Introduction

The concept of the antistructure was introduced by Wall [12] as a natural generalization of a ring with involution and as a more general object for studying Hermitian forms. In geometric applications, antistructures appear in considerations of the Browder-Livesay groups $LN_*(\pi \to G) \cong L_*(\mathbb{Z}\pi, \alpha, u)$, where $i : \pi \to G$ is an inclusion of groups of index 2 and (α, u) is the antistructure on the ring $\mathbb{Z}\pi$ (see [1, 9, 13]). The twisted quadratic extension was introduced by Ranicki [10] for construction of the algebraic version of the two-folded Wall diagram for antistructures [13]. This diagram can be written in the following form:

$$\begin{tikzcd}
L_n(\pi) \arrow[rr] \arrow[dr] & & L_n(G) \arrow[rr] \arrow[dr] & & LN_{n-2}(\pi \to G) \\
& L_n(i^!) \arrow[ur] \arrow[dr] & & L_n(i_!) \arrow[ur] \arrow[dr] & \\
LN_{n-1}(\pi \to G) \arrow[rr] \arrow[ur] & & L_{n-1}(G^-) \arrow[rr] \arrow[ur] & & L_{n-1}(\pi)
\end{tikzcd} \tag{1.1}$$

and it connects Wall groups and Browder-Livesay groups by natural maps, e.g. transfer, induced maps, and others. Recently, using diagram (1.1), Hambleton and Kharshiladze constructed a spectral sequence in surgery theory and

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described connections between differentials in the spectral sequence and the iterated Browder-Livesay invariants [2]. However, as it follows from [5, 7], there exist diagram analogues of (1.1) for the Tate cohomology groups in the case of the quadratic extension of antistructures $R \hookrightarrow S$. Then a diagram analogous to (1.1) connects Tate cohomology groups $H^n(X_R/Y_R)$ and $H^n(X_S/Y_S)$, where $Y_R \subset X_R \subset K_i(R)$, $Y_S \subset X_S \subset K_i(S)$, $i \in \{0,1\}$, are subgroups which satisfy some additional conditions. In this case Ranicki $\mathbb{L}$-spectrum [3] gives us a possibility of constructing certain spectral sequences. In the present paper we construct such spectral sequences for the Tate cohomology groups of the twisted quadratic extensions and consider some examples.

One possible motivation for such an approach is a connection between Tate cohomology groups of the ring $\widehat{\mathbb{Z}}_2\pi$, where $\widehat{\mathbb{Z}}_2$ are the 2-adic integers, and Wall surgery obstruction groups $L_n^Y(\mathbb{Z}\pi)$ (see below). So it is interesting to know which effects on surgery spectral sequence come from Tate cohomology and which ones from the relative part $L_*(\mathbb{Z}\pi \to \widehat{\mathbb{Z}}_2\pi)$. This is also interesting because Tate cohomology does not give any contribution to the group L^p. Hence, the spectral sequence of Tate cohomology groups can provide new information about distinct Wall groups with various decorations.

2. The twisted quadratic extension of a ring with antistructure

In this section we collect the necessary definitions and describe some properties of the rings with antistructures [9]. Let R be a ring with a unit. An *antistructure* is a triple (R, α, u), where $u \in R^\times$ is a unit and $\alpha : R \to R$ is an antiautomorphism of the ring R such that

$$\alpha(u) = u^{-1} \qquad \text{and} \qquad \alpha^2(x) = uxu^{-1},$$

for every $x \in R$. A ring homomorphism $f : R \to R'$ gives a *morphism* of antistructures $(R, \alpha, u) \to (R', \alpha', u')$ provided that

$$f(u) = u' \qquad \text{and} \qquad \alpha' \circ f = f \circ \alpha.$$

To define a twisted quadratic extension [10] we need a *structure* on a ring R, i.e. a pair (ρ, a), where $\rho : R \to R$ is an automorphism and $a \in R^\times$ is a unit such that

$$\rho(a) = a \qquad \text{and} \qquad \rho^2(x) = axa^{-1}$$

for every $x \in R$. Then a *twisted* quadratic extension of R is defined as the ring

$$S = R[t]/(t^2 - a)$$

with t being an independent element over R such that $\rho(x) = txt^{-1}$, for every $x \in R$. To extend an antiautomorphism α over the ring S we need the following conditions:

(i) $\alpha(t) \cdot t \in R \subset S$; and
(ii) $\alpha^2(t) = utu^{-1} \in S$, $u \in R$.

In this case, (S, α, u) is an antistructure and the inclusion $i : R \to S$ gives a morphism of antistructures $(R, \alpha, u) \to (S, \alpha, u)$, where for simplicity, we denote the extended antiautomorphism by α.

An automorphism $\rho : R \to R$ can be extended to an automorphism of the ring S by setting $\rho(x+yt) = t(x+yt)t^{-1}$, for every $x, y \in R$. There also exists an automorphism γ of the ring S over R, which is defined by $\gamma(x+yt) = x - yt$, for every $x, y \in R$. If $(R, \alpha, u) \to (S, \alpha, u)$ is a quadratic extension of antistructures there exists by [10] another quadratic extension $(R, \tilde{\alpha}, \tilde{u}) \to (S, \tilde{\alpha}, \tilde{u})$, where $\tilde{\alpha} = \rho\gamma\alpha$ and $\tilde{u} = -t\alpha(t^{-1})u$. Both quadratic extensions are equal as the quadratic extensions of the rings.

We will consider only projective right R-modules. The antiautomorphism α on the rings R, S induces involutions T on groups $K_i(R)$ and $K_i(S)$, for $i \in \{0, 1\}$. For the ring R, this involution is generated by a map $M \mapsto M^\alpha = \mathrm{Hom}_R(M, R)$, where the right multiplication $M^\alpha \times R \to M^\alpha$ is given by $(g \cdot r)(x) = \alpha(r)g(x)$, for any $x \in M$, $g \in M^\alpha$, $r \in R$, where M is an R-module. The antiautomorphism $\tilde{\alpha}$ induces an involution $\tilde{T}$ on groups $K_i(R)$ and $K_i(S)$, $i \in \{0, 1\}$, in the same way.

3. Quadratic extensions and Tate cohomology for K-groups

For every subgroup $X \subset K_j(R)$, $j \in \{0, 1\}$, invariant with respect to the involution T, decorated Wall groups $L_n^X(R, \alpha, u)$ are defined so that for any two invariant subgroups $X \subset X' \subset K_j(R)$, $j \in \{0, 1\}$, there is the Rothenberg exact sequence

$$(3.1) \quad \ldots \to L_n^X(R, \alpha, u) \to L_n^{X'}(R, \alpha, u) \to H^n(X'/X) \to L_{n-1}^X(R, \alpha, u) \to \ldots$$

The Tate cohomology in (3.1) is defined for every additively written group A with an involution T by

$$H^n(A, T) = \frac{\{a = (-1)^n Ta \mid a \in A\}}{\{b + (-1)^n Tb \mid b \in A\}}, \qquad n \in \{0, 1\}.$$

The quadratic extension of antistructures $i : (R, \alpha, u) \to (S, \alpha, u)$ gives an induced map $i_* : K_j(R) \to K_j(S)$, $j \in \{0, 1\}$, given on modules by $M \mapsto M \otimes_R S$, where M is any R-module. The map

$$i^* : K_j(S) \to K_j(R), \quad j \in \{0, 1\}$$

is given on modules as the restriction of the S-action onto $R \subset S$.

Maps i_* and i^* commute with the involution T [10] and hence they induce maps of Tate cohomology which we will denote by $i_!$ and $i^!$:

$$i_! : H^n(K_j(R), T) \to H^n(K_j(S), T)$$

and

$$i^! : H^n(K_j(S), T) \to H^n(K_j(R), T),$$

respectively. Analogously, these maps commute with the involution $\tilde{T}$. In addition, one can verify that

$$i_* T = \tilde{T} i_*, \quad i^* \tilde{T} = T i^*,$$

and

$$i_* \tilde{T} = T i_*, \quad i^* T = \tilde{T} i^*.$$

It suffices to prove only the first two results since the other two are proved analogously. Let us denote by ρ an involution of the groups $K_i(R)$, $i \in \{0,1\}$, which is given on the modules in the following way. For any R-module M we denote by ρM the module with the same additive group as M and with the right multiplication, given by the rule:

$$(m, r) = m \cdot \rho(r), \quad \text{for every } m \in M, \quad r \in R.$$

Analogously, let γ be an involution of the groups $K_i(S)$, $i \in \{0,1\}$. For an S-module M we denote by γM the module with the same additive group as M and with the right multiplication given by

$$(m, r) = m \cdot \gamma(r), \quad \text{for every } m \in M, \quad r \in R.$$

In accordance with [7], we have $i_* \rho = i_*$, $\gamma i_* = i_*$, $\rho T = \tilde{T}$ on $K_n(R)$, $n \in \{0,1\}$, and $\gamma T = \tilde{T}$ on $K_n(S)$, $n \in \{0,1\}$. Now the result follows.

Let $X \subset X' \subset K_j(R)$ and $Y \subset Y' \subset K_j(S)$, $j \in \{0,1\}$, be T and $\tilde{T}$ invariant subgroups. We suppose that $i_*(X) \subset Y$, $i_*(X') \subset Y'$, $i^*(Y) \subset X$, $i^*(Y') \subset X'$. Denote $A = X'/X$ and $B = Y'/Y$. The maps i^* and i_* induce maps of quotient groups obtained above and these maps commute with involutions T and $\tilde{T}$ analogously as in the preceding case. So we obtain induced maps of Tate cohomology groups which we will denote by $i^!$ and $i_!$ respectively. Relative Tate cohomology groups are defined as follows:

$$H^n(i_* : (A, T) \to (B, T)) = H^n(i_*^+),$$

and

$$H^n(i^* : (B, T) \to (A, T)) = H^N(i_+^*),$$

which fit into exact sequences [9]:

$$(3.2) \qquad \ldots \to H^n(A,T) \xrightarrow{i_!} H^n(B,T) \to H^n(i_*^+) \to \ldots$$

and

$$\ldots \to H^n(B,T) \xrightarrow{i^!} H^n(A,T) \to H^n(i_+^*) \to \ldots$$

Analogous exact sequences exist for other cases of pairs of involutions in which maps i_* and i^* commute with involutions on A and B. We will denote

$$H^n(i_-^*) = H^n(i^* : (B,\tilde{T}) \to (A,T))$$

and

$$H^n(i_*^-) = H^n(i^* : (A,T) \to (B,\tilde{T})).$$

Theorem 3.1. [7]. *Let $i : (R,\alpha,u) \to (S,\alpha,u)$ be a twisted quadratic extension of antistructures. Then under the hypothesis above we have a commutative braid of exact sequences*

$$(3.3) \qquad \begin{tikzcd}[column sep=small]
\ldots \to H^n(B,\tilde{T}) \arrow[rr, "i^!"] \arrow[dr] & & H^n(A,T) \arrow[rr, "i_!"] \arrow[dr] & & H^n(B,T) \arrow[rr, "i^!"] \arrow[dr] & & H^n(A,\tilde{T}) \to \ldots \\
& H^{n-1}(i_*^+) \arrow[ur] \arrow[dr] & & H^n(i_-^*) \arrow[ur] \arrow[dr] & & H^n(i_*^+) \arrow[ur] \arrow[dr] & \\
\ldots \to H^{n-1}(B,T) \arrow[rr, "i^!"] \arrow[ur] & & H^{n-1}(A,\tilde{T}) \arrow[rr, "i_!"] \arrow[ur] & & H^{n-1}(B,\tilde{T}) \arrow[rr, "i^!"] \arrow[ur] & & H^{n-1}(A,T) \to \ldots
\end{tikzcd}$$

for $n \in \{0,1\}$.

We denote the diagram above by (D). In (D) we use the isomorphisms [7]:

$$H^n(i_*^+) \cong H^{n-1}(i_* : (A,\tilde{T}) \to (B,\tilde{T})),$$

and

$$H^n(i_-^*) = H^{n-1}(i^* : (B,T) \to (A,\tilde{T})).$$

Another diagram ($\tilde{\mathrm{D}}$) is obtained if we use isomorphisms

$$H^n(i_+^*) \cong H^{n-1}(i^* : (B,\tilde{T}) \to (A,\tilde{T})),$$

and

$$H^n(i_*^-) = H^{n-1}(i_* : (A,\tilde{T}) \to (B,T)).$$

In fact, diagram (3.3) is a relative version of the diagram of Ranicki [10], since Tate cohomologies are naturally isomorphic to relative Wall groups for the identity map of ring considered with distinguished decorations.

4. Spectral sequences

Let $i : (R, \alpha, u) \to (S, \alpha, u)$ be a quadratic extension of antistructures which corresponds to the structure (ρ, a) on R. We will preserve the notation of the preceding section. There exist simplicial Ω-spectra such that all exact sequences of diagrams (D) and ($\tilde{\mathrm{D}}$) are induced by fibrations of spectra [3]. We denote by $\mathbb{H}(B,T)$ the spectrum for which $\pi_n(\mathbb{H}(B,T)) = H^n(B,T)$ and analogously for other groups in diagrams ($\tilde{\mathrm{D}}$) and (D). So exact sequences (3.2) are induced by fibrations of spectra

$$\mathbb{H}(A,T) \to \mathbb{H}(B,T) \to \mathbb{H}(i^+_*) \tag{4.1}$$

and

$$\mathbb{H}(B,T) \to \mathbb{H}(A,T) \to \mathbb{H}(i^*_+).$$

We suppose that the spectrum $\mathbb{H}$ consists of the collection of spaces $\mathbb{H}_k$ such that $\mathbb{H}_k = \Omega\mathbb{H}_{k+1}$, $k \geq 0$ [8].

Now we can use the method of Hambleton and Kharshiladze [2] to construct a spectral sequence in the considered case. Commutative diagram (D) is generated by the pull-back square of the spectra [5]:

$$\begin{tikzcd} & \mathbb{H}(B,T) \arrow[dr] & \\ \mathbb{H}(i^*_-) \arrow[ur] \arrow[dr] & & \mathbb{H}(i^+_*) \\ & \Sigma\mathbb{H}(B,\tilde{T}) \arrow[ur] & \end{tikzcd} \tag{4.2}$$

where Σ denotes a functor which is defined on each spectra $\mathbb{A} = \{\mathbb{A}_n\}$ by $((\Sigma\mathbb{A})_n) = \mathbb{A}_{n+1}$. If we consider analogous square for diagram ($\tilde{\mathrm{D}}$) we can construct a diagram, consisting of two pull-back squares:

$$\begin{tikzcd} & \mathbb{H}(B,T) \arrow[dr] & \\ \mathbb{H}(i^*_-) \arrow[ur] \arrow[dr] & & \mathbb{H}(i^+_*) \\ & \Sigma\mathbb{H}(B,\tilde{T}) \arrow[ur] \arrow[dr] & \\ \Sigma\mathbb{H}(i^*_+) \arrow[ur] \arrow[dr] & & \Sigma\mathbb{H}(i^-_*) \\ & \Sigma^2\mathbb{H}(B,T) \arrow[ur] & \end{tikzcd} \tag{4.3}$$

If we continue this procedure we obtain the vertical row of pull-back squares. The diagram consists of Ω-spectra; hence for simplicity we can write it on the space level. We will use notation analogous to that in [2]. Let

$$
\begin{aligned}
X_{0,0} &= \mathbb{H}_0(B,T), & & \\
X_{1,1} &= \mathbb{H}_1(B,\tilde{T}), & X_{2,2} &= \mathbb{H}_2(B,T), \\
X_{2k+1,2k+1} &= \mathbb{H}_{2k+1}(B,\tilde{T}), & X_{2k,2k} &= \mathbb{H}_{2k}(B,T), \\
X_{1,0} &= \mathbb{H}_0(i^*_-), & X_{2,1} &= \mathbb{H}_1(i^*_+), \\
X_{2k+1,2k} &= \mathbb{H}_{2k}(i^*_-), & X_{2k,2k-1} &= \mathbb{H}_{2k-1}(i^*_+), \\
X_{0,1} &= \mathbb{H}_0(i^+_*), & X_{1,2} &= \mathbb{H}_1(i^-_*), \\
X_{2k,2k+1} &= \mathbb{H}_{2k}(i^+_*), & X_{2k-1,2k} &= \mathbb{H}_{2k-1}(i^-_*).
\end{aligned}
$$

One should observe that the maps $X_{k+1,k} \to X_{k,k}$, $X_{k+1,k} \to X_{k+1,k+1}$, $X_{k,k} \to X_{k,k+1}$, $X_{k,k} \to X_{k-1,k}$ induce the maps of diagram (4.3).

We now extend this diagram using the pull-back squares on the left and push-out squares on the right. To obtain the space $X_{k,k-2}$ we consider the already known maps

$$X_{k-1,k-2} \to X_{k-1,k-1} \leftarrow X_{k,k-1}.$$

Then $X_{k,k-2}$ is the space which gives the pull-back square

$$
\begin{array}{ccc}
X_{k,k-2} & \to & X_{k-1,k-2} \\
\downarrow & & \downarrow \\
X_{k,k-1} & \to & X_{k-1,k-1}.
\end{array}
$$

By extending this process we obtain the diagram

$$
\begin{array}{ccccccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & X_{3,0} & \to & X_{2,0} & \to & X_{1,0} & \to & X_{0,0} & \to & X_{-1,0} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & * & \downarrow & & \downarrow & & \\
\cdots \to & & X_{3,1} & \to & X_{2,1} & \to & X_{1,1} & \to & X_{0,1} & \to & X_{-1,1} & \to & \cdots \\
 & & \downarrow & & \downarrow & * & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & X_{3,2} & \to & X_{2,2} & \to & X_{1,2} & \to & X_{0,2} & \to & X_{-1,2} & \to & \cdots \\
 & & \downarrow & * & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & X_{3,3} & \to & X_{2,3} & \to & X_{1,3} & \to & X_{0,3} & \to & X_{-1,3} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
\tag{4.4}
$$

Squares of diagram (4.3) are placed in (4.4) on the diagonal and we denote them by $*$. This diagram plays the role of diagram [2; (3)]. Hence, we can use the filtration

$$\ldots \to X_{3,0} \to X_{2,0} \to X_{1,0} \to X_{0,0} \to X_{-1,0} \to \ldots.$$

to define a spectral sequence similar to the one given in [2]. Let

$$E_1^{p,q} = \pi_{q-p}(X_{p,0}, X_{p+1,0}) = \pi_{q-p}(X_{p,i}, X_{p+1,i})$$

and let the differential

$$d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

be the natural map

$$\begin{array}{ll} \pi_{q-p}(X_{p,p}, X_{p+1,p}) & \\ \quad\downarrow \cong & \\ \pi_{q-p}(X_{p,p+1}, X_{p+1,p+1}) & \to \pi_{q-p-1}(X_{p+1,p+1}, X_{p+2,p+1}), \end{array}$$

where the vertical map is an isomorphism since we consider the pull-back (and hence push-out) squares of spectra.

From (4.1) and (4.2) we obtain that

$$E_1^{p,q} = \pi_{q-p}(X_{p,p}, X_{p+1,p}) = \pi_{q-p}(\mathbb{H}_p(A, \tilde{T})) = H^q(A, \tilde{T})$$

and

$$E_1^{p+1,q} = \pi_{q-p-1}(X_{p+1,p+1}, X_{p+2,p}) = H^n(A, \tilde{T}),$$

and that, for p even and $q \in \{0,1\}$, the map $d_1^{p,q}$ coincides with the composition

$$H^q(A, \tilde{T}) \xrightarrow{i_!} H^q(B, \tilde{T}) \xrightarrow{i^!} H^q(A, \tilde{T}),$$

where the map $i_!$ lies in diagram (D) and the map $i^!$ lies in diagram ($\tilde{\mathrm{D}}$). For p odd and $q \in \{0,1\}$, $d_1^{p,q}$ coincides with the composition

$$H^q(A, \tilde{T}) \xrightarrow{i_!} H^q(B, T) \xrightarrow{i^!} H^q(A, \tilde{T}),$$

where the map $i_!$ lies in ($\tilde{\mathrm{D}}$) and $i^!$ lies in (D). We denote by ρ the involution of the group $H^q(A, \tilde{T})$, induced by the involution ρ of $K_i(R)$, $i \in \{0,1\}$.

Theorem 4.1. *The differential $d_1^{p,q} : E_1^{p,q} \to E_1^{p,q+1}$ does not depend on p and q and it coincides with the map $1 + \rho$.*

Proof. Obviously $d_1^{p,q} = i^! \circ i_!$ and maps $i^!$ and $i_!$ of homology groups are induced by chain maps of chain complexes. Now the result follows from [10], where $i^! \circ i_! = 1 + \rho$ was obtained (see also [6]). ■

Corollary 4.2. *Let $i : (R, \alpha, u) \to (S, \alpha, u)$ be the untwisted quadratic extension, i.e. the element t commutes with all elements of ring R. Then the differential $d_1^{p,q}$ of the obtained spectral sequence is trivial.*

Proof. Clear, since in this case $\rho = 1$ and all elements of Tate cohomology groups have order 2. ■

The map $1+\rho : H^q(A,\tilde{T}) \to H^q(A,\tilde{T})$ is a differential of this group, i.e. $(1+\rho)^2 = 0$. The term $E_2^{p,q}$ of the spectral sequence is now defined as

$$E_2^{p,q} \cong \frac{\mathrm{Ker}(1+\rho)}{\mathrm{Im}(1+\rho)}$$

and it does not depend on q.

We considered the spectral sequence which corresponds to the filtration given by the horizontal rows of diagram (4.4). Analogously, there exists a spectral sequence which corresponds to the filtration given by the columns of diagram (4.4). The situation here is analogous to the considered case. We only remark that for this spectral sequence we have that $E_1^{p,q} \cong H^q(A,T)$, but $d_1^{p,q} : H^q(A,T) \to H^q(A,T)$ coincides with the map $1+\rho$ as in the case of the first spectral sequence.

In addition, two more spectral sequences are obtained analogously if, instead of (4.2), we use the following universal square of spectra:

(4.5)
$$\begin{tikzcd} & \mathbb{H}(A,T) \arrow[dr] & \\ \Sigma\mathbb{H}(i_*^+) \arrow[ur] \arrow[dr] & & \mathbb{H}(i_-^*) \\ & \Sigma\mathbb{H}(A,\tilde{T}) \arrow[ur] & \end{tikzcd}$$

In this case we obtain two spectral sequences. For the first one we have that $E_1^{p,q} \cong H^q(B,\tilde{T})$ with $d_1^{p,q} = 1+\gamma = i_! i^!$ and for the second one we have $E_1^{p,q} \cong H^q(B,T)$, with the same $d_1^{p,q}$. For the last two cases the first differential cannot vanish even for the untwisted quadratic extensions. The authors do not presently know of any example of a spectral sequence with a nontrivial second differential. (For the spectral sequence in surgery such an example was given in [2].)

We give some examples of spectral sequences with a nontrivial first differential. The main intermediate step in the calculation of surgery obstruction groups [11] is a group $L_*^Y(\widehat{\mathbb{Z}}_2\pi)$, which is connected with Tate cohomology group $H^*(K_1(\widehat{\mathbb{Z}}_2\pi/Y)$ by means of the Rothenberg exact sequence

$$\ldots \to L_*^Y(\widehat{\mathbb{Z}}_2\pi) \to L_n^K(\widehat{\mathbb{Z}}_2\pi) \to H^n(K_1(\widehat{\mathbb{Z}}_2/\pi)/Y) \to \ldots$$

Here $\widehat{\mathbb{Z}}_2$ is the ring of 2-adic integers, $Y = \{\pm\pi\} + \{\mathrm{Ker}(K_1(\widehat{\mathbb{Z}}_2\pi) \to K_1(\widehat{\mathbb{Q}}_2\pi))\}$ and involution of $K_1(\widehat{\mathbb{Z}}_2\pi)$ is induced by the standard involution of $\widehat{\mathbb{Z}}_2\pi$, given by

$$a = \Sigma n_g g \to \bar{a} = \Sigma n_g w(g) g^{-1}, \quad n_g \in \widehat{\mathbb{Z}}_2, \quad g \in \pi,$$

and $w : \pi \to \{\pm 1\}$ is an orientation homeomorphism. We will denote $K = K_1(\widehat{\mathbb{Z}}_2\pi)$. For a finite abelian 2-group π the group $L_n^K(\widehat{\mathbb{Z}}_2\pi) \cong \mathbb{Z}/2$ and hence groups $L_n^Y(\widehat{\mathbb{Z}}_2\pi)$ are almost equal to Tate cohomology groups.

Example 4.3. Let $i : \pi \to \pi \oplus \mathbb{Z}_2 = G$ be an inclusion of finite abelian 2-groups, r be the 2-rank of the group π and s be the number of summands of order 2 in π. We assume that the orientation homomorphism is trivial on G and π. The inclusion i gives a quadratic extension of rings $\widehat{\mathbb{Z}}_2\pi \to \widehat{\mathbb{Z}}_2G$ with a standard involution and $\rho = \mathrm{Id}$. Denote

$$A = (K_1(\widehat{\mathbb{Z}}_2\pi)/\{\pm\pi\}) \quad \text{and} \quad B = (K_1(\widehat{\mathbb{Z}}_2G)/\{\pm G\}).$$

Observe that in this case, $K_1(\widehat{\mathbb{Z}}_2\pi) = (\widehat{\mathbb{Z}}_2\pi)^\times$ and $K_1(\widehat{\mathbb{Z}}_2G) = (\widehat{\mathbb{Z}}_2G)^\times$. The involution $\tilde{T}$ on the group $K_1(\widehat{\mathbb{Z}}_2\pi)$ coincides with the involution T, and $\tilde{T}$ on the group $K_1(\widehat{\mathbb{Z}}_2G)$ is generated by the involution

$$\Sigma n_g g \to \Sigma n_g w'(g) g^{-1},$$

where $w' \equiv w$ on the group π and $w'(t) = -1$, for every $t \in G\backslash\pi$. All Tate cohomology groups are known [11]:

$$\begin{aligned}
&H^1(A,T) = H^1(A,\tilde{T}) = H^1(B,T) = 0\\
&H^2(B,\tilde{T}) \cong (\mathbb{Z}_2)^{2^r-s+1}\\
&H^0(A,T) = H^0(A,\tilde{T}) \cong (\mathbb{Z}_2)^{2^r}\\
&H^0(B,T) = (\mathbb{Z}_2)^{2^r+1}\\
&H^0(B,\tilde{T}) = (\mathbb{Z}_2)^{2^r-s+1}
\end{aligned}$$

Consider the spectral sequence constructed for the square (4.5) with the first differential

$$d_1 = 1+\gamma : H^q(B,T) \to H^q(B,T).$$

For q odd, d_1 is obviously trivial, and for q even, we have the following commutative diagram

$$\begin{tikzcd}
(\mathbb{Z}_2)^{2^r+1} = H^0(B,T) \arrow[rr, "d_1"] \arrow[dr, "i^!"] & & H_0(B,T) = (\mathbb{Z}_2)^{2^r+1} \\
& H^0(A,\tilde{T}) = (\mathbb{Z}_2)^{2^r} \arrow[ur, "i_!"] &
\end{tikzcd}$$

It follows from [5] that $\mathrm{Im} i^! \cong (\mathbb{Z}_2)^{s-1}$ and that $i_!$ is a monomorphism. Hence $\mathrm{Im} d_1 = (\mathbb{Z}_2)^{s-1}$ and it is nontrivial for $s \geq 2$. It is necessary to note that for the spectral sequence with $E_1^{p,q} \cong H^q(B,\tilde{T})$ the first differential d_1 is always trivial. This also follows from results in [5].

Example 4.4. Let $\pi = \mathbb{Z}_2^r \to D_{r+1}$ be an inclusion of index 2 of the cyclic group of order 2^r into the dihedral 2-group $D_{r+1} = \{x^{2^r} = y^2 = 1,\ y^{-1}xy = x^{-1}\}$, $w = 1$. As in example (4.3), we obtain a twisted quadratic extension of antistructures $\widehat{\mathbb{Z}}_2\pi \to \widehat{\mathbb{Z}}_2G$ with standard involutions, but in this case the involution ρ is given on element $x \in \pi$ by $\rho(x) = x^{-1}$. We denote $A = K_1(\widehat{\mathbb{Z}}_2\pi)/Y$ and $B = K_1(\widehat{\mathbb{Z}}_2D_{r+1})/Y$. Then $H^1(A,T) = H^1(A,\tilde{T}) = H^1(B,T) = H^1(B,\tilde{T}) = 0$, $H^0(A,T) = (\mathbb{Z}_2)^2$, $H^0(A,\tilde{T}) = (\mathbb{Z}_2)^{2^r}$, $H^0(B,T) = (\mathbb{Z}_2)^{2^{r-1}+3}$, and $H^0(B,\tilde{T}) = (\mathbb{Z}_2)^{2^{r-1}-1}$ [11, 4]. In this case we have the commutative diagram

$$\begin{array}{ccccc} (\mathbb{Z}_2)^{2^r} \cong H^0(A,\tilde{T}) & & \xrightarrow{d_1} & & H^0(A,\tilde{T}) \cong (\mathbb{Z}_2)^{2^r} \\ & \text{epi} \searrow i^! & & \nearrow i_! & \\ & & H^0(B,\tilde{T}) \cong (\mathbb{Z}_2)^{2^{r-1}-1} & & \end{array}$$

The map $i^!$ is an epimorphism (see [5]) and rank Ker $i_! \leq 2$, since this map is from the exact sequence

$$(4.6) \qquad \begin{array}{ccccccccc} \to H^0(A,T) & \xrightarrow{i_!} & H^0(B,\tilde{T}) & \xrightarrow{i^!} & H^0(A,\tilde{T}) & \xrightarrow{i_!} & H^0(B,T) & \xrightarrow{i^!} \\ \cong & & \cong & & \cong & & \cong & \\ (\mathbb{Z}_2)^2 & & (\mathbb{Z}_2)^{r^{r-1}-1} & & (\mathbb{Z}_2)^{2^r} & & (\mathbb{Z}_2)^{2^{r-1}+3} & \end{array}$$

which is a row of diagram $(\tilde{\mathrm{D}})$. The other row is trivial, since all odd dimensional Tate cohomology groups vanish. So in this case we obtain that:

$$2^{r-1} - 1 \geq \operatorname{rank}(\operatorname{Im} d_1) \geq 2^{r-1} - 3.$$

We remark that the result about d_1 can give helpful information about natural maps of Tate cohomology groups. In our case diagram (D) is described in full [4, 5], but the maps of diagram $(\tilde{\mathrm{D}})$ are unknown.

Theorem 4.5. *The map $i^! : H^0(B,T) \to H^0(A,T)$ in (4.6) is an epimorphism.*

Proof. For the calculation of $d_1 : H^0(B,T) \to H^0(B,T)$ we can use two commutative diagrams

$$(4.7) \qquad \begin{array}{ccccc} (\mathbb{Z}_2)^{2^{r-1}+3} \cong H^0(B,T) & & \xrightarrow{d_1} & & H^0(B,T) \cong (\mathbb{Z}_2)^{2^{r-1}+3} \\ & ? \searrow i^! & & \nearrow i_! \text{ mono} & \\ & & H^0(A,T) \cong (\mathbb{Z}_2)^2 & & \end{array}$$

and

$$(4.8) \qquad \begin{array}{ccccc} (\mathbb{Z}_2)^{2^{r-1}+3} \cong H^0(B,T) & & \xrightarrow{d_1} & & H^0(B,T) \cong (\mathbb{Z}_2)^{2^{r-1}+3} \\ & \text{rank Im } i^! = 2^{r-1}+1 \searrow i^! & & ? \nearrow i_! & \\ & & H^0(A,\tilde{T}) \cong (\mathbb{Z}_2)^{2r} & & \end{array}$$

In (4.7) and (4.8) we marked by ? the unknown maps from diagram $(\tilde{\mathrm{D}})$. It follows from (4.7) that rank $(\operatorname{Im} d_1) \leq 2$.

From ($\tilde{\mathrm{D}}$) we conclude that rank Im $i_! \geq 2^{r-1}+1$, hence rank Ker $i_! \leq 2^{r-1}-1$ in (4.8). But in this case it follows from (4.8) that rank Im $d_1 \geq 2$. If we compare results obtained from (4.7) and (4.8) we obtain that Im $d_1 \cong (\mathbb{Z}_2)^2$ in this case. It thus follows that the map $i^!$ in (4.7) is an epimorphism. ■

By this theorem we can immediately obtain all maps from ($\tilde{\mathrm{D}}$). Moreover, we have an exact description of the differential

$$d_1 : H^0(A,\tilde{T}) \to H^0(A,\tilde{T})$$

which is given by

$$\mathrm{Im}\ d_1 \cong \mathrm{Im}(i_! : H^0(B,\tilde{T}) \to H^0(A,\tilde{T})) \cong H^0(B,\tilde{T}) \cong (\mathbb{Z}_2)^{2^{r-1}-1}.$$

This information on maps of ($\tilde{\mathrm{D}}$) gives new information about natural maps of Wall groups [5].

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References

[1] W. Browder and G.R. Livesay, Fixed point free involutions on homotopy spheres, *Tohoku Math. J.* **25** (1973), 69–87.

[2] I. Hambleton and A.F. Kharshiladze, A spectral sequence in surgery theory, *Mat. Sbornik* **183**:9 (1992), 3–14 (in Russian); *Engl. transl. in Russian Acad. Sci. Math. Sb.* **77**:1 (1994), 1–9.

[3] I. Hambleton, A.A. Ranicki and L.R. Taylor, Round L-theory, *J. Pure Appl. Algebra* **47** (1987), 131–154.

[4] Yu. V. Muranov, The Tate cohomology and Browder-Livesay groups of dihedral groups, *Mat. Zametki* **54**:2 (1993), 44–55 (in Russian); *Engl. transl. in Math. Notes* **54**:1–2 (1993), 798–805.

[5] Yu. V. Muranov, Relative Wall groups and decorations, *Mat. Sbornik* **185**:12 (1994), 79–100 (in Russian); *Engl. transl. in Russian Acad. Sci. Math. Sb.* **83**:2 (1995), 495–514.

[6] Yu. V. Muranov, Obstruction groups to splitting along one-sided submanifolds, *Uspehi Mat. Nauk* **50**:1 (1995), 205–206 (in Russian).

[7] Yu. V. Muranov, K-groups of quadratic extensions of rings, *Mat. Zametki* **58** (1995), 272–280 (in Russian); *Engl. transl. in Math. Notes* **58**:1–2 (1995), 861–866.

[8] A.A. Ranicki, The total surgery obstruction, in: Algebraic Topology (Aarhus 1978), *Lect. Notes in Math.* **763**, Springer-Verlag, Berlin 1979, pp. 275–316.

[9] A.A. Ranicki, Exact Sequences in the Algebraic Theory of Surgery, *Math. Notes* **26**, Princeton Univ. Press, Princeton NJ, 1981.

[10] A.A. Ranicki, The L-theory of twisted quadratic extensions, *Can. J. Math.* **39** (1987), 345–364.

[11] C.T.C. Wall, Classification of Hermitian forms, VI. Group rings, *Ann. of Math.* **(2) 103** (1976), 1–80.

[12] C.T.C. Wall, On the axiomatic foundations of the theory of Hermitian forms, *Proc. Camb. Phil. Soc.* **67** (1970), 243–250.

[13] C.T.C. Wall, Surgery on Compact Manifolds, *London Math. Soc. Monogr.* **1**, Academic Press, New York, 1970.

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