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FOUR-MANIFOLDS WITH SURFACE FUNDAMENTAL GROUPS

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Abstract. We study the homotopy type of closed connected topological 4-manifolds whose fundamental group is that of an aspherical surface F. Then we use surgery theory to show that these manifolds are s-cobordant to connected sums of simply-connected manifolds with an $\mathbb{S}^2$-bundle over F.

1. Introduction

In this paper we shall study closed connected oriented topological 4-manifolds M^4 such that $\Pi_1(M) \cong \Pi_1(F)$, where F is a closed oriented aspherical surface, i.e. $F = K(\Pi_1, 1) = B\Pi_1$. The easiest examples of such manifolds are connected sums of the type $E\#M'$, where $E \to F$ is an $\mathbb{S}^2$-bundle over F and M' is a simply-connected 4-manifold. There are reasons to conjecture that any such manifold is topologically homeomorphic to some $E\#M'$. Other natural examples of 4-manifolds with surface fundamental groups are given by certain elliptic surfaces as communicated to us by Matsumoto in [9]. Recall that a compact complex manifold of complex dimension two is said to be an *elliptic surface* if it is fibered over a Riemann surface with general fiber an elliptic curve, i.e. a 2-torus $T^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$. It may admit certain (possibly multiple) singular fibers (for details see [10]). It was proved in [10] that an elliptic surface is a 4-manifold whose fundamental group is isomorphic to that of a closed surface if it has positive Euler number and does not have multiple fibers (see [10], Remark 2, p. 563).

For simplicity, we will assume that M is a spin manifold, i.e. $w_2(M) = 0$, where w_2 denotes the second Stiefel-Whitney class. As a consequence, the sphere-bundle E will be trivial. However, a condition weaker than $w_2(M) = 0$ would suffice to prove Theorem 1.1 below; in fact, $w_2(u) = 0$ is sufficient. Here $u \in H_2(M;\mathbb{Z})$ is defined in Section 2.

The referee suggested that we treat also the case $w_2(u) \neq 0$. The proof is similar to that of Theorem 1.1, but for technical reasons we will give it in the appendix.

In Section 2 we define a map of degree 1, $\psi\colon M \to F \times \mathbb{S}^2$, which gives rise to the split exact sequence

$$0 \to K_2(\psi, \Lambda) \to H_2(M; \Lambda) \to H_2(F \times \mathbb{S}^2; \Lambda) \to 0,$$

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where $\Lambda = \mathbb{Z}[\Pi_1(M)]$ is the integral group ring.

Similarly, there is a split exact sequence

$$0 \to K_2(\psi, \mathbb{Z}) \to H_2(M; \mathbb{Z}) \to H_2(F \times \mathbb{S}^2; \mathbb{Z}) \to 0.$$

The splittings respect the intersection pairings. By the result of M. Freedman (see [4] and [5]) the induced intersection form on $K_2(\psi, \mathbb{Z})$ can be realized as intersection form of a closed simply-connected topological 4-manifold M'. Let M_1 denote the connected sum of $F \times \mathbb{S}^2$ and M', and let

$$c\colon M_1 = (F \times \mathbb{S}^2) \# M' \to F \times \mathbb{S}^2$$

be the collapsing map. Since c is of degree 1, we have short split exact sequences as above; in particular,

$$0 \to K_2(c, \Lambda) \to H_2(M_1; \Lambda) \to H_2(F \times \mathbb{S}^2; \Lambda) \to 0.$$

In Section 2 we are going to construct a map from the 3-skeleton of M_1 into M. Furthermore, we prove that it can be extended over M_1 if the Λ-interesection forms on $K_2(\psi, \Lambda)$ and on $K_2(c, \Lambda)$ coincide.

More precisely, we have

Theorem 1.1. *Let M^4 be a closed connected oriented TOP 4-manifold with $w_2(M) = 0$ and $\Pi_1(M) \cong \Pi_1(F)$, where F is a closed aspherical surface. Then M is simple homotopy equivalent to the connected sum $M_1 = (F \times \mathbb{S}^2) \# M'$ if and only if the Λ-intersection forms on $K_2(\psi, \Lambda)$ and on $K_2(c, \Lambda)$ are isomorphic.*

In particular, if $\chi(M) = 2\chi(F)$, then $K_2(\psi, \Lambda) \cong 0$, hence M is simple homotopy equivalent to $F \times \mathbb{S}^2$.

We observe that in our case any homotopy equivalence is *simple* because the Whitehead group of $\Pi_1(F)$ vanishes (see [11]). Furthermore, the manifold M' is unique, up to TOP homeomorphism, because its intersection form over $\mathbb{Z}$ must be even (see for example [5]). We also note that the second part of the statement in Theorem 1.1 gives a simple alternative proof of Theorem 3 of [6].

Using recent results of Hillman ([6], [7]) and of Cochran and Habegger ([3]), we also prove that the homotopy type classifies our manifolds, up to TOP s-cobordism.

Theorem 1.2. *With the above notation, if M is simple homotopy equivalent to $E \# M'$, then M and $E \# M'$ are topologically s-cobordant.*

The assertion was first proved for the case when M is simple homotopy equivalent to E by Hillman (see [6]). We also note that TOP s-cobordant 4-manifolds M and N are stably homeomorphic (see for example [5]), i.e. $M \# k(\mathbb{S}^2 \times \mathbb{S}^2)$ is TOP homeomorphic to $N \# \ell(\mathbb{S}^2 \times \mathbb{S}^2)$ for some integers $k, \ell \geq 0$. Thus Theorem 1.2 extends a well-known result of Wall (see [12]) to the non-simply-connected case.

In a particular case, i.e. $\Pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$, the fact that the fundamental group is elementary amenable implies that s-cobordisms are always topologically products (see [5]). Thus we have the following characterization result.

Theorem 1.3. *Let M^4 be a closed connected oriented TOP 4-manifold with $\Pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let M' be the simply-connected 4-manifold defined in the discussion preceding the statement of Theorem 1.1. Then M is TOP homeomorphic to the connected sum of M' with an $\mathbb{S}^2$-bundle over the torus if and only if the homological condition of Theorem 1.1 holds.*

If further $\chi(M) = 0$, then $K_2(\psi, \Lambda) \cong 0$, hence M is homeomorphic to an $\mathbb{S}^2$-bundle over the torus.

Although we work in the topological category, we occasionally use "transversality" and "regular values". This is possible by for example [5]. Moreover, we assume that M has a CW-structure. For a general reference on combinatorial homotopy of 4-complexes see [1]. For surgery theory we refer to [2] and [13].

2. Homotopy type

Let M^4 be a manifold with the properties described in Section 1. Since F is an aspherical closed surface, we have that $F = K(\Pi_1(F), 1) = B\Pi_1(F)$. For the proof of Theorem 1.1 it will not be important which isomorphism $\Pi_1(M) \cong \Pi_1(F)$ one chooses. This isomorphism is realized by a classifying map $f : M \to F$, i.e. f classifies the universal covering $\widetilde{M}$ of M.

Lemma 2.1. *There exists a map $j : F \to M$ such that the composition*

$$f \circ j : F \to F$$

is homotopic to the identity.

Proof. There is an embedding $j_0 : F \backslash \overset{\circ}{D}{}^2 \simeq \bigvee_{2g} \mathbb{S}^1 \to M$ such that $f \circ j_0$ is homotopic to the inclusion $F \backslash \overset{\circ}{D}{}^2 \to F$. Here g denotes the genus of F. The obstruction to extending j_0 is the homotopy class $[j_0|_{\partial D^2}] \in \Pi_1(M)$, and it is mapped to the obstruction to extending $f \circ j_0$ via the isomorphism $f_* : \Pi_1(M) \underset{\cong}{\to} \Pi_1(F)$; hence it must be zero. Therefore j_0 extends to a map $j : F \to M$. It is now easy to see that $\deg(f \circ j) = 1$; hence $f \circ j$ is homotopic to the identity map of F. □

We define two elements of $H_2(M)$, by setting $u = j_*[F]$ and $v = [F']$, where $[F] \in H_2(F)$ is the fundamental class of F, $F' = f^{-1}(x_0)$ and $x_0 \in F$ is a regular value of f.

Lemma 2.2. *The homology classes u, $v \in H_2(M)$ have the following intersection numbers:*
(1) $u \circ v = 1$*; and*
(2) $v \circ v = 0$.

Proof. (1) Let $\mathrm{PD} : H^2(M) \to H_2(M)$ denote the Poincaré duality isomorphism and let $\omega_F \in H^2(F)$ be the dual class of $[F]$. Then we have that

$$\mathrm{PD}^{-1}(v) = \mathrm{PD}^{-1}[F'] = f^*(\omega_F).$$

So we obtain that

$$\begin{aligned} u \circ v = \big(\mathrm{PD}^{-1}(u) \cup \mathrm{PD}^{-1}(v)\big) \cap [M] &= \mathrm{PD}^{-1}(v) \cap j_*[F] = f^*(\omega_F) \cap j_*[F] \\ &= j^* \circ f^*(\omega_F) \cap [F] = 1, \end{aligned}$$

since $j^* \circ f^* = (f \circ j)^* = \text{identity}$.

(2) Choosing a regular value x_0' near to x_0 yields $[f^{-1}(x_0')] = [f^{-1}(x_0)] = v$. But obviously, $f^{-1}(x_0') \cap f^{-1}(x_0)$ is empty, hence $v \circ v = 0$. □

Set $a = u \circ u$. The intersection matrix of the pair (u, v) is

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.$$

The hypothesis $w_2(M) = 0$ implies that $a \equiv 0 \pmod 2$, i.e. $a = 2k$, for some integer k. The change $u \to u - kv$ produces the intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 2.3. *There exists a map $j' : F \to M$ with the following properties:*
(1) *$f \circ j'$ is homotopic to the identity; and*
(2) *$j'_*[F] = u - kv$.*

Proof. First, we represent the homology class $v = [F']$ by an immersed 2-sphere $\varphi : \mathbb{S}^2 \to M$. We choose a collection of embedded circles in F' whose homology classes form a symplectic basis for $H_1(F')$. Then from this basis we choose a single generator for each handle of F'. Next, we note that $\Pi_1(F') \to \Pi_1(M)$ is the trivial homomorphism. Therefore, by the general position each of the chosen circles bounds a 2-disc immersed into M (see [5]). We use these immersed discs to surger F' and the result is an immersed sphere Σ^2 which represents the homology class v. Then $j(F)\#k(-\Sigma^2)$ is the image of a map $j' : F \to M$ which satisfies properties (1) and (2) of the statement. If $\varphi : \mathbb{S}^2 \to M$ represents the immersed 2-sphere $\Sigma^2 \subset M$, we have $j' = j\#k\varphi$ as required. □

Remark. Obviously, we can always assume that the map $j : F \to M$ is an immersion. Thus $\Sigma^2 \subset M$ is an algebraic dual of $j(F)$.

From now on we shall assume that $j : F \to M$ is already chosen so that it satisfies the properties of the following corollary.

Corollary 2.4. *There is a map $j : F \to M$ such that:*
(1) *$f \circ j$ is homotopic to the identity; and*
(2) *the intersection matrix of the pair $u = j_*[F]$, $v = [F']$ is*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ if } w_2(u) = 0 \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ if } w_2(u) = 1.$$

Recall that $\mathrm{PD}^{-1}(v) = f^*(\omega_F)$ and $f_*(u) = [F]$. The next goal is to construct a map $g : M \to \mathbb{S}^2$ such that $g_*(v) = [\mathbb{S}^2]$ generates $H_2(\mathbb{S}^2)$. But the property $g_*(v) = [\mathbb{S}^2]$ follows from the relation $g^*(\omega_{\mathbb{S}^2}) = \mathrm{PD}^{-1}(u)$, where $\omega_{\mathbb{S}^2} \in H^2(\mathbb{S}^2)$ is the dual of $[\mathbb{S}^2]$. This holds because

$$\begin{aligned} 1 = u \circ v = (\mathrm{PD}^{-1}(u) \cup \mathrm{PD}^{-1}(v)) \cap [M] = \mathrm{PD}^{-1}(u) \cap v = g^*(\omega_{\mathbb{S}^2}) \cap v \\ = g_*(g^*(\omega_{\mathbb{S}^2}) \cap v) = \omega_{\mathbb{S}^2} \cap g_*(v), \end{aligned}$$

i.e. $g_*(v) = [\mathbb{S}^2]$ (note that $g^*(\omega_{\mathbb{S}^2}) \cap v \in H_0(M)$ and $g_* = \mathrm{Id} : H_0(M) \to H_0(\mathbb{S}^2)$).

Lemma 2.5. *There exists a map $g : M \to \mathbb{S}^2$ such that $g^*(\omega_{\mathbb{S}^2}) = \mathrm{PD}^{-1}(u)$, where $\omega_{\mathbb{S}^2}$ is the generator of $H^2(\mathbb{S}^2)$.*

Proof. Let $g' : M \to K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ be a map which represents the cohomology class $\mathrm{PD}^{-1}(u) \in H^2(M) \cong [M, K(\mathbb{Z}, 2)]$. Since M has dimension four, we can assume $g' : M \to \mathbb{C}P^2 = \mathbb{C}P^1 \cup_\eta D^4$, where $\eta : \mathbb{S}^3 \to \mathbb{C}P^1 = \mathbb{S}^2$ is the Hopf map. Now $\mathrm{PD}^{-1}(u^2) = a\omega_M = 0$, where ω_M is the dual of the fundamental class of M. Thus g' factors over $g : M \to \mathbb{C}P^1 = \mathbb{S}^2$. □

Note that the map $\psi = f \times g : M \to F \times \mathbb{S}^2$ has degree one. We use this map to prove the following result.

Proposition 2.6. *There exists a map* $\alpha : F \times \mathbb{S}^2 \backslash \overset{\circ}{D}{}^4 \to M$ *such that* $\psi \circ \alpha$ *is homotopic to the inclusion* $F \times \mathbb{S}^2 \backslash \overset{\circ}{D}{}^4 \to F \times \mathbb{S}^2$.

Proof. Recall that we have constructed $j : F \to M$ and $\varphi : \mathbb{S}^2 \to M$, i.e. we have a map $j \vee \varphi : F \vee \mathbb{S}^2 \to M$. The first obstruction to extending $j \vee \varphi$ to $F \times \mathbb{S}^2$ lies in the cohomology group $H^3(F \times \mathbb{S}^2; \Pi_2(M))$ with local coefficients. Poincaré duality now implies that $H^3(F \times \mathbb{S}^2; \Pi_2(M)) \cong H_1(F \times \mathbb{S}^2; \Pi_2(M))$. By a result of Hillman (see [6], p. 279), one has that

$$\begin{aligned}\Pi_2(M) \cong H_2(M;\Lambda) &\cong \operatorname{Ext}^2_\Lambda(H_0(M;\Lambda),\Lambda) \oplus \operatorname{Ext}^0_\Lambda(H_2(M;\Lambda),\Lambda)\\ &\cong H^2(F) \oplus \operatorname{Ext}^0_\Lambda(H_2(M;\Lambda),\Lambda),\end{aligned}$$

where the Λ-module $Q = \operatorname{Ext}^0_\Lambda(H_2(M;\Lambda),\Lambda)$ is stably Λ-free. Here Λ is as usual the group ring $\mathbb{Z}[\Pi_1(M)]$. The fact that $\operatorname{Ker}\big(\psi_* : H_2(M;\Lambda) \to H_2(F \times \mathbb{S}^2;\Lambda)\big)$ is stably Λ-free follows from [13]. Since Q is stably Λ-free, we have

$$H_1(F \times \mathbb{S}^2; Q) \cong \operatorname{Tor}_1^\Lambda(\mathbb{Z}, Q) \cong 0.$$

Hence we obtain

$$H_1(F \times \mathbb{S}^2; \Pi_2(M)) \cong H_1(F \times \mathbb{S}^2; H^2(F)) \cong H_1(F \times \mathbb{S}^2; \mathbb{Z}),$$

i.e. $H^3(F \times \mathbb{S}^2; \Pi_2(M)) \cong H^3(F \times \mathbb{S}^2; \mathbb{Z})$. Since F is aspherical, $\Pi_2(F \times \mathbb{S}^2) \cong \mathbb{Z}$ and so the map $\psi : M \to F \times \mathbb{S}^2$ induces an isomorphism

$$\psi_* : H^3(F \times \mathbb{S}^2; \Pi_2(M)) \to H^3(F \times \mathbb{S}^2; \Pi_2(F \times \mathbb{S}^2)).$$

By naturality, the image of the obstruction under ψ_* is the obstruction to extending $\psi \circ (j \vee \varphi) : F \vee \mathbb{S}^2 \to F \times \mathbb{S}^2$. But the last obstruction vanishes as $\psi \circ (j \vee \varphi)$ is homotopic to the inclusion map (use Corollary 2.4). Therefore $j \vee \varphi$ extends to the 3-skeleton $(F \times \mathbb{S}^2)^{(3)} \simeq F \times \mathbb{S}^2 \backslash \overset{\circ}{D}{}^4$, and the extension $\alpha : F \times \mathbb{S}^2 \backslash \overset{\circ}{D}{}^4 \to M$ satisfies the property $\psi \circ \alpha \simeq$ inclusion. □

Since the map $\psi : M \to F \times \mathbb{S}^2$ has degree one, it induces a splitting of the integral intersection form $\lambda_M : H_2(M) \times H_2(M) \to \mathbb{Z}$, i.e.

$$\lambda_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \lambda'.$$

By Freedman's theorems (see [4] and [5]) we can realize λ' as the intersection form of a topological simply-connected 4-manifold M', i.e. $\lambda' = \lambda_{M'}$. Recall that $H_2(M;\Lambda) \cong H_2(F) \oplus \operatorname{Ext}^0_\Lambda(H_2(M;\Lambda),\Lambda)$, where $Q = \operatorname{Ext}^0_\Lambda(H_2(M;\Lambda),\Lambda)$ is stably Λ-free. Using the universal coefficient spectral sequence

$$\operatorname{Tor}_p^\Lambda(H_q(M;\Lambda),\mathbb{Z}) \Longrightarrow H_{p+q}(M;\mathbb{Z}),$$

we obtain that

$$\begin{aligned}H_2(M;\mathbb{Z}) &\cong \operatorname{Tor}_0^\Lambda(H_2(M;\Lambda),\mathbb{Z}) \oplus \operatorname{Tor}_2^\Lambda(H_0(M;\Lambda),\mathbb{Z})\\ &\cong H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z} \oplus H_2(\Pi_1;\mathbb{Z})\\ &\cong (H_2(F;\mathbb{Z}) \oplus Q) \otimes_\Lambda \mathbb{Z} \oplus H_2(F;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus Q \otimes_\Lambda \mathbb{Z}.\end{aligned}$$

Note that $Q \otimes_\Lambda \mathbb{Z} \cong \oplus_r \mathbb{Z}$, where $r = \operatorname{rank} Q$. In particular, we have

$$Q \otimes_\Lambda \mathbb{Z} \cong H_2(M';\mathbb{Z}),$$

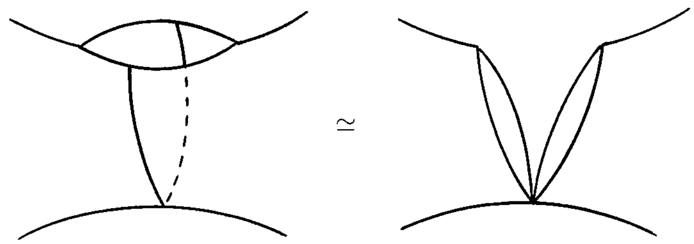

FIGURE 1

and the above decomposition of $H_2(M;\mathbb{Z})$ into a direct sum gives the splitting

$$\lambda_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \lambda_{M'}$$

of the intersection form over $\mathbb{Z}$. In summary, we have

$$\Pi_2(M) \otimes_\Lambda \mathbb{Z} \cong (\mathbb{Z} \oplus Q) \otimes_\Lambda \mathbb{Z} \cong \mathbb{Z} \oplus H_2(M'),$$

i.e. the r generators of $H_2(M')$ can be represented by maps of 2-spheres. In other words, we have a map $\beta : M' \backslash \mathring{D}^4 \simeq \bigvee_r \mathbb{S}^2 \to M$. Now we observe that $((F \times \mathbb{S}^2)\#M')\backslash \mathring{D}^4$ is homotopy equivalent to the wedge $(F \times \mathbb{S}^2 \backslash \mathring{D}^4) \vee (M' \backslash \mathring{D}^4)$, as shown in Figure 1.

Thus the map $\alpha\#\beta \simeq \alpha \vee \beta : (F \times \mathbb{S}^2 \# M')\backslash \mathring{D}^4 \to M$ induces isomorphisms on Π_1 and on $H_2(\cdot;\mathbb{Z})$. Let us denote $M_1 = F \times \mathbb{S}^2 \# M'$. The above arguments also imply that the Λ-ranks of $H_2(M;\Lambda)$ and $H_2(M_1;\Lambda)$ coincide. Next we want to extend $\alpha\#\beta : M_1 \backslash \mathring{D}^4 \to M$ to a map $M_1 \to M$. The obstruction for extending $\alpha\#\beta$ is

$$\theta = [\partial(M_1 \backslash \mathring{D}^4) \xrightarrow{\alpha\#\beta} M] \in \Pi_3(M),$$

i.e. θ is the homotopy class of the restriction of $\alpha\#\beta$ to the boundary of $M_1 \backslash \mathring{D}^4$.

Obviously, θ is the image of the generator of

$$\Pi_4(M_1, M_1 \backslash \mathring{D}^4) \cong H_4(M_1, M_1 \backslash \mathring{D}^4; \Lambda) \cong \Lambda$$

under the composition

$$\Pi_4(M_1, M_1 \backslash \mathring{D}^4) \xrightarrow{\partial_*} \Pi_3(M_1 \backslash \mathring{D}^4) \xrightarrow{(\alpha\#\beta)_*} \Pi_3(M).$$

Therefore the existence of an extension $h : M_1 \to M$ of $\alpha\#\beta$ follows from the following result.

Proposition 2.7. *With the above notation, the composition $(\alpha\#\beta)_* \circ \partial_*$ is the trivial homomorphism.*

Using this proposition, we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Since the obstruction θ is zero, there exists a map $h \colon M_1 \to M$ which extends $\alpha\#\beta$. Obviously, h induces an isomorphism on Π_1. It suffices to prove that $h_* : H_q(M_1; \Lambda) \to H_q(M; \Lambda)$ is an isomorphism for $q = 2, 3, 4$. Since

$$h_* : H_2(M_1; \mathbb{Z}) \underset{\cong}{\to} H_2(M; \mathbb{Z})$$

and $H_2(M; \mathbb{Q}) \neq 0$, the map h has degree one if one chooses the appropriate orientations. Hence $h_* : H_q(M_1; \Lambda) \to H_q(M; \Lambda)$ is onto. The kernel $K_2(h, \Lambda)$ of $h_* : H_2(M_1; \Lambda) \to H_2(M; \Lambda)$ is Λ-projective (see [13]); in fact, it is stably Λ-free. Since the Λ-ranks of $H_2(M_1; \Lambda)$ and $H_2(M; \Lambda)$ coincide, the Λ-rank of $K_2(h, \Lambda)$ is zero. Therefore $K_2(h, \Lambda) \cong 0$, by Kaplansky' s lemma (see for example [6] and [8]). By Poincaré duality we obtain isomorphisms for all q, i.e. h is a homotopy equivalence, as asserted. □

Proof of Proposition 2.7. Note first that $\alpha \# \beta : M_1 \backslash \overset{\circ}{D^4} \to M$ factors over the 3-skeleton of M, i.e.

$$\alpha \# \beta : M_1 \backslash \overset{\circ}{D^4} \to M \backslash \overset{\circ}{D^4} \subset M.$$

Here we have used the identifications $M \backslash \overset{\circ}{D^4} = M^{(3)}$ and $M_1 \backslash \overset{\circ}{D^4} = M_1^{(3)}$, where $M^{(q)}$ and $M_1^{(q)}$ denote the q-skeletons of M and M_1, respectively. We can also assume that $\alpha \# \beta$ is a cellular map. Consider the following diagram

$$(/) \qquad \begin{array}{ccccccc}
\Pi_4(M_1, M_1 \backslash \overset{\circ}{D^4}) & \xrightarrow{\partial_*} & \Pi_3(M_1 \backslash \overset{\circ}{D^4}) & = & \Pi_3(M_1 \backslash \overset{\circ}{D^4}) & \longrightarrow & \Pi_3(M_1) \\
 & & \downarrow{\scriptstyle (\alpha \# \beta)_*} & & \downarrow{\scriptstyle (\alpha \# \beta)_*} & & \downarrow{\scriptstyle \gamma} \\
\Pi_4(M, M \backslash \overset{\circ}{D^4}) & \longrightarrow & \Pi_3(M \backslash \overset{\circ}{D^4}) & \longrightarrow & \Pi_3(M) & = & \Pi_3(M).
\end{array}$$

The proof will be completed once we construct a homomorphism

$$\gamma : \Pi_3(M_1) \to \Pi_3(M)$$

such that the diagram (/) commutes. For this, we consider the Whitehead exact sequence for a 4-dimensional CW-complex X (see [1], [14] and [15]):

$$H_4(\widetilde{X}) \longrightarrow \Gamma(\Pi_2(X)) \longrightarrow \Pi_3(X) \longrightarrow H_3(\widetilde{X}) \longrightarrow 0.$$

This sequence is natural with respect to maps $X \to Y$. Here $\widetilde{X}$ is the universal covering of X, $\Pi_3(X) \to H_3(\widetilde{X})$ is the Hurewicz homomorphism and Γ denotes the quadratic functor on abelian groups. We recall that $\Gamma(\Pi_2(X))$ is equal to $\mathrm{Im}\left(\Pi_3(X^{(2)}) \to \Pi_3(X^{(3)})\right)$. In our case, we have $H_4(\widetilde{M}) \cong H_4(\widetilde{M_1}) \cong 0$ because $\Pi_1(M) \cong \Pi_1(M_1)$ is an infinite group. Moreover,

$$H_3(\widetilde{M}) \cong H_3(M; \Lambda) \cong H^1(M; \Lambda) \cong H^1(\Pi_1; \Lambda) \cong H^1(F; \Lambda) \cong H_1(F; \Lambda) \cong 0$$

as F is an aspherical surface. Similarly, $H_3(\widetilde{M_1}) \cong 0$. Hence the above sequence implies that $\Gamma(\Pi_2(M)) \underset{\cong}{\longrightarrow} \Pi_3(M)$ and $\Gamma(\Pi_2(M_1)) \underset{\cong}{\longrightarrow} \Pi_3(M_1)$. Now the map $\alpha \# \beta : M_1 \backslash \overset{\circ}{D^4} \to M$ induces $(\alpha \# \beta)_* : \Pi_2(M_1) \cong \Pi_2(M_1 \backslash \overset{\circ}{D^4}) \to \Pi_2(M)$, hence $(\alpha \# \beta)_{**} : \Gamma(\Pi_2(M_1)) \to \Gamma(\Pi_2(M))$. Then the homomorphism γ is defined by the following diagram:

$$\begin{array}{ccc} \Gamma(\Pi_2(M_1)) & \xrightarrow{(\alpha\#\beta)_{**}} & \Gamma(\Pi_2(M)) \\ {\scriptstyle \text{iso}}\downarrow & & \downarrow{\scriptstyle \text{iso}} \\ \Pi_3(M_1) & \xrightarrow[\gamma]{} & \Pi_3(M). \end{array}$$

The commutativity of (/) follows from the second interpretation of $\Gamma(\Pi_2)$ looking at the diagram shown below:

$$\begin{array}{ccccccc} \Pi_3(M_1^{(2)}) & \longrightarrow & \Pi_3(M_1\backslash \overset{\circ}{D}{}^4) & \longrightarrow & \Pi_3(M_1) & \xleftarrow{\cong} & \mathrm{Im}(\Pi_3(M_1^{(2)}) \to \Pi_3(M_1^{(3)})) \\ \downarrow{\scriptstyle (\alpha\#\beta)_*} & & \downarrow{\scriptstyle (\alpha\#\beta)_*} & & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle (\alpha\#\beta)_{**}} \\ \Pi_3(M^{(2)}) & \longrightarrow & \Pi_3(M\backslash \overset{\circ}{D}{}^4) & \longrightarrow & \Pi_3(M) & \xleftarrow[\cong]{} & \mathrm{Im}\left(\Pi_3(M^{(2)}) \to \Pi_3(M^{(3)})\right) \end{array}$$

This completes the proof. ☐

Remarks. (1) As a corollary we obtain that in the decomposition $\Pi_2(M) \cong \mathbb{Z} \oplus Q$, the Λ-module Q is actually free. This improves the result of Hillman [6].

(2) The proof of Proposition 2.7 shows that $\gamma : \Pi_3(M_1) \to \Pi_3(M)$ is an isomorphism, and hence the sequence

$$\Pi_4(M_1, M_1\backslash \overset{\circ}{D}{}^4) \xrightarrow{\partial_*} \Pi_3(M_1\backslash \overset{\circ}{D}{}^4) \xrightarrow{(\alpha\#\beta)_*} \Pi_3(M) \longrightarrow 0$$

is exact.

(3) The proof of Proposition 2.7 can be most easily seen as follows. We write the obstruction $\theta = \theta_1 + \theta_2 + \theta_3$ according to the splitting

$$\Pi_3(M) \cong \Gamma(\Pi_2(F \times \mathbb{S}^2)) \oplus \Pi_2(F \times \mathbb{S}^2) \otimes K_2(\psi, \Lambda) \oplus \Gamma(K_2(\psi, \Lambda))$$

induced by $H_2(M; \Lambda) \cong H_2(F \times \mathbb{S}^2; \Lambda) \oplus K_2(\psi, \Lambda)$.

Now $\theta_1 \in \Gamma(\Pi_2(F \times \mathbb{S}^2))$ is zero because it is the obstruction for extending $\psi \circ \alpha$, hence vanishes by Proposition 2.6.

The addendum $\theta_2 \in \Pi_2(F \times \mathbb{S}^2) \otimes K_2(\psi, \Lambda)$ is determined by intersection numbers of elements of the submodule A, generated by $\mathrm{Im}(\alpha_*) \subset H_2(M; \Lambda)$, and elements of $K_2(\psi, \Lambda)$. But they are all zero by construction.

Finally, $\theta_3 \in \Gamma(K_2(\psi, \Lambda))$ is zero by hypothesis.

3. s-cobordism type

In this section we are going to prove Theorem 1.2. In Section 2 we have constructed a simple homotopy equivalence $h : M \to F \times \mathbb{S}^2 \# M'$. To obtain Theorem 1.2, it suffices to prove the following two results.

Proposition 3.1. *The pair (M, h) is normally cobordant to a self-homotopy equivalence*

$$g : F \times \mathbb{S}^2 \# M' \to F \times \mathbb{S}^2 \# M'.$$

The following is well-known (see [6], Lemma 6, p. 282).

Proposition 3.2. *The surgery obstruction map*

$$\theta : [(F \times \mathbb{S}^2 \# M') \times I, (F \times \mathbb{S}^2 \# M') \times \partial I, G/\mathrm{TOP}] \to L_5(\Pi_1)$$

is surjective.

Now one can use the 5-dimensional surgery theory to construct an s-cobordism between M and $M_1 = F \times \mathbb{S}^2 \# M'$. In fact, let $W \to M_1 \times I$ be a normal cobordism between (M, h) and (M_1, g) guaranteed by Proposition 3.1, i.e. the normal invariants of (M, h) and (M_1, g) coincide. Using the surgery sequence (see [5] and [13]) and Proposition 3.2, it follows that M_1 and M are topologically s-cobordant. This proves Theorem 1.2.

Since Proposition 3.2 is well-known, it only remains to prove Proposition 3.1. In the case $M' \cong \mathbb{S}^4$, the result was proved by Hillman (see [7]). To prove Proposition 3.1 we use this result and "paste it together" with the corresponding result for simply-connected topological 4-manifolds (see [3]).

Let us first recall the description of normal invariants (for more details we refer to [2]). Let $\delta : M_1 = F \times \mathbb{S}^2 \# M' \to \mathrm{BTOP}$ be the classifying map of the stable normal (micro) bundle of M_1 and let $\rho : \mathrm{BTOP} \to \mathrm{BG}$ be the principal fibration with fiber G/TOP. Here BG is the classifying space of stable spherical fibrations, i.e. $\xi = \rho \circ \delta : M_1 \to \mathrm{BG}$ classifies the Spivak fibration of the Poincaré 4-complex M_1. Any normal cobordism class of normal maps $N \to M_1$ is determined by a linearization of ξ, i.e. by a lifting δ' of $\xi = \rho \circ \delta$

$$\begin{array}{ccc} M_1 & \xrightarrow{\delta'} & \mathrm{BTOP} \\ \| & & \downarrow \rho \\ M_1 & \xrightarrow[\xi]{} & \mathrm{BG} \end{array}$$

via the Thom construction. This means, fixing the lifting δ', that the normal cobordism classes of normal maps correspond uniquely to the elements of $[M_1, G/\mathrm{TOP}]$, i.e. $\delta'(x) = g(x)\delta(x)$ with $g : M_1 \to G/\mathrm{TOP}$. Let $\Sigma^3 \subset M_1 = F \times \mathbb{S}^2 \# M'$ be the 3-sphere along which the manifolds $F \times \mathbb{S}^2$ and M' are glued together. Then $[g|_{\Sigma^3}] \in \Pi_3(G/\mathrm{TOP}) = 0$. Consequently, $g|_{F \times \mathbb{S}^2 \setminus \mathring{D}^4}$ and $g|_{M' \setminus \mathring{D}^4}$ extend to maps $g_1 : F \times \mathbb{S}^2 \to G/\mathrm{TOP}$ and $g_2 : M' \to G/\mathrm{TOP}$, respectively. The values of g_1 and g_2 coincide on the 4-ball D^4. Two extensions of $g|_{\Sigma^3}$ over the 4-ball D^4 differ by an element of $\Pi_4(G/\mathrm{TOP}) \cong \mathbb{Z}$. We use the unique extension of $g|_{\Sigma^3}$ such that the surgery obstruction of g_2 is zero. In other words, we have constructed a map

$$\mu : [F \times \mathbb{S}^2 \# M', G/\mathrm{TOP}] \to [F \times \mathbb{S}^2, G/\mathrm{TOP}] \oplus [M', G/\mathrm{TOP}]$$

which sends $[g]$ into $([g_1], [g_2])$.

On the other hand, attaching a 4-ball D^4 to Σ^3 yields a map

$$t : F \times \mathbb{S}^2 \# M' \to F \times \mathbb{S}^2 \# M' \cup_{\Sigma^3} D^4 \simeq F \times \mathbb{S}^2 \vee M'$$

which induces

$$\begin{aligned} t_* : [F \times \mathbb{S}^2 \vee M', G/\mathrm{TOP}] &\cong [F \times \mathbb{S}^2, G/\mathrm{TOP}] \oplus [M', G/\mathrm{TOP}] \\ &\to [F \times \mathbb{S}^2 \# M', G/\mathrm{TOP}]. \end{aligned}$$

Now it is very easy to see that $t_* \circ \mu$ is the identity, hence t_* is surjective. On the other hand, the connected sum with (M', g_2) gives the following commutative diagram

$$(//)\qquad \begin{array}{ccc} [F\times \mathbb{S}^2, G/\operatorname{TOP}] & \xrightarrow{\theta_1} & L_4(\Pi_1) \\ {\scriptstyle \#(M',g_2)}\big\downarrow & & \big\| \\ [F\times \mathbb{S}^2\# M', G/\operatorname{TOP}] & \xrightarrow[\theta]{} & L_4(\Pi_1). \end{array}$$

The map induced on $L_4(\Pi_1)$ is the identity because the surgery obstruction of (M', g_2) is zero. If $g : F \times \mathbb{S}^2\#M' \to G/\operatorname{TOP}$ is the normal invariant of a given (simple) homotopy equivalence $h : M \to F \times \mathbb{S}^2\#M'$ and $\mu([g]) = ([g_1], [g_2])$, then $\theta_1(g_1) = 0$. This follows from the diagram (//) and the fact that $\theta_2(g_2) = 0$, where $\theta_2 : [M', G/\operatorname{TOP}] \to L_4(1)$.

In summary, we have proved the following result.

Proposition 3.3. *Any element* $[g] \in [F \times \mathbb{S}^2\#M', G/\operatorname{TOP}]$*, coming from a (simple) homotopy equivalence* $h : M \to F \times \mathbb{S}^2\#M'$*, belongs to* $\operatorname{Im} t_*$*.*

More precisely, there are elements

$$[g_1] \in \operatorname{Ker}(\theta_1 : [F \times \mathbb{S}^2, G/\operatorname{TOP}] \to L_4(\Pi_1)),$$
$$[g_2] \in \operatorname{Ker}(\theta_2 : [M', G/\operatorname{TOP}] \to L_4(1))$$

such that $t_*([g_1], [g_2]) = [g]$*.*

To finish the proof of Proposition 3.1 we recall that the elements of $\operatorname{Ker}(\theta_1)$ and $\operatorname{Ker}(\theta_2)$ come from elements of $\mathrm{HE}_{\mathrm{Id}}(F \times \mathbb{S}^2)$ and $\mathrm{HE}_{\mathrm{Id}}(M')$, respectively (see [3] and [7]). Here $\mathrm{HE}_{\mathrm{Id}}$ denotes the set of homotopy classes of simple self-homotopy equivalences inducing the identities on Π_1 and on H_*. More precisely, the proofs of the results of the quoted papers show that there are representatives in $\mathrm{HE}_{\mathrm{Id}}$ leaving a 4-ball fixed. Therefore, if $h_1 : F \times \mathbb{S}^2 \to F \times \mathbb{S}^2$ and $h_2 : M' \to M'$ are such representatives of g_1 and g_2, then $h_i|_{D^4_i} =$ identity for $i = 1, 2$. Thus we can form the map $h_1\#h_2 : M_1 \to M_1$. Obviously, h and $h_1\#h_2$ have the same normal invariants. This proves Proposition 3.1.

In this section we did not use the hypothesis that $w_2(M) = 0$. In fact, our arguments prove the following more general result.

Theorem 3.4. *Let* M^4 *be a closed connected oriented (TOP)* 4*-manifold homotopy equivalent to* $E\#M'$*, where* E *is an* $\mathbb{S}^2$*-bundle over a closed oriented aspherical surface* F *and* M' *is a simply-connected* 4*-manifold. Then* M *is topologically s-cobordant to* $E\#M'$*.*

4. Appendix

As announced in the introduction, here we will treat the case $w_2(u) \neq 0$. First recall that there is only one twisted $\mathbb{S}^2$-bundle over an oriented closed surface F, denoted by $F \underset{\sim}{\times} \mathbb{S}^2$, because these bundles are determined by the first and second Stiefel-Whitney classes. It can be obtained from $(F\backslash \overset{\circ}{D}{}^2) \times \mathbb{S}^2$ by attaching $D^2 \times \mathbb{S}^2$ with a map $\alpha : \partial D^2 \times \mathbb{S}^2 \to \partial D^2 \times \mathbb{S}^2$ associated to the generator of $\Pi_1(\mathrm{SO}(3))$. The intersection matrix of $F \underset{\sim}{\times} \mathbb{S}^2$ is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to x, $y \in H_2(F \underset{\sim}{\times} \mathbb{S}^2)$, where y represents the fiber and $\pi_*(x) = [F]$, $\pi : F \underset{\sim}{\times} \mathbb{S}^2 \to F$ being the fiber projection.

Proposition 4.1. *Let M^4 be a closed connected oriented 4-manifold with $\Pi_1(M) \cong \Pi_1(F)$. Assume that $w_2(u) \neq 0$ (notation as in Section 2). Then there is a map $\phi : M \to F \underset{\sim}{\times} \mathbb{S}^2$ of degree 1.*

Proof. Let $f : M \to F$ and $g' : M \to \mathbb{C}P^\infty$ be as in the proof of Lemma 2.5. Then the restriction

$$f \times g'|_{M \setminus \overset{\circ}{D^4}} : M \setminus \overset{\circ}{D^4} \to F \times \mathbb{C}P^\infty$$

factors as follows:

$$\begin{array}{ccc} M\setminus \overset{\circ}{D^4} & \xrightarrow{f\times g'|_{M\setminus \overset{\circ}{D^4}}} & F \times \mathbb{C}P^\infty \\ \phi' \downarrow & & \uparrow \\ [F \times \mathbb{C}P^\infty]^{(3)} & = & [F \times \mathbb{C}P^\infty]^{(3)}. \end{array}$$

But note that $[F \times \mathbb{C}P^\infty]^{(3)} = F \times \mathbb{S}^2 \setminus \overset{\circ}{B^4}$, B^4 being a 4-ball. Hence we have a map

$$\phi' : M \setminus \overset{\circ}{D^4} \to F \times \mathbb{S}^2 \setminus \overset{\circ}{B^4}.$$

Obviously ϕ' extends to $\phi : M \to (F \times \mathbb{S}^2 \setminus \overset{\circ}{B^4}) \cup_\lambda D^4$, where $\lambda = \phi'|_{\partial D^4}$. Therefore it remains to show that $(F \times \mathbb{S}^2 \setminus \overset{\circ}{B^4}) \cup_\lambda D^4$ is homotopy equivalent to $F \underset{\sim}{\times} \mathbb{S}^2$. If F were $\mathbb{S}^2$, then $(F \times \mathbb{S}^2 \setminus \overset{\circ}{B^4}) \cup_\lambda D^4$ is a Poincaré complex with intersection matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is homotopy equivalent to $\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^2$. The general case can be reduced to that of a sphere $\mathbb{S}^2$ by considering the collapsing map

$$c : F \underset{\sim}{\times} \mathbb{S}^2 \to (F/(F \setminus \overset{\circ}{D^2_1})) \underset{\sim}{\times} \mathbb{S}^2 \cong \mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^2.$$

Here $D^2_1 \subset F$ is a 2-disc which contains in its interior the 2-disc $D^2 \subset F$ used at the beginning of the section to describe

$$F \underset{\sim}{\times} \mathbb{S}^2 \cong ((F \setminus \overset{\circ}{D^2}) \times \mathbb{S}^2) \cup_\alpha (D^2 \times \mathbb{S}^2).$$

The general case will follow from the fact that ϕ' can be homotoped in a collar of the boundary of $M \setminus \overset{\circ}{D^4}$ such that $\phi'(\partial D^4) \subset \overset{\circ}{D^2_1} \times \mathbb{S}^2$. To extend $\phi'|_{\partial D^4}$ over D^4 we need to reglue $\overset{\circ}{D^2} \times \mathbb{S}^2 \subset D^2_1 \times \mathbb{S}^2$ by the twist $\alpha : \partial D^2 \times \mathbb{S}^2 \to \partial D^2 \times \mathbb{S}^2$, i.e. we have to form

$$F \underset{\sim}{\times} \mathbb{S}^2 \cong ((F \setminus \overset{\circ}{D^2}) \times \mathbb{S}^2) \cup_\alpha (D^2 \times \mathbb{S}^2).$$

To see that we can assume $\phi'(\partial D^4) \subset \overset{\circ}{D}{}_1^2 \times \mathbb{S}^2$ we consider the short exact homotopy sequence (recall that F is now aspherical, so $\Pi_3(F \times \mathbb{S}^2) \cong \mathbb{Z}$):

$$0 \to \Pi_4(F \times \mathbb{S}^2, F \times \mathbb{S}^2 \backslash \overset{\circ}{B}{}^4) \cong \Lambda \to \Pi_3(F \times \mathbb{S}^2 \backslash \overset{\circ}{B}{}^4) \to \Pi_3(F \times \mathbb{S}^2) \cong \mathbb{Z} \to 0.$$

This sequence splits because

$$\mathrm{Ext}^1_\Lambda(\mathbb{Z}, \Lambda) \cong H^1(F; \Lambda) \cong H_1(F; \Lambda) \cong 0.$$

Then we have $[\lambda] = [\lambda_1] + [\lambda_2] \in \Lambda \oplus \mathbb{Z}$. Therefore $[\lambda_2] = k[\eta]$, where $k \in \mathbb{Z}$, $\eta : \mathbb{S}^3 \to \{*\} \times \mathbb{S}^2$ is the Hopf map and $* \in \overset{\circ}{D}{}_1^2$. It follows that $\lambda_2(\partial \overset{\circ}{D}{}^4) \subset \overset{\circ}{D}{}_1^2 \times \mathbb{S}^2$. On the other hand we choose $B^4 = D^2 \times D^2_- \subset F \times \mathbb{S}^2$, where D^2_- is the lower hemisphere. Hence a generator τ of $\Lambda \subset \Pi_3(F \times \mathbb{S}^2 \backslash \overset{\circ}{B}{}^4)$ has image in $D_1^2 \times \mathbb{S}^2$. Since $[\lambda_1] = a\tau$, where $a \in \Lambda$, the image of λ_1 belongs to $D_1^2 \times \mathbb{S}^2$, up to some arcs running through $(F \backslash \overset{\circ}{D}{}^2) \times \mathbb{S}^2$. This completes the proof. □

Since the other arguments are the same as in the case $w_2(u) = 0$, we have completed Therorem 1.1 with the following result involving twisted $\mathbb{S}^2$-bundles over aspherical surfaces.

Theorem 1.1′. *Let M^4 be a closed connected oriented 4-manifold with $\Pi_1(M) \cong \Pi_1(F)$. Assume that $w_2(u) \neq 0$ (notation as in Section 2). Then M is simple homotopy equivalent to the connected sum $M_1 = (F \underset{\sim}{\times} \mathbb{S}^2) \# M'$, where M' is the simply-connected 4-manifold defined in the discussion preceding the statement of Theorem 1.1, if and only if the Λ-intersection forms on $K_2(\phi, \Lambda)$ and on $K_2(c', \Lambda)$ are isomorphic, where c' denotes the collapsing map from M_1 to $F \underset{\sim}{\times} \mathbb{S}^2$. Moreover, the manifolds M and $(F \underset{\sim}{\times} \mathbb{S}^2) \# M'$ are topologically s-cobordant.*

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