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On manifolds with nonhomogeneous factors

Research Article

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Abstract: We present simple examples of finite-dimensional connected homogeneous spaces (they are actually topological manifolds) with nonhomogeneous and nonrigid factors. In particular, we give an elementary solution of an old problem in general topology concerning homogeneous spaces.

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1. Introduction

A topological space X is said to be *homogeneous* if for every pair of points $x, y \in X$ there exists a homeomorphism $h\colon (X, \{x\}) \to (X, \{y\})$. This very classical topological notion became very important when in the 1960's Bing and Borsuk proved that in dimensions below 3, homogeneity can actually detect topological manifolds among all finite-dimensional absolute neighborhood retracts (ANR's).

Bing and Borsuk also conjectured that this is true in all dimensions, and this conjecture remains a formidable open problem (in dimension 3 it implies the Poincaré Conjecture). Recently, homogeneity has gained renewed attention among geometric topologists, since it turned out that the so-called Busemann G-spaces (which have also been conjectured to be topological manifolds) possess homogeneity among other key properties [9, 17].

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It was in our recent investigations of Bing–Borsuk and Busemann Conjectures that we came upon some observations on the homogeneity of products of nonhomogeneous spaces which we have collected in the present paper. In particular, we give (uncountably many) connected nonrigid finite-dimensional positive answers to the following question from [4, § 1.7] listed on p. 125: "Is there a nonhomogeneous (compact) space whose square is homogeneous?" Clearly, there is a dimensional restriction to such examples, namely $n \geq 3$.

Several positive answers to this question are already known. In 2003 a nonconnected example was given by Rosicki [26] in the realm of topological groups. Earlier, an infinite-dimensional connected rigid example was constructed by van Mill [21] in 1981, whereas in 1983 Ancel and Singh [2] constructed finite-dimensional rigid examples of X with $\dim X \geq 4$ and Ancel, Duvall and Singh [1] produced such an example also for the case $\dim X = 3$. Recall that a space is said to be *rigid* if it does not have any self-homeomorphism other than the identity.

The results in this paper provide alternative finite-dimensional answers to the question above which are easier to construct than the rigid ones, as a straightforward application of the theory of decompositions of manifolds.

In a more general setting, we say that a space X is *k-homogeneous*, $k \geq 2$, if for any given k-element sets $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$, there exists a homeomorphism $h\colon X \to X$ such that $h(x_i) = y_i$ for $i = 1, \dots, n$. The case $k = 2$ is simply referred to as *bihomogeneity*. Our results can also be used to get examples of manifold factors that fail to be bihomogeneous and ($k \geq 3$)-homogeneous since any of them implies homogeneity.

2. Wild cells of arbitrary dimensions and codimensions

In geometric topology, a *wild k-cell* in $\mathbb{R}^n$, $0 < k < n$, is a topological embedding of the unit k-ball of $\mathbb{R}^k$ which cannot be mapped onto the canonical embedding $B^k \subset \mathbb{R}^k \subset \mathbb{R}^n$ by a homeomorphism of $\mathbb{R}^n$ onto itself. First examples of wild 1-cells in $\mathbb{R}^3$ (called *wild arcs*) were constructed by Artin and Fox [15]. In fact, there are uncountably many wild arcs [16, 20, 22]. In [22] different arcs were distinguished by the fundamental groups of their complements: i.e., two arcs α, β are not equivalent if and only if $\pi_1(S^3 \setminus \alpha) \ncong \pi_1(S^3 \setminus \beta)$. Recall that two arcs are called *equivalent* if there is a self-homeomorphism of $\mathbb{R}^3$ taking one arc to the other. Notice that we can consider these arcs to be wild also in S^3.

Well-known methods based on elementary properties of the suspension of a space lead to the construction of wild cells in arbitrary dimensions. To illustrate this we shall give some details. Let $\mathcal{F}_{3,1}$ be any uncountable family of wild arcs in S^3 such that their complements in S^3 are not simply connected (for instance, the one given in [22]), and let $\alpha \in \mathcal{F}_{3,1}$. Then for each $k \geq 1$ one can construct from α a sequence of wild arcs $(\alpha_k) \subset S^{3+k}$. Indeed, if we already have a wild arc α_{k-1} in $S^{3+(k-1)}$, then by [11, Corollary 2.6.4], the sphere S^{3+k} is homeomorphic to the suspension of the quotient space $S^{3+(k-1)}/\alpha_{k-1}$.

Now, using [11, Lemma 2.7.2], let α_k be the arc in S^{3+k} that corresponds to the suspension of the class of points of α_{k-1} in the quotient $S^{3+(k-1)}/\alpha_{k-1}$. Notice that $S^{3+k} \setminus \alpha_k$ is homotopically equivalent to $S^{3+(k-1)} \setminus \alpha_{k-1}$, hence α_k is wild. So for each $n \geq 3$, there is a family $\mathcal{F}_{n,1}$ of uncountably many distinct wild arcs in S^n.

Given $n \geq 3$ and $0 < k < n$, let α be a wild arc in S^{n-k+1} from the collection $\mathcal{F}_{n-k+1,1}$. By [11, Lemma 1.4.1], the $(k-1)$-th suspension $\Sigma^{k-1}\alpha$ of α is a wild k-cell in S^n. Hence, for each $n \geq 3$ and $0 < k < n$, there is a family $\mathcal{F}_{n,k}$ of uncountably many wild k-cells embedded in S^n. Notice that the k-cells in $\mathcal{F}_{n,k}$ are cell-like non-cellular sets, for $n \geq 3$ and $0 < k < n$ (the failure of cellularity follows from [11, Exercise 2.7.4] and [27, Exercise 2.6.2 (a)]). As above, these k-cells can be taken to be embedded either in $\mathbb{R}^n$ or in S^n.

3. Products of generalized manifolds

A generalized n-manifold X is defined as a finite-dimensional Euclidean neighborhood retract (ENR) whose local $\mathbb{Z}$-homology groups agree with those of the Euclidean n-space, i.e.

$$H_*(X, X \setminus \{x\}; \mathbb{Z}) \cong H_*(\mathbb{R}^n; \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \qquad \text{for all} \quad x \in X.$$

By [24, Theorem 6] and [7], if $X^k \times Y^l$ is a homology n-manifold then X^k and Y^l are homology k- and l-manifolds, respectively, where $k + l = n$; see also [6, Theorem 2].

Also, by [19, Problem K.3, p. 30], $X \times Y$ is a metrizable ANR if and only if X and Y are metrizable ANR's. Combining both results we get the following:

Proposition 3.1.
If $X^k \times Y^l$ is a generalized $(k + l)$-manifold then X^k and Y^l are generalized k- and l-manifolds, respectively.

4. Examples

Unless otherwise stated, all manifolds in this section will be assumed to be connected.

Theorem 4.1.
There exist uncountably many distinct topological n-manifolds M^n such that $M^n = X \times Y$, where X is a nonhomogeneous nonmanifold factor, if and only if $n \geq 4$.

Proof. ($\Leftarrow$) If $n \geq 4$, then by [3, Theorem 1] it suffices to pick any wild arc $\alpha \subset \mathbb{R}^{n-1}$ such that $\mathbb{R}^{n-1} \setminus \alpha$ is not simply connected, from the family $\mathcal{F}_{n,1}$ defined in Section 2, and consider the quotient space $X^{n-1} = \mathbb{R}^{n-1}/\alpha$. Then X^{n-1} is a generalized $(n-1)$-manifold with one singular point (hence a nonhomogeneous space), since the space X^{n-1} fails to be locally Euclidean at $\pi(\alpha) \in X$, where $\pi\colon \mathbb{R}^{n-1} \to X^{n-1} = \mathbb{R}^{n-1}/\alpha$ is the quotient map. On the other hand, $X \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^n$, so X^{n-1} is an n-manifold factor. By letting α range over the uncountable family $\mathcal{F}_{n,1}$, we obtain uncountably many distinct examples. Moreover, if one wants to obtain a closed manifold M, one just observes that $(S^{n-1}/\alpha) \times \mathbb{R}$ is an n-manifold, and so $(S^{n-1}/\alpha) \times S^1$ is a closed n-manifold.

In a similar way, but in a more general setting, one can apply [8, Theorem 1.1]: Let D be any k-cell in the family $\mathcal{F}_{n,k}$ defined in Section 2, with $n \geq 4$ and $0 < k < n$. Then $(\mathbb{R}^{n-1}/D) \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^n$. Again, the space $X = \mathbb{R}^{n-1}/D$ is not a manifold since it has a unique singular point which makes the space nonhomogeneous. By varying D in $\mathcal{F}_{n,k}$ we get uncountably many distinct examples.

($\Rightarrow$) Let $n \leq 3$ and $M^n = X^k \times Y^l$ so that $k + l = n$. Then by Proposition 3.1, X^k and Y^l are generalized k- and l-manifolds, respectively. Since M is connected, we may assume that $0 < k$ and $l < 3$. Hence these low-dimensional generalized manifolds X and Y are actually genuine manifolds, see [25] or [29, Chapter IX], and are therefore also homogeneous. □

Remark 4.2.
For $n \geq 5$, more examples can be constructed. Given a topological $(n-2)$-manifold M^{n-2} choose any cell-like usc-decomposition $\mathcal{G}$ such that $X^{n-2} = M^{n-2}/\mathcal{G}$ is finite-dimensional. (This dimensionality condition is necessary due to examples of Dranishnikov [12] and Dydak–Walsh [13].) By [10, Theorem 26.8], $X^{n-2} \times \mathbb{R}^2$ has the Disjoint Disks Property and hence, by Edwards' Theorem, see e.g. [17, Theorem 2.2],

$$N^n = X^{n-2} \times \mathbb{R}^2$$

is a topological n-manifold, whereas X is nonhomogeneous if one assumes that the singular set is not dense in X, i.e. $\overline{S(X)} \neq X$. Here $S(X)$ denotes the *singular set* of X, i.e., the set of all points in X having no Euclidean neighborhood.

Theorem 4.3.
There exist uncountably many topological $2n$-dimensional manifolds M^{2n} such that $M^{2n} = X \times X$, where X is a nonhomogeneous manifold factor, if and only if $n \geq 3$.

Proof. ($\Leftarrow$) For $n \geq 3$, we apply [5, Corollary 3], see also [28], to a cell-like decomposition $\mathcal{G}$ of a manifold M of dimension $\dim M \geq 3$ in order to obtain that $(M/\mathcal{G}) \times (M/\mathcal{G})$ is homeomorphic to $M \times M$. Hence, it is enough to take $M = S^n$ and $\mathcal{G}$ to be the cell-like decomposition, whose only nondegenerate element is one of the k-cells from the family $\mathcal{F}_{n,k}$ defined in Section 2.

($\Rightarrow$) Let $n \leq 2$. Given $N^{2n} = X \times X$, then (as above) it follows by Proposition 3.1 that X is a generalized n-manifold. Therefore, for $n = 2$, X is a generalized 2-manifold and hence a surface; while for $n = 1$, X is a generalized 1-manifold and hence a circle. Recall that ($n < 3$)-dimensional homology manifolds are genuine manifolds [29, Chapter IX]. □

Remark 4.4.
In fact, Bass' result used in the proof of Theorem 4.3 shows that given two manifolds of dimensions greater than or equal to 3 and cell-like decompositions $\mathcal{G}$ and $\mathcal{G}'$ of M and N, respectively, satisfying certain mild conditions, it follows that $M \times N \cong (M/\mathcal{G}) \times (N/\mathcal{G}')$. Hence, these constructions provide affirmative answers in all dimensions greater than or equal to 6 to the second question from [4, § 1.7, p. 125]: "Can the product of two nonhomogeneous spaces be homogeneous?"

5. Epilogue

Question 5.1.
What can one say about homogeneous continua with nonhomogeneous factors in arbitrary dimensions? More explicitly, we state the following questions:

(1) Can an ($n \leq 3$)-dimensional homogeneous continuum K be written as a product $K = X \times Y$, where at least one of the factors X and Y is not homogeneous?

(2) Can an ($n \leq 5$)-dimensional homogeneous continuum K be written as a product of two nonhomogeneous factors?

(3) Can an ($n \leq 5$)-dimensional homogeneous continuum K be written as $K = X \times X$, where X is not homogeneous?

Question 5.2.
Does the Logarithmic Law hold for homogeneous compact ANR's, i.e. does the following equality hold:

$$\dim (X \times Y) = \dim X + \dim Y?$$

According to the proof sketched in [14], the so-called *Pontryagin surfaces* T_p are homogeneous. Recall that these celebrated compacta, which have the property that $\dim T_p = 2$ for all prime p, but $\dim (T_p \times T_q) = 3$, whenever $p \neq q$, show that the Logarithmic Law fails if X and Y are not ANR's. Recent work [9] was believed to lead to a positive answer to Question 5.2, see [14]. However, last year Bryant discovered a serious gap in the proof of [9, Theorem 2].

Remark 5.3.
The most famous problem still open in decomposition theory is the classical R.L. Moore Problem from the 1930's, concerning the characterization of topological n-manifolds. It asks if that every (finite-dimensional) cell-like decomposition $\mathbb{R}^n/\mathcal{G}$ of $\mathbb{R}^n$ is a topological factor of $\mathbb{R}^{n+1}$, i.e.

$$(\mathbb{R}^n/\mathcal{G}) \times \mathbb{R} \cong \mathbb{R}^{n+1}$$

(see [18] for a recent survey on this difficult problem).

In connection with the Moore Problem we mention [23, Problem 9.5], which asks if the product of a homology manifold and $\mathbb{R}$ is always homogeneous? Many examples (in particular, those in Theorems 4.1 and 4.3) give partial affirmative answers to both of these questions, but there are still far more examples to be considered.

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References

[1] Ancel F.D., Duvall P.F., Singh S., Rigid 3-dimensional compacta whose squares are manifolds, Proc. Amer. Math. Soc., 1983, 88(2), 330–332
[2] Ancel F.D., Singh S., Rigid finite-dimensional compacta whose squares are manifolds, Proc. Amer. Math. Soc., 1983, 87(2), 342–346
[3] Andrews J.J., Curtis M.L., n-space modulo an arc, Ann. of Math., 1962, 75, 1–7
[4] Arhangelskii A.V., Pearl E., Problems from A.V. Arhangelskii's *Structure and classification of topological spaces and cardinal invariants*, Topology Atlas, Problems from Topology Proceedings, 2003, 123–134, available at http://at.yorku.ca/i/a/a/z/05.htm
[5] Bass C.D., Some products of topological spaces which are manifolds, Proc. Amer. Math. Soc., 1981, 81(4), 641–646
[6] Brahana T.R., Products of generalized manifolds, Illinois J. Math., 1958, 2, 76–80
[7] Bredon G.E., Wilder manifolds are locally orientable, Proc. Nat. Acad. Sci. U.S.A., 1969, 63, 1079–1081
[8] Bryant J.L., Euclidean space modulo a cell, Fund. Math., 1968, 63, 43–51
[9] Bryant J.L., Reflections on the Bing–Borsuk conjecture, In: Abstracts of talks presented at the 19th Annual Workshop in Geometric Topology, Grand Rapids, June 13–15, 2002, 2–3, available at http://www.calvin.edu/~venema/workshop/proceedingspapers/bryant.pdf
[10] Daverman R.J., Decompositions of Manifolds, AMS Chelsea, Providence, 2007
[11] Daverman R.J., Venema G.A., Embeddings in Manifolds, Grad. Stud. Math., 106, American Mathematical Society, Providence, 2009
[12] Dranishnikov A.N., On a problem of P.S. Aleksandrov, Mat. Sb., 1988, 135(177)(4), 551–557
[13] Dydak J., Walsh J.J., Infinite-dimensional compacta having cohomological dimension two: an application of the Sullivan conjecture, Topology, 1993, 32(1), 93–104
[14] Fedorchuk V.V., On homogeneous Pontryagin surfaces, Dokl. Akad. Nauk, 2005, 404(5), 601–603
[15] Fox R.H., Artin E., Some wild cells and spheres in three-dimensional space, Ann. of Math., 1948, 49, 979–990
[16] Fox R.H., Harrold O.G., The Wilder arcs, In: Topology of 3-Manifolds and Related Topics, The Univ. of Georgia Institute, 1961, Prentice-Hall, Englewood Cliffs, 1962, 184–187
[17] Halverson D.M., Repovš D., The Bing–Borsuk and the Busemann conjectures, Math. Commun., 2008, 13(2), 163–184
[18] Halverson D.M., Repovš D., Survey on the Generalized R.L. Moore problem, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia (in press), preprint available at http://arxiv.org/abs/1201.3897
[19] Hu S., Homotopy Theory, Pure Appl. Math., 8, Academic Press, New York, 1959
[20] Lomonaco S.J., Uncountably many mildly wild non-Wilder arcs, Proc. Amer. Math. Soc., 1968, 19(4), 895–898
[21] van Mill J., A rigid space X for which $X \times X$ is homogeneous; an application of infinite-dimensional topology, Proc. Amer. Math. Soc., 1981, 83(3), 597–600
[22] Myers R., Uncountably many arcs in S^3 whose complements have non-isomorphic, indecomposable fundamental groups, J. Knot Theory Ramifications, 2000, 9(4), 505–521
[23] Quinn F., Problems on homology manifolds, In: Exotic Homology Manifolds, Oberwolfach, June 29 – July 5, 2003, Geom. Topol. Monogr., 9, Geometry & Topology Publications, Coventry, 2006, 87–103
[24] Raymond F., Separation and union theorems for generalized manifolds with boundary, Michigan Math. J., 1960, 7(1), 7–21
[25] Repovš D., Detection of higher-dimensional topological manifolds among topological spaces, In: Seminari di Geometria. Giornate di Topologia e Geometria delle Varietá, Bologna, September 27–29, 1990, Universitá degli Studi di Bologna, Dipartimento di Matematica, Bologna, 1992, 113–143

[26] Rosicki W., On the uniqueness of the decomposition of continua into Cartesian products, Bull. Polish Acad. Sci. Math., 2003, 51(3), 247–250
[27] Rushing T.B., Topological Embeddings, Pure Appl. Math., 52, Academic Press, New York–London, 1973
[28] Smith B.J., Products of decompositions of E^n, Trans. Amer. Math. Soc., 1973, 184, 31–41
[29] Wilder R.L., Topology of Manifolds, Amer. Math. Soc. Colloq. Publ., 32, American Mathematical Society, Providence, 1979