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On the Equivalent Spines Problem.

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Sunto. – *Si costruiscono esempi di 3-varietà compatte, connesse ed orientabili (regolarmente immerse nella 3-sfera* S^3*) con bordo connesso di genere* ≥ 2 *che non sono omeomorfe ma ammettono la stessa spina. Questo risolve un problema posto in* [4].

1. – Introduction.

It was shown in [2], [3] and [14], using very different arguments and techniques, that there exist nonhomeomorphic compact 3-manifolds $M_i \subset S^3$, $i = 1, 2$, with boundary $\partial M_i \cong S^1 \times S^1$ such that M_i collapses onto the same spine.

The following question has been posed in [4].

QUESTION. – *Do there exist nonhomeomorphic compact connected 3-manifolds* M_i *in the 3-sphere* S^3, $i = 1, 2$, *with connected boundary* $\partial M_1 \cong \partial M_2$ *of genus* ≥ 2 *such that* M_1 *and* M_2 *have the same spine and the fundamental group* $\Pi_1(M_1) \cong \Pi_1(M_2)$ *is not a nontrivial free product?*

Note that in general one cannot get examples by simply drilling extra holes in the 3-manifolds with genus one boundary constructed in the quoted papers since the resulting spaces may become homeomorphic. Furthermore, these changes might yield manifolds whose fundamental groups are nontrivial free products so the question is open in the general case.

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Also note that the case of genus zero (suitably reformulated) is related to the unresolved status of the Poincaré conjecture (for more details see [2] and [18]).

This paper is devoted to giving an affirmative answer to the above question, and is organized as follows. In Section 2 we shall study the homeomorphism type of certain classes of compact 3-manifolds with connected boundary, called *θ-manifolds*, and answer our question. Section 3 deals with the combinatorial representation of bordered 3-manifolds by spines corresponding to finite group presentations. In Section 4 we shall construct 2-complexes that are spines of precisely two 3-manifolds of the previous type.

Concepts and notations from piecewise-linear (PL) topology are standard, and can be found for example in [20]. The prefix PL will be omitted. For a general reference on 3-manifold topology see [6], [7], [8] and [12]. All manifolds will be connected and compact. For classical knot theory we refer to [1], [11] and [19].

2. - θ-manifolds.

Let K be an oriented knot in the oriented 3-sphere S^3. Then $\overline{K}$ will denote the image of K under an orientation reversing homeomorphism of S^3. In particular, $\overline{K}$ can be obtained as the *mirror-image* of K by reflecting K across a plane. The result of reversing the orientation of K is called the *inverted knot* of K, written rK. A knot K is said to be *invertible* if $K = rK$ (here = means *equivalence* of knots by means of ambient isotopy in S^3), and *amphicheiral* if $K = \overline{K}$. Recall that the *exterior* X of K is the closure of the complement of a regular neighbourhood of K in S^3, and that K is called *simple* if X is atoroidal, i.e. any incompressible torus in X is boundary parallel.

Let $\theta(K_1, K_2, K_3)$ be the oriented θ-curve embedded in S^3 with its three arcs knotted according to the oriented knots K_1, K_2 and K_3, as shown in Figure 1.

Let $M(K_1, K_2, K_3)$ be the closure of the complement of a regular neighbourhood of this graph in S^3. Then $M(K_1, K_2, K_3)$ is an irreducible 3-manifold, with boundary of genus two, inheriting an orientation from S^3. Any such manifold will be called a *θ-manifold.*

The Torus Decomposition Theorem (which is part of the theory of characteristic varieties) for a compact irreducible 3-manifold M asserts the following (see [8], [9] and [10]). In M there exists a collection of incompressible tori that separate M into atoroidal or Seifert

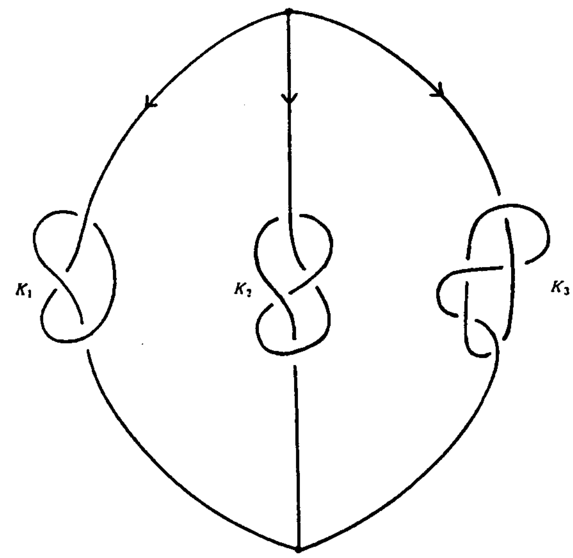

Figure 1. – The oriented θ-curve $\theta(K_1,K_2,K_3)$.

fibered components. The collection is minimal with respect to this property, and is unique up to ambient isotopy.

Suppose that K_1, K_2 and K_3 are simple oriented knots. Then the decomposing tori for the θ-manifold $M(K_1,K_2,K_3)$ are T_1, T_2 and T_3 as shown in Figure 2.

These tori separate $M(K_1,K_2,K_3)$ into four components. Three of them, X_1, X_2 and X_3 say, are copies of the exteriors of the three knots and the fourth is a genus two orientable handlebody less three standard unknotted solid tori. The components X_i, $i=1,2,3$, are given as atoroidal. The fourth component is easily seen to be atoroidal and it is not Seifert fibered (as it has a genus two boundary component).

THEOREM 2.1. – *Let K_i and K_i', $i=1,2,3$, be oriented simple knots in the oriented 3-sphere S^3 and let $M(K_1,K_2,K_3)$ and $M(K_1',K_2',K_3')$ be the corresponding θ-manifolds. Suppose that*

$$h: M(K_1,K_2,K_3) \to M(K_1',K_2',K_3')$$

is an orientation preserving homeomorphism.

Then either

$$\{K_1,K_2,K_3\} = \{K_1',K_2',K_3'\} \quad or \quad \{K_1,K_2,K_3\} = \{rK_1',rK_2',rK_3'\}.$$

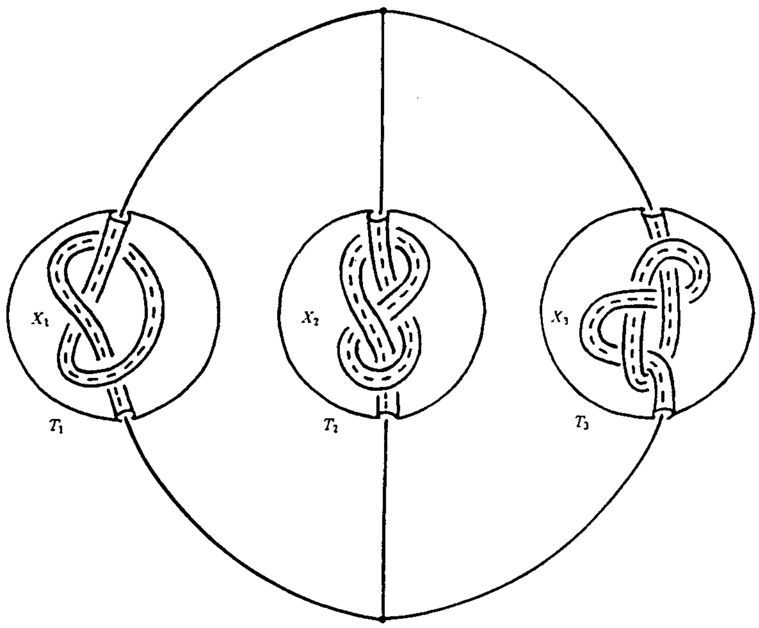

Figure 2. - The decomposing tori for the θ-manifold $M(K_1, K_2, K_3)$.

PROOF. - After an ambient isotopy (if necessary) we can assume that $h(T_i) = T_j'$, for some j, where T_1', T_2' and T_3' are the decomposing tori for the θ-manifold $M(K_1', K_2', K_3')$. This is an immediate consequence of the uniqueness part of the above-mentioned theorem. It may be also assumed that $j = i$, changing the notation if necessary. Of course, it follows that $h(X_i) = X_i'$, where X_i' is the copy of the exterior of K_i' in $M(K_1', K_2', K_3')$. Now, the result that knots are determined by their complements (see [5]) asserts that the unoriented meridian and longitude of a knot complement are well-defined. Thus h maps the meridian of X_i to that of X_i', $i = 1, 2, 3$, either preserving or reversing orientation. However the oriented meridians of X_1, X_2 and X_3 represent elements of the first integral homology group of $M(K_1, K_2, K_3)$ that add up to zero and satisfy no other relation. The meridians of X_1', X_2' and X_3' have a similar property. Thus if h reverses the direction of one meridian, then it reverses them all. As h is an orientation preserving homeomorphism, if it reverses a meridian, then it also reverses the corresponding longitude. Thus either $K_i = K_i'$ or $K_i = rK_i'$ for each $i = 1, 2, 3$. This completes the proof. ■

REMARKS. – (1) The converse of the theorem is clearly true; if the triples of knots are the same, then so are the oriented θ-manifolds.

(2) If h is permitted to be orientation reversing, then the result can be applied to $\{\overline{K}_1,\overline{K}_2,\overline{K}_3\}$. The given θ-manifolds are equivalent by an orientation reversing homeomorphism if $\{\overline{K}_1,\overline{K}_2,\overline{K}_3\}$ is equal to $\{K_1',K_2',K_3'\}$ or $\{rK_1',rK_2',rK_3'\}$.

(3) It is not necessary to insist that the knots K_i be simple. If they are not, then the decomposing tori of the components X_i must be briefly mentioned, but $h(\partial X_i) = \partial X_j$, for some j, as pieces with a genus two boundary component must correspond under h.

(4) The argument of the theorem generalizes to consideration of n knots inserted into the arcs of a graph of two vertices joined by n edges.

(5) The full power of [5] is only needed if general knots are to be considered. Specific examples (see Section 4) can be constructed using knots known, by elementary means, to be determined by their complements.

COROLLARY 2.2. – *Let K be a simple oriented knot in the oriented 3-sphere S^3. Let M_1, M_2, M_3 and M_4 be the θ-manifolds constructed as above from the following sets:*

(i) $\{K,K,K\}$,

(ii) $\{K,K,r\overline{K}\}$,

(iii) $\{K,r\overline{K},r\overline{K}\}$,

(iv) $\{r\overline{K},r\overline{K},r\overline{K}\}$.

If $K=r\overline{K}$, then these manifolds are trivially all homeomorphic. Otherwise, up to orientation preserving homeomorphism,

$$M_4 \neq M_2 \neq M_1 \neq M_3\,,$$

and $M_1 \cong M_4$ and $M_2 \cong M_3$ if and only if $K=\overline{K}$. If orientation reversing homeomorphism is also permitted, then

$$M_1 \cong M_4 \neq M_2 \cong M_3.$$

In particular, for a trefoil knot K, the θ-manifolds $M(K,K,K)$ and $M(K,K,\overline{K})$ are not homeomorphic since K is invertible but nonamphicheiral. However these manifolds have homeomorphic 2-dimensional spines as shown in Theorem 2.3 below. They are also irreducible, have incompressible boundaries and hence their fundamental groups are not nontrivial free products (see Theorem 7.1 of [6]). This answers the initial question.

THEOREM 2.3. – *Let K_1, K_2 and K_3 be oriented knots in the oriented 3-sphere* S^3. *Then the θ-manifolds* $M(K_1,K_2,K_3)$ *and* $M(K_1,K_2,r\overline{K}_3)$ *both collapse to homeomorphic copies of the same 2-dimensional polyhedron.*

PROOF. – Let U denote the unknot in S^3 and, as usual, let X_3 be the exterior of K_3. The θ-manifold $M(K_1,K_2,K_3)$ can be regarded as $M(K_1,K_2,U)\cup X_3$, with $M(K_1,K_2,U)\cap X_3$ being an annulus in the boundary of each part. However, that collapses to a copy of $M(K_1,K_2,U)\cup X_3$ with $M(K_1,K_2,U)\cap X_3$ being a simple closed curve δ, a meridian of X_3 and the core of the annulus, in the boundary of each part. Similarly, if X_3' is the exterior of $r\overline{K}_3$, then $M(K_1,K_2,r\overline{K}_3)$ collapses to a copy of $M(K_1,K_2,U)\cup X_3'$ with $M(K_1,K_2,U)\cap X_3'$ being the same simple closed curve δ in $\partial M(K_1,K_2,U)$. There is a homeomorphism $X_3\to X_3'$ that is the identity on δ; it is, essentially, reflection across the plane of δ. That extends, by the identity map, to a homeomorphism

$$M(K_1,K_2,U)\cup X_3\to M(K_1,K_2,U)\cup X_3'$$

(this referring to the copies in which the intersections are just the simple closed curves). Of course further collapsing will reduce this polyhedron to dimension two as requested. ∎

3. – Group presentations and spines.

For a standard 2-complex Σ^2, an algorithm was exhibited in [15] for recognizing Σ^2 as a spine of some closed connected orientable 3-manifold. The procedure was extended in [2] to the boundary case.

Let $\mathcal{P}$ be a group presentation with g generators $a_1,a_2,\dots,a_g$ and k relators r_1, $r_2,\dots,r_k$, where $g\geq k$. Let $\Sigma_{\mathcal{P}}$ denote the canonical 2-complex of $\mathcal{P}$, with one vertex v, g 1-cells and k 2-cells. Then $\Pi_1(\Sigma_{\mathcal{P}})$ is presented by $\mathcal{P}$.

It is well-known that $\Sigma_{\mathcal{P}}$ has a 3-dimensional thickening if and only if it is possible to draw 2-sided curves α_j on the boundary of a handlebody V with g handles (one handle for each generator a_i) such that α_j reads r_j on V (see for example [7]). Obviously, the $(k+1)$-tuple $(V;\alpha_1,\alpha_2,\dots,\alpha_k)$ is a *Heegaard diagram* of the bordered 3-manifold $M^3=M^3(\Sigma_{\mathcal{P}})$ (see [6] for the theory of Heegaard splittings).

If we cut each handle of V, then the curves α_j give rise to arcs running on a 2-sphere with $2g$ holes $D_i, \overline{D}_i$, $i=1,2,\dots,g$. Let e_i^β (resp. $\bar{e}_i^\beta$), $\beta=1,2,\dots,\gamma(i)$, denote the intersection points of ∂D_i (resp. $\partial\overline{D}_i$)—the boundary of the thickened i-th cutting disc—with the curves α_j. For each i, they are assumed to be ordered clockwise (resp. counterclockwise) according to an orientation of the 2-sphere. Identifying D_i with $\overline{D}_i$ such that each e_i^β covers $\bar{e}_i^\beta$ yields the previous Heegaard diagram of M^3.

Let E denote the set consisting of all points e_i^β and $\bar{e}_i^\beta$, and define the following three permutations A, B and C on E:

(i) A is the product of the disjoint transpositions interchanging the endpoints of the arcs arising from α_j;

(ii) B is the involutive permutation given by setting $B(e_i^\beta)=\bar{e}_i^\beta$;

(iii) $C=\prod_{i=1}^{g}(e_i^1\dots e_i^{\gamma(i)})(\bar{e}_i^{\gamma(i)}\dots\bar{e}_i^1)$.

If p_s is a permutation of E $(s=1,2,\dots,t)$, then the symbol $|p_1,p_2,\dots,p_t|$ represents the number of the orbits of the group generated by $p_1, p_2,\dots,p_t$.

The following result was proved in [15] for closed 3-manifolds and extended in [2] to the boundary case.

THEOREM 3.1. – *Let $\mathcal{P}$ be a group presentation with g generators and k relators, where $g\geqslant k$, and let $\Sigma_{\mathcal{P}}$ be the canonical 2-complex of $\mathcal{P}$. Then $\Sigma_{\mathcal{P}}$ is a spine of a compact connected orientable 3-manifold $M=M(A,B,C)$ with nonvoid boundary if and only if*

$$|A|-|C|+2=|AC|\,.$$

The number of components of ∂M equals $|AC,BC|$. If ∂M is connected, then it is the closed orientable surface of genus $g-k$.

The permutations A and B are uniquely determined by the group presentation $\mathcal{P}$. The search (possibly by computer) of all permutations C's satisfying the relations of Theorem 3.1 yields all bordered orientable connected 3-manifolds (possibly with repetitions) having the same spine $\Sigma_{\mathcal{P}}$.

4. - Equivalent spines: examples.

Let K be the trefoil knot in S^3 and let $M = M(K, K, K)$ be the corresponding θ-manifold defined as in Section 2. Applying the Wirtinger algorithm gets a presentation for $\Pi_1(M)$ with six generators and four relators (use Figure 3):

$$\Pi_1(M) \cong \langle x_i, y_i, i = 1,2,3: \quad y_i x_i y_i x_i^{-1} y_i^{-1} x_i^{-1} = 1, \quad i = 1,2,3, \\ y_1 y_2 y_3 = 1 \rangle .$$

By setting as usual $a_i = x_i y_i$ and $b_i = x_i y_i^2$, $i = 1,2,3$, we obtain the following equivalent presentation $\mathcal{P}$ for $\Pi_1(M)$:

$$\mathcal{P} = \langle a_i, b_i, i = 1,2,3: \quad a_i^3 b_i^{-2} = 1, \quad i = 1,2,3, \\ a_1^{-1} b_1 a_2^{-1} b_2 a_3^{-1} b_3 = 1 \rangle .$$

It is easily seen that $\mathcal{P}$ arises from the *RR-system* shown in Figure 4 (for the theory of *RR*-systems we refer to [16] and [17]). This confirms that the canonical 2-complex $\Sigma_{\mathcal{P}}$ corresponds to a spine of a connected orientable 3-manifold with boundary. In particular, it corresponds to a spine of the θ-manifold $M(K, K, K)$ by construction.

Now we apply the algorithm discussed in Section 3 to determine

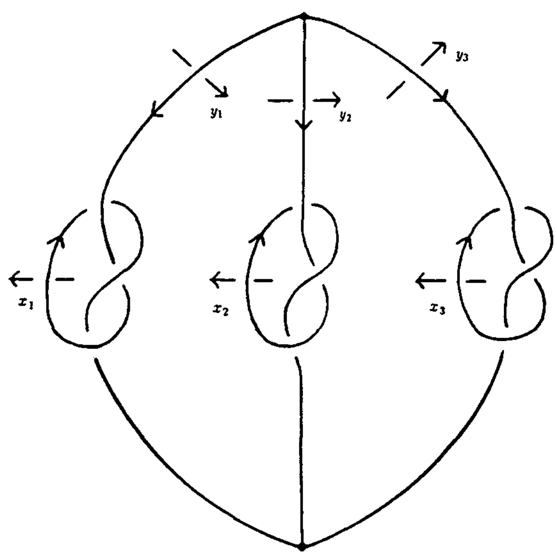

Figure 3. - The oriented θ-curve $\theta(K, K, K)$, K thefoil knot.

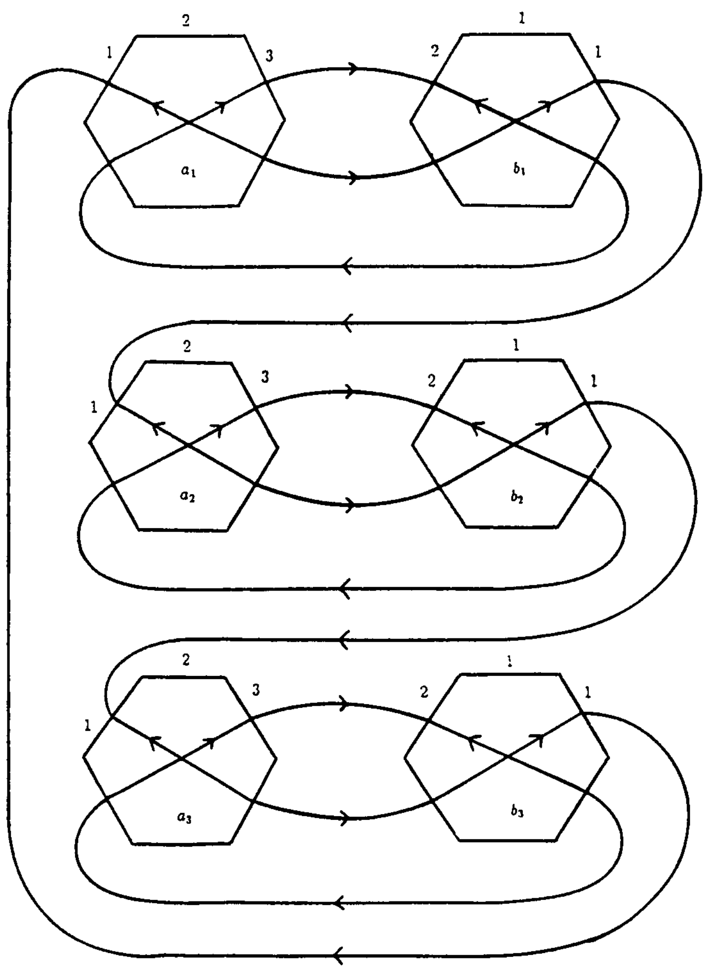

Figure 4. – An RR-system inducing the group presentation $\mathcal{P}$.

all (nonhomeomorphic) orientable prime 3-manifolds with boundary which admit $\Sigma_{\mathcal{P}}$ as spine.

THEOREM 4.1. – *Let $\mathcal{P}$ be the finite group presentation*

$$\mathcal{P} = \langle a_i, b_i, i=1,2,3: \quad a_i^3 b_i^{-2} = 1, \quad i=1,2,3, \\ a_1^{-1} b_1 a_2^{-1} b_2 a_3^{-1} b_3 = 1 \rangle,$$

and let $\Sigma_{\mathcal{P}}$ be the canonical 2-complex of $\mathcal{P}$. Then any orientable connected 3-manifold with nonvoid connected boundary having $\Sigma_{\mathcal{P}}$

as spine is homeomorphic to one of the two θ-manifolds $M(K,K,K)$ and $M(K,K,\bar{K})$, where K is the trefoil knot and $\bar{K}$ is its mirror-image.

REMARKS. – (1) One can immediately produce other examples substituting the trefoil knot by an arbitrary invertible nonamphicheiral knot.

(2) From Theorem 4.1 we also construct examples with boundary of arbitrarily large genus. It suffices to use the group presentation

$$\mathcal{P} = \langle a_i, b_i, i=1,2,\ldots,g:\quad a_i^3 b_i^{-2} = 1\,,\quad i=1,2,\ldots,g\,,\quad a_1^{-1} b_1 a_2^{-1} b_2 \ldots a_g^{-1} b_g = 1\rangle\,.$$

PROOF OF THEOREM 4.1. – Let us denote the oriented 1-cells of $\Sigma_{\mathcal{P}}$ by a_i, b_i, $i=1,2,3$, and the 2-cells of $\Sigma_{\mathcal{P}}$ by c_i, $i=1,2,3$, and d. Then ∂c_i and ∂d are attached to the wedge

$$\Sigma_{\mathcal{P}}^{(1)} = \bigvee_{i=1}^{3} a_i \vee \bigvee_{i=1}^{3} b_i$$

by the words $a_i^3 b_i^{-2}$ and $a_1^{-1} b_1 a_2^{-1} b_2 a_3^{-1} b_3$, respectively. The set E consists of 42 elements, two for each occurrence of a generator in the relators of $\mathcal{P}$. Suppose we number these elements as shown in Figure 5 so E is identified with the set

$$E \equiv \{1,2,\ldots,21,\bar{1},\bar{2},\ldots,\overline{21}\}\,.$$

Then we have

$$A = (\bar{1}\ \ 2)(\bar{2}\ \ 3)(\bar{3}\ \ \bar{5})(5\ \ \bar{6})(6\ \ 1)(13\ \ 8)(\bar{8}\ \ 9)(\bar{9}\ \ 10)(\overline{10}\ \ \overline{12})(12\ \ \overline{13})(\overline{15}\ \ 16)$$
$$(\overline{16}\ \ 17)(\overline{17}\ \ \overline{19})(19\ \ \overline{20})(20\ \ 15)(4\ \ 7)(\bar{7}\ \ \overline{11})(11\ \ 14)(\overline{14}\ \ \overline{18})(18\ \ 21)(\overline{21}\ \ \bar{4}),$$

$$B = \prod_{i=1}^{21} (i\ \ \bar{i})$$

and C acts cyclically on the orbit sets

$$\{1,2,3,4\} \subset \partial D_1, \qquad e_1 = a_1 \cap D_1,$$
$$\{\bar{1},\bar{2},\bar{3},\bar{4}\} \subset \partial \bar{D}_1, \qquad \bar{e}_1 = a_1 \cap \bar{D}_1,$$
$$\{8,9,10,11\} \subset \partial D_2, \qquad e_2 = a_2 \cap D_2,$$
$$\{\bar{8},\bar{9},\overline{10},\overline{11}\} \subset \partial \bar{D}_2, \qquad \bar{e}_2 = a_2 \cap \bar{D}_2,$$
$$\{15,16,17,18\} \subset \partial D_3, \qquad e_3 = a_3 \cap D_3,$$

$\{\overline{15},\overline{16},\overline{17},\overline{18}\} \subset \partial \overline{D}_3, \quad \bar{e}_3 = a_3 \cap \overline{D}_3,$

$\{5,6,7\} \subset \partial D_1', \quad e_1' = b_1 \cap D_1',$

$\{\bar{5},\bar{6},\bar{7}\} \subset \partial \overline{D}_1', \quad \bar{e}_1' = b_1 \cap \overline{D}_1',$

$\{12,13,14\} \subset \partial D_2', \quad e_2' = b_2 \cap D_2',$

$\{\overline{12},\overline{13},\overline{14}\} \subset \partial \overline{D}_2', \quad \bar{e}_2' = b_2 \cap \overline{D}_2',$

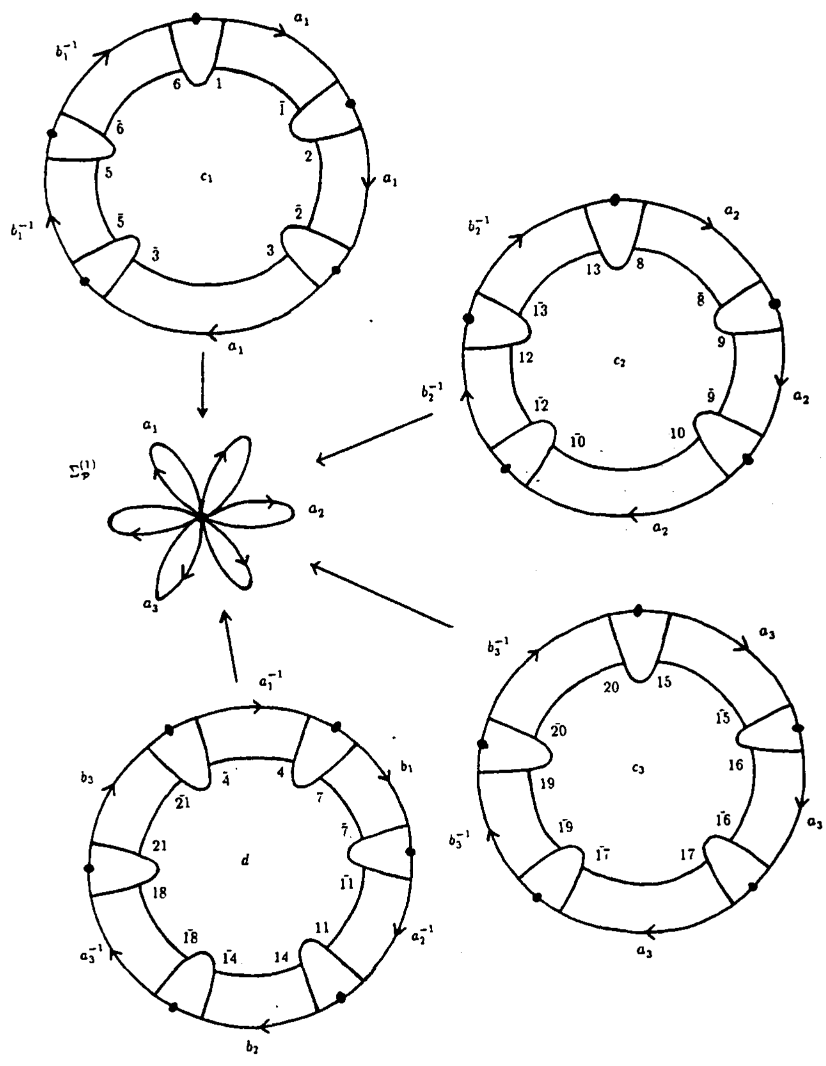

Figure 5. – The canonical 2-complex $\Sigma^2_{\mathcal{P}}$.

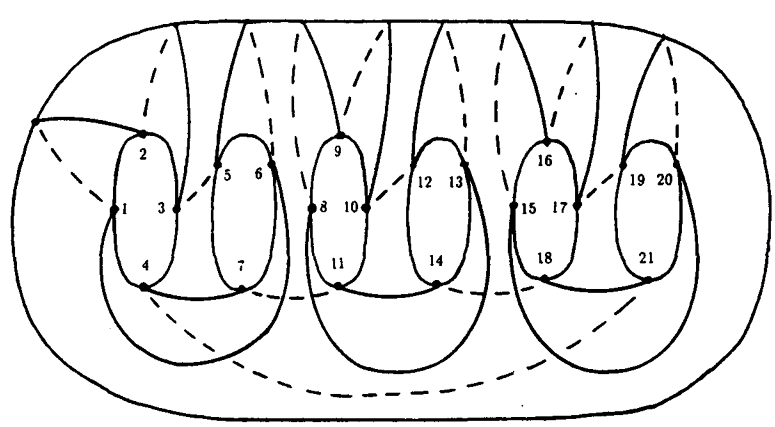

Figure 6. – An Heegaard diagram for the θ-manifold $M(K,K,K)$, K trefoil knot.

$$\{19,20,21\} \subset \partial D_3', \qquad e_3' = b_3 \cap D_3',$$

$$\{\overline{19},\overline{20},\overline{21}\} \subset \partial \overline{D}_3', \qquad \overline{e}_3' = b_3 \cap \overline{D}_3'.$$

Since the cyclic orderings induced by C on ∂D_i, $\partial D_i'$, $\partial \overline{D}_i$ and $\partial \overline{D}_i'$ must be opposite, we have exactly $6^3 \cdot 2^3 = 1728$ cases for C. By a computer program, we have verified that the permutations C's satisfying the formula

$$|A| - |C| + 2 = 21 - 12 + 2 = 11 = |AC|$$

are exactly the following four (and their inverses which yield the

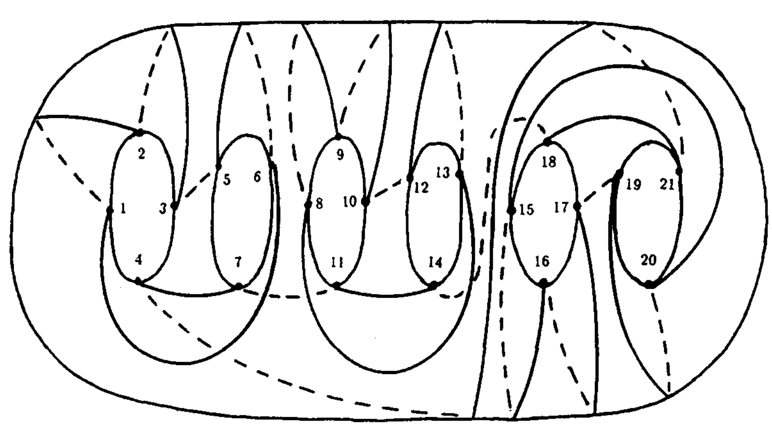

Figure 7. – An Heegaard diagram for the θ-manifold $M(K,K,\overline{K})$, K trefoil knot and $\overline{K}$ its mirror-image.

same manifolds) according to Corollary 2.2:

$$C_1 = (1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 9\ 10\ 11)(12\ 13\ 14)(15\ 16\ 17\ 18)(19\ 20\ 21)$$
$$(\bar{4}\ \bar{3}\ \bar{2}\ \bar{1})(\bar{7}\ \bar{6}\ \bar{5})(\overline{11}\ \overline{10}\ \bar{9}\ \bar{8})(\overline{14}\ \overline{13}\ \overline{12})(\overline{18}\ \overline{17}\ \overline{16}\ \overline{15})(\overline{21}\ \overline{20}\ \overline{19}),$$

$$C_2 = (1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 9\ 10\ 11)(12\ 13\ 14)(15\ 18\ 17\ 16)(19\ 21\ 20)$$
$$(\bar{4}\ \bar{3}\ \bar{2}\ \bar{1})(\bar{7}\ \bar{6}\ \bar{5})(\overline{11}\ \overline{10}\ \bar{9}\ \bar{8})(\overline{14}\ \overline{13}\ \overline{12})(\overline{16}\ \overline{17}\ \overline{18}\ \overline{15})(\overline{20}\ \overline{21}\ \overline{19}),$$

$$C_3 = (1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 11\ 10\ 9)(12\ 14\ 13)(15\ 16\ 17\ 18)(19\ 20\ 21)$$
$$(\bar{4}\ \bar{3}\ \bar{2}\ \bar{1})(\bar{7}\ \bar{6}\ \bar{5})(\bar{9}\ \overline{10}\ \overline{11}\ \bar{8})(\overline{13}\ \overline{14}\ \overline{12})(\overline{18}\ \overline{17}\ \overline{16}\ \overline{15})(\overline{21}\ \overline{20}\ \overline{19}),$$

$$C_4 = (1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 11\ 10\ 9)(12\ 14\ 13)(15\ 18\ 17\ 16)(19\ 21\ 20)$$
$$(\bar{4}\ \bar{3}\ \bar{2}\ \bar{1})(\bar{7}\ \bar{6}\ \bar{5})(\bar{9}\ \overline{10}\ \overline{11}\ \bar{8})(\overline{13}\ \overline{14}\ \overline{12})(\overline{16}\ \overline{17}\ \overline{18}\ \overline{15})(\overline{20}\ \overline{21}\ \overline{19}).$$

Hence there exist only two distinct θ-manifolds, i.e. $M(K,K,K)$ and $M(K,K,\overline{K})$, where K is the trefoil knot and $\overline{K}$ is its mirror-image (use Corollary 2.2). Now it is easily checked that A and C_i generate a transitive group and that AC_i and BC_i also generate a transitive group, so $\partial M(A,B,C_i)$ is the orientable closed connected surface of genus two. The induced Heegaard diagrams (full outside) of the θ-manifolds $M(A,B,C_1) = M(K,K,K)$ and $M(A,B,C_2) = M(K,K,\overline{K})$ are drawn in Figures 6 and 7, respectively. They are related by a *twist move* of the type considered in [12] and [13]. This completes the proof. ■

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