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ON A CERTAIN SURGERY SPECTRAL SEQUENCE

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Abstract

Given a morphism between twisted quadratic extensions of antistructures, we construct a spectral sequence which generalizes, in a natural way, the surgery spectral sequence introduced by Hambleton and Kharshiladze in [Mat. Sb. 183(9) (1992), 3-14 (in Russian); English Transl. in Russian Acad. Sci. Sb. Math. 77(1) (1994), 1-9]. Our spectral sequence allows us to obtain some additional information about the Browder-Livesay invariants. Then we study the relations between the above-mentioned spectral sequences, and discuss some examples and applications.

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1. Introduction

In [5] Cappell and Shaneson settled the question of the existence of a spectral sequence in surgery theory related to the splitting problem in the case of one sided submanifolds. Such a spectral sequence was constructed by Hambleton and Kharshiladze in [18] as a natural algebraic object closely related to the Browder-Livesay invariants and the problem of determining which elements of surgery obstruction groups arise from normal maps of closed manifolds.

Let $X \subset Y$ be a one-sided codimension 1 submanifold of an $(n+1)$-dimensional manifold Y such that the inclusion map induces an isomorphism between the fundamental groups. We set $\pi = \pi_1(Y \setminus X)$, and $G = \pi_1(Y)$. Denote by $hT(Y)$ the set of homotopy triangulations of the manifold Y (we work in the piecewise-linear category), which consists of all simple homotopy equivalences $f : M \to Y$ with the following equivalence relation. Two simple homotopy equivalences $f_1 : M_1 \to Y$ and $f_2 : M_2 \to Y$ are said to be *equivalent* if there exists a piecewise-linear homeomorphism $g : M_1 \to M_2$ such that the composite map $f_2 \circ g$ is homotopic to f_1.

The Browder-Livesay group $LN_n(\pi \to G, w)$ is the group of obstructions for splitting a simple homotopy equivalence $f : M \to Y$ along X for any manifold M (see [3], [4], [14], [22], [31], and [36]). The homomorphism $i : \pi \to G$ is induced by the natural inclusion, and $w : G = \pi_1(Y) \to \{\pm 1\}$ is the orientation homomorphism. The manifold X is equipped with the orientation character $w^- : G \to \{\pm 1\}$ which is changed only on the element $t \in G \setminus \pi$. We denote by G^- the group G with this orientation.

There exist two important maps between the Browder-Livesay groups and the Wall surgery obstruction groups (for recent results on the (stable) classification of closed 4-manifolds by using surgery obstruction groups we refer to [6], [7], [8], and [9]).

The first map

$$ti_* : LN_n(\pi \to G) \to L_n(G^-)$$

can be described in the following way. Suppose that the obstruction for splitting the map f along X is the element

$$x = \sigma(f) \in LN_n(\pi \to G, w).$$

Then the element

$$y = ti_*(x) \in L_n(G^-)$$

is represented by the normal map $f^{-1}(X) \to X$.

To describe the second map

$$i^!t^{-1} : L_{n+2}(G) \to LN_n(\pi \to G),$$

we must realize any element $x \in L_{n+2}(G)$ as a normal map $F : Z \to Y \times [0, 1]$ between manifolds with boundaries (see [36]). We set $\partial_0(Z) = F^{-1}(Y \times 0)$ and $\partial_1(Z) = F^{-1}(Y \times 1)$.

The restriction

$$(F|_{\partial_1(Z)} : \partial_1(Z) \to Y \times 1) \in hT(Y)$$

of the map F is a homotopy triangulation of the manifold Y. The restriction $F|_{\partial_0(Z)}$ is the identity map, and the obstruction for splitting the simple homotopy equivalence

$$F|_{\partial_1(Z)} : \partial_1(Z) \to Y \times 1$$

along the submanifold X is the element $i^!t^{-1}(x) \in LN_n(\pi \to G)$.

There are algebraic definitions of these maps (see [14], [30], and [31]) which give effective methods of computing them (see [15], [23], [24], and [26]).

The following deep result about the map $i^!t^{-1}$ was obtained by Cappell and Shaneson in [4] (see also [14] and [15]).

Theorem 1.1. *Let $i : \pi \to G$ be an inclusion of index 2 and let x be an element of $L_{n+1}(G)$. If $i^! t^{-1}(x) \neq 0$ in $LN_{n-1}(\pi \to G)$, then x cannot be realized by a normal map of closed manifolds. Furthermore, it never acts trivially on any element of the set $hT(Y)$ of homotopy triangulations of any closed connected manifold Y with $\pi_1(Y) \cong G$.*

Thus the map $i^! t^{-1}$ represents the first obstruction for realizing an element of the Wall group $L_{n+1}(G)$ by a normal map of closed manifolds. Sometimes, it is called the *Browder-Livesay invariant* since the case of the index 2 inclusion $1 \to \mathbb{Z}/2$ was first considered in [3] (see also [22]). The Browder-Livesay invariant fits in the following braid of exact sequences (see [14], [20], [26], [32], [33], and [36]):

$$
\begin{array}{ccccccccccc}
\to & & L_{n+1}(\pi) & & \to & & L_{n+1}(G) & & \overset{i^! t^{-1}}{\to} & & LN_{n-1}(\pi \to G) & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & L_{n+1}(i_-') & & \downarrow \Gamma & & L_{n+1}(i_*) & & & & (D) \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & LN_n(\pi \to G) & & \to & & L_n(G^-) & & \to & & L_n(\pi) & &
\end{array}
$$

where the vertical map Γ yields isomorphisms between the homology groups of upper and lower rows. This diagram has period 4. The map Γ can be described in the following way. Let $\dim Y = n$, $\pi_1(Y) = G$, and suppose that a normal map

$$F : Z \to Y \times [0, 1]$$

realizes an element $x \in L_{n+1}(G)$. If $i^! t^{-1}(x) = 0$, then the map

$$F|_{F^{-1}(X \times [0,1])} : F^{-1}(X \times [0, 1]) \to X \times [0, 1]$$

is a simple homotopy equivalence on the boundary

$$\partial F^{-1}(X \times [0, 1]) = F^{-1}(X \times 0) \cup F^{-1}(X \times 1),$$

and hence it represents the element $\Gamma(x)$ of $L_n(G^-)$.

We compare the index 2 inclusions $\pi \to G$ and $\pi \to G^-$ of groups with orientations. There exists a diagram similar to (*D*) arising from the inclusion $\pi \to G^-$. The last one contains the map ti_*, and it will be denoted by (D^-).

By using diagrams (*D*) and (D^-), there was constructed in [18] a surgery exact sequence with

$$E_1^{p,q} = LN_{q-2p-2}(\pi \to G^{(-)^p}) = LN_{q+2}(\pi \to G).$$

The first differential

$$d_1 : E_1^{p,q} \to E_1^{p+1,q}$$

of this spectral sequence coincides with the composition

$$\begin{array}{ll} LN_{q-2p-2}(\pi \to G^{(-)^p}) \overset{ti_*}{\to} L_{q-2p-2}(G^{(-)^{p+1}}) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow i^!t^{-1} & \\ \qquad\qquad LN_{q-2p}(\pi \to G^{(-)^{p+1}}) = LN_{q-2p-2}(\pi \to G^{(-)^p}), & \end{array}$$

which is the map $1 \pm \Phi$ for the involution Φ on the groups $LN_*(\pi \to G)$ (see [14] and [18]). The higher differentials of this spectral sequence can be written down in the form $i^!t^{-1} \circ \Gamma^k \circ ti_*$, and thus they coincide with the iterated Browder-Livesay invariants.

In [18] there was given an example of the nontrivial second differential of the spectral sequence. However, it follows from [23] and [24] that all second differentials are trivial in the case of L'-groups for finite abelian 2-groups. The problem about the existence of nontrivial higher differentials is still open.

Afterwards in [10] there was constructed spectral sequences in *K*-theory for a twisted quadratic extension of antistructures. These spectral sequences are simpler than the surgery spectral sequence, and give more effective methods for computing natural maps between Tate cohomology groups, and, hence, natural maps between decorated L-groups ([16], [35], and [36]). As follows from [10], [11], [18], [26], and [32], it is possible to construct a spectral sequence similar to those discussed above for any quadratic extension of antistructures.

We remark that in geometric applications of the splitting problem for a one-sided codimension 1 submanifold, the groups $LS_n(F)$ arise in the case when the map $\pi_1(X) \to G = \pi_1(Y)$, induced by the inclusion $X \subset Y$, is only an epimorphism. The group $LS_n(F)$ of the obstructions for the splitting problem does not depend on the particular pair of manifolds, but it depends functorially on the push-out square F of fundamental groups with orientation:

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(Y \setminus X) \\ \downarrow & & \downarrow \\ \pi_1(X) & \to & \pi_1(Y) \end{pmatrix}, \tag{1.1}$$

where U is a tubular neighborhood of X in Y, and on dimension n (mod 4) (see [1], [2], [12], [13], [19], [21], [25], [27], [28], [31], [33], and [36]). If the map i in square (1.1) is an isomorphism, then we have

$$LS_n(F) \cong LN_n(\pi \to G).$$

In the case of LS_* groups, there exists a natural map

$$\Theta : L_{n+1}(G) \to LS_{n-1}(F),$$

which is similar to the map $i^! t^{-1}$, and it is called the *generalized Browder-Livesay invariant* (see [1] and [21]).

For an n-dimensional manifold Y we have the surgery exact sequence (see [36])

$$\xrightarrow{\sigma_*} L_{n+1}(\pi_1(Y)) \to hT(Y) \to [Y, G/PL] \xrightarrow{\sigma_*} L_n(\pi_1(Y)) \tag{1.2}$$

which is the main tool for computing the set $hT(Y)$ for the closed manifold Y. The group $\pi_1(Y)$ is equipped with an orientation $w : \pi_1(Y) \to \{\pm 1\}$ and G denotes the monoid of classes of stable homotopy equivalences of the standard n-sphere S^n. It follows from the definition of the exact sequence (1.2) that the elements of the group $L_n(\pi_1(Y))$, which lie in the image of the map σ_*, are realized by normal maps of closed manifolds (see [14], [15], [31], [33] and [36]). Additionally, let $X \subset Y$ be a closed one-sided submanifold of the manifold Y (codim $X = 1$). Then we have the following commutative diagram (see [31], and [33]):

$$\begin{array}{ccccccccc} \xrightarrow{\sigma_*} & L_{n+1}(\pi_1(Y)) & \to & hT(Y) & \to & [Y, G/PL] & \xrightarrow{\sigma_*} & L_n(\pi_1(Y)) \\ & \| & & \downarrow & & \downarrow & & \| \\ \to & L_{n+1}(\pi_1(Y)) & \xrightarrow{\Theta} & LS_{n-1}(F) & \to & LP_{n-1}(F) & \to & L_n(\pi_1(Y)) \end{array} \tag{1.3}$$

The second row of diagram (1.3) is the exact sequence appearing in the diagram similar to (D) (see Section 2 below). It follows from diagram (1.3) that any element $x \in L_{n+1}(\pi_1(Y))$ with $\Theta(x) \neq 0$ acts nontrivially on the set $hT(Y)$. In the considered case we have also the Browder-Livesay invariant

$$L_{n+1}(\pi_1(Y)) \to LN_{n-1}(\pi_1(Y \setminus X) \to \pi_1(Y))$$

which is the composition

$$L_{n+1}(\pi_1(Y)) \xrightarrow{\Theta} LS_{n-1}(F) \to LN_{n-1}(\pi_1(Y \setminus X) \to \pi_1(Y)),$$

where the second map is induced by the natural map of squares

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(Y \setminus X) \\ \downarrow & & \downarrow \\ \pi_1(X) & \xrightarrow{i} & \pi_1(Y) \end{pmatrix} \to \begin{pmatrix} \pi_1(Y \setminus X) & \xrightarrow{=} & \pi_1(Y \setminus X) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \xrightarrow{=} & \pi_1(Y) \end{pmatrix}. \tag{1.4}$$

Thus the generalized Browder-Livesay invariant can give deeper information about the set $hT(Y)$ than the Browder-Livesay invariant.

From an algebraic point of view, the natural generalization of square (1.1) is given by a *geometric diagram of antistructures* (see [19], [25], and

[26]). This is a commutative square of antistructures [34] in which the vertical maps are quadratic extensions of antistructures [32], and the horizontal maps are epimorphisms with some additional natural properties. A square of antistructures of this type arises from square (1.1) by considering the group rings with standard involutions (see [19], [25], and [28]).

In the present paper, we construct a spectral sequence for any geometric diagram of antistructures, and study its algebraic properties. In the case when LS_*-groups are isomorphic to the Browder-Livesay groups, our spectral sequence coincides with the spectral sequence constructed by Hambleton and Kharshiladze in [18]. Then we study the relations with the previous spectral sequences, and discuss some examples and applications.

2. Geometric Diagrams of Antistructures

We first recall some basic definitions and results on the topic (see [11], [25], [32], and [34] for more details). Let R be a ring with unity 1. An *antistructure* is a triple (R, α, u), where $u \in R^*$ is a unit and $\alpha : R \to R$ is an anti-automorphism such that $\alpha(u) = u^{-1}$ and $\alpha^2(x) = uxu^{-1}$ for every $x \in R$. The anti-automorphism α induces involutions on the groups $K_i(R)$ for any $i = 0, 1$. A ring homomorphism $f : R \to R'$ defines a *morphism* of antistructures, also denoted by f,

$$f : (R, \alpha, u) \to (R', \alpha', u')$$

if $f(u) = u'$, and $\alpha' \circ f = f \circ \alpha$.

For any antistructure (R, α, u) and for any subgroup $X \subset K_i(R)$ or $X \subset \widetilde{K}_i(R)$, $i = 0, 1$, which is invariant under the induced involutions, the decorated Wall groups were defined (see [16], [31], [34], and [35]).

In this section, we fix one of the possible decorations, namely "s", "h", or "p" (see also [15] and [36]). So all L-groups are equipped with a fixed decoration which we shall not mention whenever this does not lead to any confusion.

The *structure* on the ring R is given by the pair (ρ, a), where $\rho : R \to R$ is an automorphism and $a \in R$ is a unit such that $\rho(a) = a$ and $\rho^2(x) = axa^{-1}$ for every $x \in R$. In this case, a twisted quadratic extension of the antistructure (R, α, u) can be defined. This is the antistructure (S, α, u), where $S = R[t]/(t^2 - a)$. The element t is independent over R, and $\rho(x)t = tx$ for every $x \in R$. In addition, we suppose that $\alpha(t)t \in R \subset S$. Then the anti-automorphism α can be extended on the ring S (see [32]). The natural inclusion yields the morphism of antistructures

$$i : (R, \alpha, u) \to (S, \alpha, u).$$

The automorphism ρ is extended on the ring S by formula $\rho(x + yt) = t(x + yt)t^{-1}$ for every $x, y \in R$. We can define another antistructure $(S, \tilde{\alpha}, \tilde{u})$, where $\tilde{u} = -t\alpha(t^{-1})u$, $\tilde{\alpha} = \rho\gamma\alpha$, and $\gamma(x + yt) = x - yt$ for every $x, y \in R$. Since $\tilde{u} \in R$ and the ring R is $\tilde{\alpha}$-invariant, the antistructure $(R, \tilde{\alpha}, \tilde{u})$ is defined. Thus we have the quadratic extension

$$\tilde{i} : (R, \tilde{\alpha}, \tilde{u}) \to (S, \tilde{\alpha}, \tilde{u}),$$

which coincides with i as quadratic extension of rings.

Let (R, α, u) and (P, β, v) be antistructures with structures (ρ, a) and (ρ', a'), respectively. Let us consider the commutative diagram of antistructures

$$F = \begin{pmatrix} (R, \alpha, u) & \xrightarrow{f} & (P, \beta, v) \\ \downarrow i & & \downarrow j \\ (S, \alpha, u) & \xrightarrow{g} & (Q, \beta, v) \end{pmatrix} = \begin{pmatrix} R & \to & P \\ \downarrow & & \downarrow \\ S & \to & Q \end{pmatrix}, \tag{2.1}$$

where the horizontal maps are ring epimorphisms, the vertical maps are the quadratic extensions of the antistructures corresponding to the previous structures, and $g(t) = t'$ for $t^2 = a$, and $t'^2 = a'$ (see [11], [25], and [26]). Diagram (2.1) with these properties is called a *geometric* *diagram of antistructures*, or simply a *geometric diagram*. This diagram represents a natural generalization of diagram (1.1) arising from the splitting problem for a one-sided submanifold.

By using the automorphism γ and the quadratic extensions $\tilde{i}$ and $\tilde{j}$, we can construct the following geometric diagrams:

$$\tilde{F} = \begin{pmatrix} (R, \tilde{\alpha}, \tilde{u}) & \xrightarrow{\tilde{f}} & (P, \tilde{\beta}, \tilde{v}) \\ \downarrow \tilde{i} & & \downarrow \tilde{j} \\ (S, \tilde{\alpha}, \tilde{u}) & \xrightarrow{\tilde{g}} & (Q, \tilde{\beta}, \tilde{v}) \end{pmatrix} = \begin{pmatrix} \tilde{R} & \to & \tilde{P} \\ \downarrow & & \downarrow \\ \tilde{S} & \to & \tilde{Q} \end{pmatrix}, \tag{2.2}$$

$$\tilde{F}_\gamma = \begin{pmatrix} (R, \tilde{\alpha}, \tilde{u}) & \xrightarrow{\tilde{f}} & (P, \tilde{\beta}, \tilde{v}) \\ \downarrow \tilde{i}_\gamma & & \downarrow \tilde{j}_\gamma \\ (S, \gamma\tilde{\alpha}, \tilde{u}) & \xrightarrow{\tilde{g}} & (Q, \gamma\tilde{\beta}, \tilde{v}) \end{pmatrix} = \begin{pmatrix} \tilde{R} & \to & \tilde{P} \\ \downarrow & & \downarrow \\ \tilde{S}_\gamma & \to & \tilde{Q}_\gamma \end{pmatrix}, \tag{2.3}$$

and

$$F_\gamma = \begin{pmatrix} (R, \alpha, u) & \xrightarrow{f} & (P, \beta, v) \\ \downarrow i_\gamma & & \downarrow j_\gamma \\ (S, \gamma\alpha, u) & \xrightarrow{g_\gamma} & (Q, \gamma\beta, v) \end{pmatrix} = \begin{pmatrix} R & \to & P \\ \downarrow & & \downarrow \\ S_\gamma & \to & Q_\gamma \end{pmatrix}. \tag{2.4}$$

For square (2.1), the groups $LS_*(F)$ and $LP_*(F)$ were defined in [25] and [28] (see also [11], and [27]).

These groups fit in the following braid of exact sequences:

$$\begin{array}{ccccccccccc} \to & L_{n+1}(P, \beta, v) & & \to & & L_{n+1}(Q, \beta, v) & & \xrightarrow{\Theta} & & LS_{n-1}(F) & \to \\ & \nearrow & \searrow & & & \nearrow & \searrow & & & \nearrow \quad \searrow & \\ & & & LP_n(F) & & \downarrow \Gamma & & L_{n+1}(j) & & & \\ & \searrow & \nearrow & & & \searrow & \nearrow & & & \searrow \quad \nearrow & \\ \to & LS_n(F) & & \to & & L_n(S, \gamma\alpha, u) & & \to & & L_n(P, \beta, v) & \to \end{array} \tag{D1}$$

Diagram $(D1)$ coincides with diagram (D) if square (2.1) is generated by square (1.1) for the Browder-Livesay pair by considering the group rings over $\mathbb{Z}$ with standard involution. By using square (2.4), we get, in a similar way, the following commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
\to & L_{n+1}(P,\beta,\upsilon) & & \to & L_{n+1}(Q,\gamma\beta,\upsilon) & & \overset{\Theta^-}{\to} & LS_{n-1}(F_\gamma) & \to \\
 & \nearrow \quad \searrow & & & \nearrow \quad \searrow & & & \nearrow \quad \searrow & \\
 & & LP_n(F_\gamma) & & \downarrow\Gamma^- & & L_{n+1}(j_\gamma) & & \\
 & \searrow \quad \nearrow & & & \searrow \quad \nearrow & & & \searrow \quad \nearrow & \\
\to & LS_n(F_\gamma) & & \to & L_n(S,\alpha,u) & & \to & L_n(P,\beta,\upsilon) & \to
\end{array}
\qquad (D1^-)
$$

since γ^2 is the identity map. As remarked above, this diagram generalizes diagram (D^-), discussed in Section 1.

3. Spectral Sequences

A surgery spectral sequence was first constructed in [10]. For this, the case of one-sided submanifolds and the Browder-Livesay groups in diagram (1.1) were considered. From an algebraic point of view, diagram (1.1) can be constructed for a pair $\pi \subset G$, where π is an index 2 subgroup of the group G equipped with an orientation homomorphism.

The realization of diagram (1.1) on the spectra level is the main tool used to construct the spectral sequence, mentioned above. By using this construction, spectral sequences of Tate cohomology groups of K-groups for quadratic extensions of antistructures were obtained in [3]. As a consequence, first differentials in these spectral sequences were completely described.

In this section, we discuss the surgery spectral sequences constructed by using diagrams $(D1)$ and $(D1^-)$ for a one-sided submanifold of codimension 1.

We maintain all notations considered in the previous section. According to [25] and [28] (see also [16], [18], and [36]), diagrams $(D1)$ and $(D1^-)$ can be realized on the spectra level. This means that homotopy long exact sequences of the maps in the central squares of these diagrams (written down on the spectra level) give rise to full diagrams. Let us consider the central square of diagram $(D1)$, which is realized on the spectra level, i.e.,

$$\begin{array}{ccccc} & & \mathbb{L}(Q, \beta, \upsilon) & & \\ & \nearrow & & \searrow & \\ \Sigma\mathbb{LP}(F) & & & & \mathbb{L}(j) \\ & \searrow & & \nearrow & \\ & & \Sigma\mathbb{L}(S, \gamma\alpha, u) & & \end{array} \tag{3.1}$$

where Σ denotes a functor from the category of spectra into itself. For any spectrum $\mathbb{A} = \{\mathbb{A}_n\}$, this functor is characterized by the condition $(\Sigma\mathbb{A})_n = \mathbb{A}_{n+1}$. Diagram (3.1) is a homotopy push-out (and pull-back) square of spectra in which the fibers and the cofibers of parallel maps are naturally homotopy equivalent. There exists a similar square of spectra constructed for diagram $(D1^-)$, i.e.,

$$\begin{array}{ccccc} & & \Sigma\mathbb{L}(Q, \gamma\beta, \upsilon) & & \\ & \nearrow & & \searrow & \\ \Sigma^2\mathbb{LP}(F_\gamma) & & & & \Sigma\mathbb{L}(j_\gamma) \\ & \searrow & & \nearrow & \\ & & \Sigma^2\mathbb{L}(S, \alpha, u) & & \end{array} \tag{3.2}$$

which is considered in dimension one less than the previous square. In turn, by using squares (3.1) and (3.2), in appropriate dimensions, we can construct the following diagram of spectra:

$$\begin{array}{ccccc} & & \mathbb{L}(Q, \beta, \upsilon) & & \\ & \nearrow & & \searrow & \\ \Sigma\mathbb{LP}(F) & & & & \mathbb{L}(j) \\ & \searrow l_\gamma & & \nearrow & \\ & & \Sigma\mathbb{L}(S, \gamma\alpha, u) & & \\ & & \downarrow (g_\gamma)_* & & \\ & & \Sigma\mathbb{L}(Q, \gamma\beta, \upsilon) & & \\ & \nearrow & & \searrow & \\ \Sigma^2\mathbb{LP}(F_\gamma) & & & & \Sigma\mathbb{L}(j_\gamma) \\ & \searrow l & & \nearrow & \\ & & \Sigma^2\mathbb{L}(S, \alpha, u) & & \\ & & \downarrow g_* & & \end{array} \tag{3.3}$$

$$
\begin{array}{ccccc}
 & & \Sigma^2\mathbb{L}(Q,\beta,v) & & \\
 & \nearrow & & \searrow & \\
\Sigma^3\mathbb{LP}(F) & & & & \Sigma^2\mathbb{L}(j) \\
 & \searrow l_\gamma & & \nearrow & \\
 & & \Sigma^3\mathbb{L}(S,\gamma\alpha,u) & & \\
 & & \downarrow (g_\gamma)_* & & \\
 & & \Sigma^3\mathbb{L}(Q,\gamma\beta,v) & & \\
 & \nearrow & & \searrow & \\
\Sigma^4\mathbb{LP}(F_\gamma) & & & & \Sigma^3\mathbb{L}(j_\gamma) \\
 & & \cdots & &
\end{array}
$$

From diagram (3.3), we get the following column of spectra and natural maps:

$$
\begin{array}{ccc}
 & & \mathbb{L}(Q,\beta,v) \\
 & \nearrow & \\
\Sigma\mathbb{LP}(F) & & \\
 & \searrow d_\gamma & \\
 & & \Sigma\mathbb{L}(Q,\gamma\beta,v) \\
 & \nearrow & \\
\Sigma^2\mathbb{LP}(F_\gamma) & & \\
 & \searrow d & \\
 & & \Sigma^2\mathbb{L}(Q,\beta,v) \\
 & \nearrow & \\
\Sigma^3\mathbb{LP}(F) & & \\
 & \searrow d_\gamma & \\
 & & \Sigma^3\mathbb{L}(Q,\gamma\beta,v) \\
 & \nearrow & \\
\Sigma^4\mathbb{LP}(F_\gamma) & & \\
\cdots & \cdots & \cdots
\end{array}
\tag{3.4}
$$

in which the maps $d = g_* \circ l$ and $d_\gamma = (g_\gamma)_* \circ l_\gamma$ are the compositions of the maps shown in diagram (3.3).

We now introduce the following notations:

$$
\begin{array}{ll}
X_{0,0} = \mathbb{L}(Q, \beta, \upsilon) & X_{1,1} = \Sigma\mathbb{L}(Q, \gamma\beta, \upsilon) \\
X_{2,2} = \Sigma^2\mathbb{L}(Q, \beta, \upsilon) & X_{3,3} = \Sigma^3\mathbb{L}(Q, \gamma\beta, \upsilon) \\
\dots & \dots \\
X_{2k,2k} = \Sigma^{2k}\,\mathbb{L}(Q, \beta, \upsilon) & X_{2k+1,2k+1} = \Sigma^{2k+1}\mathbb{L}(Q, \gamma\beta, \upsilon) \\
\dots & \dots \\
X_{1,0} = \Sigma\mathbb{LP}(F) & X_{2,1} = \Sigma^2\mathbb{LP}(F_\gamma) \\
X_{3,2} = \Sigma^3\mathbb{LP}(F) & X_{4,3} = \Sigma^4\mathbb{LP}(F_\gamma) \\
\dots & \dots
\end{array}
\tag{3.5}
$$

For every k, we can define the spectrum $X_{k,k-2}$ as a spectrum that fits in the following pull-back square of spectra:

$$
\begin{array}{ccc}
X_{k,k-2} & \to & X_{k-1,k-2} \\
\downarrow & & \downarrow \\
X_{k,k-1} & \to & X_{k-1,k-1}
\end{array}
$$

We continue this process to define spectra $X_{k,k-j}$ for all k and $j \geq 3$, and an infinite homotopy commutative diagram of spectra:

$$
\begin{array}{ccccccc}
 & & & & & & X_{0,0} \\
 & & & & & \nearrow & \\
 & & & & X_{1,0} & & \\
 & & & \nearrow & & \searrow & \\
 & & X_{2,0} & & & & X_{1,1} \\
 & \nearrow & & \searrow & & \nearrow & \\
 & & & & X_{2,1} & & \\
 & & & \nearrow & & \searrow & \\
 & & & & & & X_{2,2} \\
 & & & & & \nearrow &
\end{array}
\tag{3.6}
$$

To construct the surgery spectral sequence, we consider the following filtration (see [18]):

$$\cdots \to X_{3,0} \to X_{2,0} \to X_{1,0} \to X_{0,0}. \tag{3.7}$$

Then we define the first term of the spectral sequence

$$E_1^{p,s} = \pi_{s-p}(X_{p,0}, X_{p+1,0}).$$

From the construction of the spectral sequence, we get isomorphisms

$$\pi_{s-p}(X_{p,0}, X_{p+1,0}) \cong \pi_{s-p}(X_{p,1}, X_{p+1,1}) \cong \cdots \cong \pi_{s-p}(X_{p,p}, X_{p+1,p}).$$

The differential

$$d_1^{p,s} : E_1^{p,s} = \pi_{s-p}(X_{p,p}, X_{p+1,p}) \to E_1^{p+1,s} = \pi_{s-p-1}(X_{p+1,p+1}, X_{p+2,p})$$

is the composition of the natural map

$$\pi_{s-p}(X_{p,p}, X_{p+1,p}) \xrightarrow{\partial} \pi_{s-p-1}(X_{p+1,p}, X_{p+2,p})$$

from the exact sequence of the triple $X_{p+2,p} \to X_{p+1,p} \to X_{p,p}$ with the natural isomorphism

$$\pi_{s-p-1}(X_{p+1,p}, X_{p+2,p}) \to \pi_{s-p-1}(X_{p+1,p+1}, X_{p+2,p+1}).$$

Theorem 3.1. *Under the hypotheses above, we have isomorphisms*

$$E_1^{p,s} \cong LS_{s-2p-2}(F_{(\gamma^p)})$$

for any $p, s \geq 0$.

Proof. The homotopy long exact sequence of the pair $(X_{p,p}, X_{p+1,p})$ is naturally isomorphic to the following homotopy long exact sequence of the cofibration of spectra:

$$\to \pi_n(\Sigma^{p+1}\mathbb{LS}(F_{(\gamma^p)})) \to \pi_n(\Sigma^{p+1}\mathbb{LP}(F_{(\gamma^p)})) \to$$

$$\pi_n(\Sigma^p\mathbb{L}(Q, (\gamma^p)\beta, v)) \to \pi_n(\Sigma^{p+2}\mathbb{LS}(F_{(\gamma^p)})) \to \dots.$$

So we can identify $\pi_n(X_{p,p}, X_{p+1,p})$ with $\pi_n(\Sigma^{p+2}\mathbb{LS}(F_{(\gamma^p)}))$. Now, the result follows from these isomorphisms

$$\begin{aligned} E_1^{p,s} &= \pi_{s-p}(X_{p,p}, X_{p+1,p}) \\ &\cong \pi_{s-p}(\Sigma^{p+2}\mathbb{LS}(F_{(\gamma^p)})) \\ &\cong \pi_{s-p-p-2}(\mathbb{LS}(F_{(\gamma^p)})). \end{aligned}$$

Theorem 3.2. *The first differential $d_1^{p,s} : E_1^{p,s} \to E_1^{p+1,s}$ coincides with the composition*

$$\begin{aligned} LS_{s-2p-2}(F_{(\gamma^p)}) \to L_{s-2p-2}(S, (\gamma^{p+1})\alpha, u) \\ \downarrow g_{(\gamma^{p+1})} \\ L_{s-2p-2}(Q, (\gamma^{p+1})\beta, u) \to LS_{s-2p}(F_{(\gamma^{p+1})}) \end{aligned}$$

where the first map lies in diagram $(D1^{(-)^p})$, and the last one lies in diagram $(D1^{(-)^{p+1}})$.

Proof. By the definition, the differential $d_1^{p,s}$ is given by the composition

$$\begin{aligned} \pi_{s-p}(X_{p,p}, X_{p+1,p}) \xrightarrow{\partial} \pi_{s-p-1}(X_{p+1,p}, X_{p+2,p}) \\ \to \pi_{s-p-1}(X_{p+1,p+1}, X_{p+2,p+1}). \end{aligned} \qquad (3.8)$$

We can decompose the first map ∂ in the following way:

$$\pi_{s-p}(X_{p,p}, X_{p+1,p}) \to \pi_{s-p}(X_{p+1,p}) \xrightarrow{\partial_1} \pi_{s-p-1}(X_{p+1,p}, X_{p+2,p}).$$

Now, we have a commutative diagram

$$\begin{array}{ccc}
\pi_{s-p}(X_{p+1,p}) & \overset{\partial_1}{\to} & \pi_{s-p-1}(X_{p+1,p}, X_{p+2,p}) \\
\downarrow & & \downarrow \\
\pi_{s-p}(X_{p+1,p+1}) & \to & \pi_{s-p-1}(X_{p+1,p+1}, X_{p+2,p+1})
\end{array}$$

where the right vertical map coincides with the second map in (3.7). Hence we have the following decomposition of the first differential

$$\pi_{s-p}(X_{p,p}, X_{p+1,p}) \to \pi_{s-p}(X_{p+1,p}) \to \pi_{s-p}(X_{p+1,p+1}) \\ \to \pi_{s-p-1}(X_{p+1,p+1}, X_{p+2,p+1}).$$

By notations (3.5) of spectra $X_{k,l}$ and diagram (3.6) one can easily verify that this decomposition is induced by the following maps of spectra:

$$\Sigma^{p+2}\mathbb{L}\mathbb{S}(F_{(\gamma^p)}) \to \Sigma^{p+2}\mathbb{L}\mathbb{P}(F_{(\gamma^p)}) \to \Sigma^{p+2}\mathbb{L}(Q, (\gamma^{p+1})\beta, v) \\ \to \Sigma^{p}\mathbb{L}\mathbb{S}(F_{(\gamma^{p+1})}).$$

The middle map of this composition is $g_{(\gamma^{p+1})} \circ l_{(\gamma^{p+1})}$. Now, the result follows from diagrams $(D1)$, $(D1^-)$, and (3.3).

The problem of describing the first differential in algebraic terms on the level of rings with antistructures, analogously to the case of Browder-Livesay groups, remains still open. But the above-obtained description gives us good possibilities of computation.

Remark. It follows from Theorem 3.2 that the first differential coincides with the generalized Browder-Livesay invariant on the image of the map

$$LS_n(F_{(\gamma^p)}) \to L_n(Q, (\gamma^{p+1})\alpha, u).$$

Theorem 3.3. *The differential* $d_r^{p,s} : E_r^{p,s} \to E_r^{p+r,s+r-1}$ $(r \geq 2)$ *coincides with the composition*

$$
\begin{array}{l}
LS_{s-2p-2}(F_{(\gamma^p)}) \to L_{s-2p-2}(S, (\gamma^{p+1})\alpha, u) \\
\qquad\qquad\qquad\qquad \downarrow g_{(\gamma^{p+1})} \\
\qquad\qquad\qquad L_{s-2p-2}(Q, (\gamma^{p+1})\alpha, u) \\
\qquad\qquad\qquad\qquad \downarrow \Gamma \\
\qquad\qquad\qquad L_{s-2p-3}(S, (\gamma^{p+2})\alpha, u) \\
\qquad\qquad\qquad\qquad \downarrow g_{(\gamma^p)} \\
\qquad\qquad\qquad L_{s-2p-3}(Q, (\gamma^{p+2})\alpha, u) \\
\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad\qquad \vdots \\
\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad L_{s-2p-r-1}(Q, (\gamma^{p+r})\alpha, u) \to LS_{s-2p-r+1}(F_{(\gamma^{p+r})})
\end{array}
$$

where the maps Γ *denote the corresponding isomorphisms of homology groups from diagrams* $(D1)$ *and* $(D1^-)$, *and the groups* E_r *are considered as corresponding subfactor groups of the groups* LS.

Proof. If the map g is an isomorphism, then the result was obtained in [18]. An arbitrary pair of maps

$$
\begin{array}{ccc}
 & & \Sigma^r \mathbb{L}(Q, (\gamma^r)\beta, v) \\
 & \nearrow & \\
\Sigma^{r+1}\mathbb{LP}(F_{(\gamma^r)}) & & \\
 & \searrow d_\gamma & \\
 & & \Sigma^{r+1}\mathbb{L}(Q, (\gamma^{r+1})\beta, v)
\end{array} \tag{3.9}
$$

from diagram (3.4) defines a braid of exact sequences. First, it is necessary to construct a push-out square of spectra for diagram (3.9), and then write down homotopy long exact sequences of the maps in this square. Consider the following part of obtained diagrams in which the maps Υ denote the isomorphisms of homology groups in the corresponding members:

$$\begin{array}{ccccc} \to & \pi_n(\mathbb{L}(Q, \beta, \upsilon)) & (= L_n(Q, \beta, \upsilon)) & \to & \\ & \downarrow \Upsilon & & & \\ \to & \pi_n(\Sigma\mathbb{L}(Q, \gamma\beta, \upsilon)) & (= L_{n-1}(Q, \gamma\beta, \upsilon)) & \to & \end{array} \tag{3.10}$$

It follows now from [18] and diagram (3.10) that the differential $d_r^{p,s}$ coincides with the composition

$$\begin{array}{l} LS_{s-2p-2}(F_{(\gamma^p)}) \to L_{s-2p-2}(Q, (\gamma^{p+1})\alpha, u) \\ \qquad\qquad \downarrow \Upsilon \\ \qquad\qquad L_{s-2p-3}(Q, (\gamma^{p+2})\alpha, u) \\ \qquad\qquad \downarrow \\ \qquad\qquad \vdots \\ \qquad\qquad \downarrow \\ \qquad\qquad L_{s-2p-r-1}(Q, (\gamma^{p+r})\alpha, u) \to LS_{s-2p-r+1}(F_{(\gamma^{p+r})}) \end{array} \tag{3.11}$$

There exists a natural map Λ of diagram ($D1$) to the corresponding diagram (3.10). Hence we can write down the following diagram:

$$\begin{array}{ccccccc} & & L_{n+1}(Q, \beta, \upsilon) & \overset{\mathrm{Id}}{\to} & & L_{n+1}(Q, \beta, \upsilon) & \\ & \nearrow & & & \nearrow & & \\ LP_n(F) & & \downarrow \Gamma & LP_n(F) & & \downarrow \Upsilon & \\ & \searrow & & & \searrow & & \\ & & L_n(S, \gamma\alpha, u) & \overset{g_\gamma}{\to} & & L_n(Q, \gamma\beta, \upsilon) & \end{array} \tag{3.12}$$

where the map Id is the identity map. The map Λ induces homomorphisms of the corresponding homology groups, which we denote by Λ_*. Let x be an element of the group $L_{n+1}(Q, \beta, \upsilon)$ representing the class $[x]$ of the homology group. It follows now from diagram (3.12) that the class $\Upsilon \circ \Lambda_*([x])$ is represented by the element $\Upsilon \circ \mathrm{Id}(x) = \Upsilon(x) = g_\gamma\Gamma(x)$. We have a similar result for the map Γ^-, too. Now, diagram (3.11) and Theorem 3.2 imply the statement.

4. Natural Maps of Spectral Sequences

In this section, we describe some natural maps of spectral sequences which connect our surgery spectral sequence with that introduced by Hambleton and Kharshiladze in [18]. Denote by

$$\psi = \begin{pmatrix} (R, \alpha, u) & \overset{\mathrm{Id}}{\to} & (R, \alpha, u) \\ \downarrow i & & \downarrow i \\ (S, \alpha, u) & \overset{\mathrm{Id}}{\to} & (S, \alpha, u) \end{pmatrix}$$

and

$$\Phi = \begin{pmatrix} (P, \beta, \upsilon) & \overset{\mathrm{Id}}{\to} & (P, \beta, \upsilon) \\ \downarrow j & & \downarrow j \\ (Q, \beta, \upsilon) & \overset{\mathrm{Id}}{\to} & (Q, \beta, \upsilon) \end{pmatrix}$$

the geometric diagrams constructed by using geometric diagram (2.1). Then there exist natural maps of these diagrams

$$\Psi \overset{\sigma}{\to} F \overset{\varepsilon}{\to} \Phi. \tag{4.1}$$

By using the construction of Section 3, we can obtain surgery spectral sequences for every square from (4.1). In this section, we shall denote these spectral sequences as $\{{}_{\Psi}E_r^{p,q},\ {}_{\Psi}d_r\}$, $\{{}_{F}E_r^{p,q},\ {}_{F}d_r\}$, and $\{{}_{\Phi}E_r^{p,q},\ {}_{\Phi}d_r\}$, respectively. The spectral sequences $\{{}_{\Psi}E_r^{p,q},\ {}_{\Psi}d_r\}$ and $\{{}_{\Phi}E_r^{p,q},\ {}_{\Phi}d_r\}$ coincide with the restricted spectral sequence of Hambleton and Kharshiladze [18] in the case when the diagram F is the diagram of antistructures arising from the splitting problem for an one-sided submanifold.

Diagrams $(D1)$ and $(D1^-)$ are natural under morphisms of squares of antistructures (2.1), and the maps in (4.1) induce the maps of the corresponding push-out squares (3.1) and (3.2) of spectra. Hence the maps in (4.1) induce the maps of diagrams (3.4) which we can write down as the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{L}(S,\alpha,u) & \to & \mathbb{L}(Q,\beta,v) & \to & \mathbb{L}(Q,\beta,v) \\
\nearrow & & \nearrow & & \nearrow \\
\Sigma\mathbb{LP}(\Psi) & \to & \Sigma\mathbb{LP}(F) & \to & \Sigma\mathbb{LP}(\Phi) \\
\searrow & & \searrow & & \searrow \\
\Sigma\mathbb{L}(S,\gamma\alpha,u) & \to & \Sigma\mathbb{L}(Q,\gamma\beta,v) & \to & \Sigma\mathbb{L}(Q,\gamma\beta,v) \\
\nearrow & & \nearrow & & \nearrow \\
\Sigma^2\mathbb{LP}(\Psi_\gamma) & \to & \Sigma^2\mathbb{LP}(F_\gamma) & \to & \Sigma^2\mathbb{LP}(\Phi_\gamma) \\
\searrow & & \searrow & & \searrow \\
\Sigma^2\mathbb{L}(S,\alpha,u) & \to & \Sigma^2\mathbb{L}(Q,\beta,v) & \to & \Sigma^2\mathbb{L}(Q,\beta,v) \\
\cdots & & \cdots & & \cdots
\end{array} \tag{4.2}$$

Each column of diagram (4.2) gives the homotopy commutative diagram of spectra corresponding to diagram (3.6). We shall denote the spectra fitting in these diagrams by $Y_{k,l}$, $X_{k,l}$, and $Z_{k,l}$, respectively. The pull-back construction is natural. Hence we obtain the natural maps

$$Y_{k,l} \overset{\sigma_*}{\to} X_{k,l} \overset{\varepsilon_*}{\to} Z_{k,l}$$

induced by maps in (4.2), which give the natural maps of commutative diagrams (3.6). In particular, we obtain two morphisms of filtrations. In fact, the map σ induces a map of the filtration

$$\cdots \to Y_{3,0} \to Y_{2,0} \to Y_{1,0} \to Y_{0,0} \tag{4.3}$$

into the filtration

$$\cdots \to X_{3,0} \to X_{2,0} \to X_{1,0} \to X_{0,0}, \tag{4.4}$$

and the map ε induces a map of filtration (4.4) into the filtration

$$\cdots \to Z_{3,0} \to Z_{2,0} \to Z_{1,0} \to Z_{0,0}. \tag{4.5}$$

From this the following result follows.

Proposition 4.1. *The maps in* (4.1) *induce the morphisms of spectral sequences*

$$\{{}_\Psi E_r^{p,q},\ {}_\Psi d_r\} \overset{\sigma_*}{\to} \{{}_F E_r^{p,q},\ {}_F d_r\} \overset{\varepsilon}{\to} \{{}_\Phi E_r^{p,q},\ {}_\Phi d_r\}.$$

5. Morphisms of Quadratic Extensions

From an algebraic point of view the assumption that the horizontal maps in (2.1) are epimorphic is not natural. Really, it is possible to construct some spectral sequences for an arbitrary morphism of twisted quadratic extensions of antistructures. In this section, we consider square (2.1) without the condition that the horizontal maps are epimorphisms. In this case the horizontal maps give a morphism of quadratic extensions (see [25], [26], and [29]). The square $\widetilde{F}$ in (2.2) is defined, and hence we can construct the following homotopy commutative diagram of spectra

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{L}(\widetilde{R}) & \longrightarrow & \mathbb{L}(\widetilde{P}) & \longrightarrow & \mathbb{L}(\widetilde{f}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{L}(\widetilde{S}) & \longrightarrow & \mathbb{L}(\widetilde{Q}) & \longrightarrow & \mathbb{L}(\widetilde{g}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{L}(\widetilde{i}) & \longrightarrow & \mathbb{L}(\widetilde{j}) & \longrightarrow & \mathbb{L}(\widetilde{F}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{5.1}$$

The rows and columns of this diagram are cofibrations. For the spectra in (5.1) we have

$$\pi_i(\mathbb{L}(\widetilde{R})) = L_i(R, \widetilde{\alpha}, \widetilde{u}), \qquad \pi_i(\mathbb{L}(\widetilde{S})) = L_i(S, \widetilde{\alpha}, \widetilde{u}),$$

$$\pi_i(\mathbb{L}(\widetilde{P})) = L_i(P, \widetilde{\beta}, \widetilde{v}), \qquad \pi_i(\mathbb{L}(\widetilde{Q})) = L_i(Q, \widetilde{\beta}, \widetilde{v}).$$

Now, we can define a spectrum $\mathbb{L}S(F)$ as the homotopy cofiber of any of the following maps (see [29]):

$$\Omega^2\mathbb{L}(\widetilde{F}) \longrightarrow \mathbb{L}(\widetilde{R}), \ \ \Omega\mathbb{L}(\widetilde{S}) \longrightarrow \Omega\mathbb{L}(\widetilde{j}), \ \ \Omega\mathbb{L}(\widetilde{P}) \longrightarrow \Omega\mathbb{L}(\widetilde{g}) \tag{5.2}$$

which are the compositions of maps in diagram (5.1). We remark here that this definition formally coincides with that of splitting obstruction groups, but it has no relations with geometry. In a similar way (using the naturality of the transfer map), it is possible to construct the following homotopy commutative diagram of spectra (see [29]):

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{L}(S_\gamma) & \longrightarrow & \mathbb{L}(Q_\gamma) & \longrightarrow & \mathbb{L}(g_\gamma) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{L}(R) & \longrightarrow & \mathbb{L}(P) & \longrightarrow & \mathbb{L}(f) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{L}(i_\gamma^!) & \longrightarrow & \mathbb{L}(j_\gamma^!) & \longrightarrow & \mathbb{L}(F^!) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{5.3}$$

in which $i_\gamma^!$ and $j_\gamma^!$ are transfer maps. Now, the spectrum $\mathbb{L}P(F)$ is defined as the homotopy fiber of any of the following maps:

$$\Omega\mathbb{L}(f) \longrightarrow \mathbb{L}(i_\gamma^!),\ \mathbb{L}(S_\gamma) \longrightarrow \mathbb{L}(P),\ \Omega\mathbb{L}(j_\gamma^!) \longrightarrow \mathbb{L}(g_\gamma) \tag{5.4}$$

which is obtained as composition of maps in diagram (5.3).

If we begin our considerations from the square (F_γ) and then construct the square $(F_\gamma)_\gamma \cong F$, we shall obtain the spectra $\mathbb{L}P(F_\gamma)$ and $\mathbb{L}S(F_\gamma)$.

Proposition 5.1. *For the spectra considered above, pull-back diagrams* (3.1) *and* (3.2) *of spectra are also defined.*

Proof. The existence of the first diagram was proved in [35]. The result for the other diagram can be obtained in a similar way.

We can now repeat the construction of Section 3 to obtain the following:

Theorem 5.2. *For any morphism* (2.1) *of quadratic extensions of antistructures there exists a spectral sequence with*

$$E_1^{p,s} = \pi_{s-2p-2}(\mathbb{L}S(F_{(\gamma^p)})),$$

where the spectrum $\mathbb{L}S$ *is defined above. Furthermore, the differentials of this spectral sequence can be described in the same way as done in Section* 3.

Let $i : \pi \to G$ be an inclusion of index 2 between groups with orientation. If R and R' are rings with unity, then any ring homomorphism $\varphi : R \to R'$ defines a morphism of quadratic extensions

$$\begin{pmatrix} R[\pi] & \to & R'[\pi] \\ \downarrow & & \downarrow \\ R[G] & \to & R'[G] \end{pmatrix} \tag{5.5}$$

where any group ring is equipped with the standard antistructure, i.e., $u = 1$ and the anti-automorphisms coincide with the standard involutions. Hence, the spectral sequence is defined for square (5.5). It should be interesting to understand which properties of the maps φ and i can be described by this spectral sequence. In fact, a square of type (5.5) arises naturally in the computation of Wall's groups, where $R = \mathbb{Z}$ and $R' = \hat{\mathbb{Z}}_2$ (see [34] and [35]). In this case, the corresponding spectral sequence is closely related to the computation of L-groups and natural maps in L-theory.

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