ON THE CONSTRUCTION OF $4k$–DIMENSIONAL GENERALIZED MANIFOLDS

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ABSTRACT. We construct $4k$–dimensional generalized manifolds, $k > 1$, which have no resolutions. The construction proceeds as in a paper of Bryant, Ferry, Mio and Weinberger (see [1]) but does not use their controlled (ϵ, δ)–surgery sequence. The controlled surgery sequence is believed to be true. Recently, Pedersen, Quinn and Ranicki have given a proof of this sequence in the case of trivial local fundamental groups (see [4]).

1. Exposition of the construction.

Generalized manifolds have been the first time systematically constructed in [1]. Beginning with a simply connected n–dimensional manifold M^n, with $n \geq 5$, Bryant, Ferry, Mio and Weinberger constructed a sequence of

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Poincaré duality complexes $\{X_i\}$, $i = 0, 1, 2, \dots$, and maps $p_i : X_i \to X_{i-1}$, where $X_{-1} = M$, which satisfy the following conditions:

(1) all maps p_i are UV^1;

(2) X_i is an η_i–Poincaré complex of dimension n over X_{i-1};

(3) for any $i \geq 1$, the map $p_i : X_i \to X_{i-1}$ is a ζ_i–homotopy equivalence over X_{i-2};

(4) there is a regular neighbourhood W_0 of X_0 embedded in a sufficiently large Euclidean space $\mathbb{R}^L$ and there are embeddings $X_i \to W_0$ and retractions

$$r_i : W_0 \to X_i$$

satisfying $d(r_i, r_{i-1}) < \zeta_i$, for any $i \geq 1$.

Here d is the metric on W_0 induced from $\mathbb{R}^L$. Moreover, the sequences of positive numbers $\{\eta_i\}$ and $\{\zeta_i\}$ are given and subject to the conditions

(/) $\sum_i \eta_i < \infty$

(//) (ζ_i, h)–cobordisms over X_{i-1} of dimension L admit δ_i–product structures; such ζ_i exist by the thin h–cobordism theorem of Quinn (see [5], Theorem 2.7). Moreover, we require $\sum_i \delta_i < \infty$. Since we also assume $\zeta_i < \delta_i$, we have $\sum_i \zeta_i < \infty$.

A construction of the spaces X_i is indicated at the end of this section.

We can choose small regular neighbourhoods W_i of X_i in W_0 with projection maps $\pi_i : W_i \to X_i$ such that $W_{i+1} \subset \operatorname{int} W_i$ for all $i = 0, 1, \dots$. Moreover, the choice can be made so that $W_i \setminus \operatorname{int} W_{i+1}$ is a (ζ_{i+1}, h)–cobordism with respect to the restriction of $r_{i+1} : W_0 \to X_{i+1}$.

We define $X := \cap_{i=1}^{\infty} W_i$ and show that it is a generalized manifold. For any $x \in W_0$, $r(x) = \lim_{i \to \infty} r_i(x)$ is well–defined by the properties $d(r_i, r_{i+1}) < \zeta_i$ and $\sum_i \zeta_i < \infty$. Obviously, we have

$$\lim_{i \to \infty} r_i(x) = r(x) \in X.$$

We observe that X can be defined as the inverse limit of the complexes $\{X_i\}$, that is, $X = \varprojlim_i X_i$, since the W_i' s become smaller and smaller regular neighbourhoods as i goes to infinity. In particular, we have $r(x) = x$ for any $x \in X$, i.e., X is an ANR–space. The proof that X is a homology manifold relies to the following result due to Daverman and Hush (see [3]).

Theorem 1.1. *Let $p : M \to B$ be a proper map which is an approximate fibration of the connected m–manifold (without boundary) M onto an ANR–space B. Then B is a k–dimensional generalized manifold. Moreover, if M is orientable, then the fiber of p has the shape of a Poincaré duality space of formal dimension $m - k$.*

To apply this criterion to our case we also need Proposition 4.5 of [1] (see Proposition 3.6 in Section 3 below). We define a retraction $\rho_i : W_0 \to X_i$ by composing $\pi_i : W_i \to X_i$ with the deformation given by the thin h–cobordisms

$$W_0 \backslash \operatorname{int} W_i = (W_0 \backslash \operatorname{int} W_1) \cup (W_1 \backslash \operatorname{int} W_2) \cup \cdots \cup (W_{i-1} \backslash \operatorname{int} W_i)$$

to ∂W_i. We can form the limit as $i \to \infty$ to get a new retraction (see Remark 1.1 below) $\rho : W_0 \to X$. It follows from Proposition 3.6 in Section 3 that given $\delta > 0$, then for sufficiently large i, the restriction

$$\pi_i|_{\partial W_i} : \partial W_i \to X_i$$

has the δ–lifting property (because X_i has an η_i–Poincaré structure with η_i very small as i becomes large). The composed h–cobordisms give a homeomorphism $\partial W_0 \cong \partial W_i$, hence $\rho_i|_{\partial W_0} : \partial W_0 \to X_i$ has the δ–lifting property, too. It follows that in the limit $i \to \infty$ one can obtain a δ–approximative fibration $\rho : \partial W_0 \to X$ for any $\delta > 0$, i.e., an approximative fibration. Thus X is a homology manifold.

Remark 1.1. The δ_i–thin h–cobordisms $W_i \backslash \operatorname{int} W_i$ are needed to construct the limit of the maps ρ_i, i.e., $\rho = \lim_{i \to \infty} \rho_i : \partial W_0 \to X$. We have homeomorphisms $h_i : \partial W_i \times [\tau_i, \tau_{i+1}] \to W_i \backslash \operatorname{int} W_i$ such that the diameter of the set

$$\{\pi_i \circ h_i(x, t) : t \in [\tau_i, \tau_{i+1}]\}$$

is less than δ_i. For any $x \in \partial W_0$, we follow these lines beginning with $W_0 \backslash \operatorname{int} W_1$ by using h_0, then with $W_1 \backslash \operatorname{int} W_2$ by using h_1, and so on. This gives a curve beginning in x and converging to $\rho(x) \in \cap_{i=1}^{\infty} W_i$. This map is continuous. Recall that ∂W_0, W_0 and X are included in $\mathbb{R}^L$. Given $\epsilon > 0$, we choose a sufficiently large number i so that $\sum_{j=0}^{\infty} \delta_{i+j} < \epsilon/4$. The first $(i+1)$–product structures of $W_0 \backslash \operatorname{int} W_1, \ldots, W_{i-1} \backslash \operatorname{int} W_i$ define a continuous map $\theta_i : \partial W_0 \to \partial W_i$ (in fact, a homeomorphism). The map ρ is the composition of θ_i with a map $\theta_i' : \partial W_i \to X$ defined by the product structures of $W_i \backslash \operatorname{int} W_{i+1}$, $W_{i+1} \backslash \operatorname{int} W_{i+2}$, ..., which are δ_k controlled with $k = i, i+1, \ldots$. Hence, if x', $y' \in \partial W_i$ and $\| x' - y' \| < \alpha$, then

$$\| \theta'(x') - \theta'(y') \| < \alpha + 2 \sum_{j=0}^{\infty} \delta_{i+j} < \alpha + \frac{\epsilon}{2}.$$

Now we choose $\delta > 0$ so that for any x, $y \in \partial W_0$ and $\| x - y \| < \delta$ implies $\| \theta_i(x) - \theta_i(y) \| < \epsilon/2$. Then we have

$$\| \rho(x) - \rho(y) \| = \| \theta_i' \circ \theta_i(x) - \theta_i' \circ \theta_i(y) \| = \| \theta_i'(\theta_i(x) - \theta_i(y)) \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that ρ is continuous. Note that the above construction defines a map $\rho : W_0 \to X$ which is a (deformation) retraction.

Our construction of the η_i–Poincaré complexes begins with an element $\sigma \in H_{4k}(M, \mathbb{L})$. If we have chosen an appropriate σ, then our resulting generalized manifold X has no resolution.

It follows a brief description of the spaces X_i and maps $p_i : X_i \to X_{i-1}$. More details are given in Section 4. Let $n = 4k$, and let $\sigma \in H_n(M, \mathbb{L})$ be given. We decompose $M = B \cup_D C$, where B is a regular neighbourhood of the 2–skeleton of M, C is the closure of its complement and $D = \partial B = \partial C$ is its boundary. Using results from Sections 2 and 3 we realize σ by a degree 1 normal map $F_\sigma : V \to D \times I$ with $F_\sigma|_{\partial_0 V} = \mathrm{Id} : \partial_0 V = D \to D$, and $f_\sigma = F_\sigma|_{\partial_1 V} : \partial_1 V = D' \to D$ a controlled homotopy equivalence over M. Then we put $X_0 = B \cup_D V \cup_{f_\sigma} C$. There is an obvious map $p : X_0 \to M$ of degree 1. Next we build the manifold $M_0 = B \cup_D V \cup_{D'} (-V) \cup_D C$. There is an obvious map $g_0 : M_0 \to X_0$. We decompose $M_0 = B_1 \cup_{D_1} C_1$ in the same way as M and realize σ by $F_{1,\sigma} : V_1 \to D_1 \times I$ with $F_{1,\sigma}|_{\partial_0 V_1} = \mathrm{Id} : \partial_0 V_1 = D_1 \to D_1$ and $f_{1,\sigma} = F_{1,\sigma}|_{\partial_1 V_1} : \partial_1 V_1 = D_1' \to D_1$ a controlled homotopy equivalence over M. Then we consider $X_1' = B_1 \cup_{D_1} V_1 \cup_{f_{1,\sigma}} C_1$. There is an obvious map $f_1' : X_1' \to X_0$. The composition $g_0 \circ f_1' : X_1' \to X_0$ is a degree 1 normal map with vanishing controlled surgery obstruction. One can do surgery outside the singular set to get a controlled homotopy equivalence $p_1 : X_1 \to X_0$ over M. Note that X_0 and X_1 are not homotopy equivalent to M. Nevertheless, the composition $p \circ g_0 : M_0 \to X_0 \to M$ is of degree 1. Using $\mathbb{L}$–Poincaré duality for the manifolds M_0 and M (see [7]), it follows that there is an element $\overline{\sigma} \in H_n(X_0, \mathbb{L})$ with $p_*(\overline{\sigma}) = \sigma$. Let now $\overline{\overline{\sigma}} \in H_n(X_1, \mathbb{L})$ be such that $p_{1*}(\overline{\overline{\sigma}}) = \overline{\sigma}$. Taking a degree 1 normal map $g_1 : M_1 \to X_1$ with controlled surgery obstruction $-\overline{\overline{\sigma}}$ we can proceed to construct a map $p_2 : X_2 \to X_1$ which is a controlled UV^1–homotopy equivalence over X_0, and so on.

Remark 1.2. The proof given in [1] relies very much on their (ϵ, δ)–surgery sequence displayed in Theorem 2.4 (to be more precise on their Theorem 2.8: see for instance the conclusion on p.454). By the time of the ICTP–Conference, the proofs of these theorems were not yet published, but Pedersen, Quinn and Ranicki announced a proof of the controlled surgery sequence which is now available (see [4]). We will work instead with non–singular associated even symmetric bilinear forms over compact

ANR–spaces introduced by Quinn in [6]. We use the theorems of this paper for our construction. Therefore, our construction is restricted to the $4k$–dimensional simply connected case with $k > 1$. Moreover, our proof uses some results proved in Section 4 of [1]. We recall these in Section 3. The results of Quinn are summarized in Section 2.

2. A review on Quinn's results.

In this section all manifolds and Poincaré complexes will have dimension $4k$, for $k > 1$. As announced in Section 1, we shall restate here the main results of [6] for control maps over compact metric ANR–spaces X. Suppose that K is a Poincaré complex and $p : K \to X$ is proper (that is, K is compact). Let $f : M \to K$ be a *surgery problem* (possibly with boundary), i.e., f is a degree 1 normal map, and let $\epsilon > 0$ be given.

Definition 2.1. An ϵ–form (A, λ) over X is said to be *associated* to the surgery problem

$$M \xrightarrow{f} K \xrightarrow{p} X$$

(considered over X), where K is an ϵ–Poincaré complex over X, and p is $(\epsilon, 1)$–connected, if the following conditions are satisfied:

(1) A is a geometric module over X;

(2) $\lambda : A \times A \to \mathbb{Z}$ is an ϵ–form, i.e., if $d(a,b) \geq \epsilon$, then $\lambda(a,b) = 0$ (here d is the metric on X);

(3) there is a normal bordism of f rel. ∂M to

$$M'' \xrightarrow{f''} K \xrightarrow{p} X;$$

(4) there is a CW–pair (K', M'') with cells only in dimension $2k+1$ (recall that $\dim K = 4k$) such that $C_{2k+1}(K', M'') = A$;

(5) there exists an ϵ–equivalence $(K', M'') \to (K, M'')$ over X;

(6) the form λ is given by the intersection numbers in M'' of the images of A under the homomorphism $A = C_{2k+1}(K, M'') \to C_{2k}(M'')$ (some details are given in the proof of Proposition 2.2).

Remark 2.1. Here (K', M'') is the pair defined by f'', as usual. The space K' is roughly constructed as follows. One does a controlled surgery on $f : M \to K$ over X to obtain a $(\epsilon, 2k-1)$–connected map

$$f' : M' \to K.$$

Then one can replace the pair (K, M') by a pair (K', M') such that

$$C_q(K', M') \cong A$$

for $q = 2k+1$, and vanishing otherwise (use Proposition 2.4 of [6]).

The following is Theorem 2.1 of [6].

Theorem 2.1. *Assume that $p : K \to X$ is UV^1 and that K is a $4k$–dimensional δ–Poincaré complex over X for all $\delta > 0$. Then we have:*

(i) For all $\epsilon > 0$ there exist non-singular symmetric even ϵ–bilinear forms (G, λ) associated to a surgery problem $f : M \to K$.

(ii) For all $\alpha > 0$, there is a real number $\epsilon > 0$ so that for any associated even symmetric non-singular ϵ–bilinear form (G, λ) (with respect to $f : M \to K$), which is ϵ–bordant to the trivial one (see definition below), there exists a normal bordism of $f : M \to K$ over X to an α–homotopy equivalence $f' : M' \to K'$ over X. In the relative case, we have $\partial M' = \partial M$.

(iii) Given $\gamma > 0$, there is a real number ϵ with $0 < \epsilon < \gamma$ such that if

$$g : (N, \partial_0 N, \partial_1 N) \to (P, \partial_0 P, \partial_1 P)$$

is a normal bordism with $P \to X$ $(\epsilon, 1)$–connected and P a (relative) ϵ–Poincaré complex over X, then ϵ–associated forms to $g|_{\partial N_0}$ and $g|_{\partial N_1}$ are γ–bordant.

Here we have used the following notion. Two forms (A_1, λ_1) and (A_2, λ_2) over X are said to be ϵ–*bordant* if there are a geometric module H over X and an ϵ–isomorphism from $(A_1, \lambda_1) \oplus (A_2, -\lambda_2) \oplus (H \oplus H, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ to a hyperbolic form over X. For instance, if (A, λ) is an ϵ–form as above, then $(A, \lambda) \oplus (A, -\lambda)$ is ϵ–isomorphic to a hyperbolic form over X.

We need the following proposition which is not proved in [6].

Proposition 2.2. *Let $p : K \to X$ be a δ–Poincaré complex over X for all $\delta > 0$, and suppose that the map p is UV^1. Let $f_1 : M_1^{4k} \to K$ and $f_2 : M_2^{4k} \to M_1^{4k}$ be normal maps of degree 1. Let (G_1, λ_1) and (G_2, λ_2) be non-singular symmetric even ϵ_i–forms, $i = 1, 2$, associated to the surgery problems*

$$M_1 \xrightarrow{f_1} K \xrightarrow{p} X$$

and

$$M_2 \xrightarrow{f_2} M_1 \xrightarrow{p \circ f_1} X,$$

respectively. Then $(G_1, \lambda_1) \oplus (G_2, \lambda_2)$ is an ϵ_3–form associated to

$$M_2 \xrightarrow{f_1 \circ f_2} K \xrightarrow{p} X,$$

where ϵ_3 depends on ϵ_1 and ϵ_2. In particular, ϵ_3 is small whenever ϵ_1 and ϵ_2 are small.

Proof. Suppose for simplicity that f_1 and f_2 are already $2k$–connected, hence so is the composition $f_1 \circ f_2$. Therefore, $H_*(K, M_1)$ and $H_*(M_1, M_2)$

are zero except for $* = 2k+1$ (see Lemma 2.5 of [6]). Our application will have these properties. Thus, $H_*(K, M_2) \cong 0$ for $* \neq 2k+1$ and

$$H_{2k+1}(K, M_2) \cong H_{2k+1}(K, M_1) \oplus H_{2k+1}(M_1, M_2).$$

Following the proof of Theorem 2.1 in [6] there are complexes $A_{i,*}$, $i = 1, 2$, with $A_{i,q} \cong 0$ for $q \leq 2k$, and δ_i'–chain equivalences over X

$$C_*(K, M_1) \to A_{1,*} \qquad \text{and} \qquad C_*(M_1, M_2) \to A_{2,*}.$$

Moreover, we can assume that $A_{i,*}$ is of the form $\partial_i : B_{i,2k+1} \to B_{i,2k}$ (i.e., it is concentrated in dimension $2k$ and $2k+1$). It follows that there is a δ'–equivalence over X

$$C_*(K, M_2) \to A_{1,*} \oplus A_{2,*},$$

where $\delta' = \delta_1' + \delta_2'$ (see [8], Proposition 2.3).

The complexes

$$B_{i,2k+1} \xrightarrow{\partial_i} B_{i,2k}$$

are constructed by folding, so they come with splittings

$$s_i : B_{i,2k} \to B_{i,2k+1}.$$

Then one does surgeries in M_1 on small $(2k)$–spheres given by a basis of $B_{1,2k}$. One gets a normal map $f_1'' : M_1'' \to K$ which is normally cobordant to $f_1 : M_1 \to K$. Similarly, one does surgeries in M_2 on small $(2k)$–spheres given by a basis of $B_{2,2k}$. This produces a normal map $f_2'' : M_2'' \to M_1$ which is normally cobordant to $f_2 : M_2 \to M_1$. The above surgeries have the effect that the complexes $A_{i,*}$ change to complexes concentrated in dimension $2k+1$ of the form $B_{i,2k+1} \oplus B_{i,2k}$, i.e., there are $\widetilde{\delta}_i$–chain equivalences

$$C_*(K, M_1'') \to B_{1,2k+1} \oplus B_{1,2k} \qquad \text{and} \qquad C_*(M_1, M_2'') \to B_{2,2k+1} \oplus B_{2,2k}.$$

Here $\widetilde{\delta}_i$ depends on δ_i' and the "small" surgeries on the $(2k)$–spheres. In particular, $\widetilde{\delta}_i$ can be made arbitrarily small if δ_i' is small enough. Then one applies Proposition 2.4 of [6] to construct CW pairs (K', M_1'') and (P', M_2'') which are δ_1''– and δ_2''–homotopy equivalent to (K, M_1'') and (M_1, M_2''), respectively. Moreover, they have cells (relatively) only in dimension $2k+1$ which correspond to generators in the module $B_{i,2k+1} \oplus B_{i,2k}$. Then $G_i = B_{i,2k+1} \oplus B_{i,2k}$, and the intersection forms λ_i are defined as follows. Let a, $b \in G_1$. They correspond to $(2k+1)$–cells in K' rel. M_1''. Then one defines $\lambda_1(a, b)$ to be the intersection number of their attaching spheres (similarly, for λ_2). Then λ_i is $(4\delta_i'')$–non–singular for any $i = 1, 2$ (see [6], p.273). Setting $\epsilon_i = 4\delta_i''$ yields the non–singular ϵ_i–forms (G_i, λ_i). Of course, δ_i'' depends on $\widetilde{\delta}_i$, hence on δ_i', i.e., δ_i'' is small if δ_i' is. Now we construct an associated form (G, λ) of the composite map $f_1 \circ f_2 : M_2 \to K$, and compare it with the form $(G_1 \oplus G_2, \lambda_1 \oplus \lambda_2)$. We begin with the δ'–equivalence $C_*(K, M_2) \to A_{1,*} \oplus A_{2,*}$, where $\delta' = \delta_1' + \delta_2'$. We lift the small $(2k)$–spheres in M_1 (corresponding to the elements of a basis of $B_{1,2k}$) to small $(2k)$–spheres in M_2 via the map $f_2 : M_2 \to M_1$, and do surgeries on them. Then we obtain a normal map $g : N \to K$ which is normally cobordant to $f_1 \circ f_2 : M_2 \to K$. Obviously, g factors over M_1'', i.e., we have a diagram of normal maps

$$\begin{array}{ccc} N & \xrightarrow{\ g\ } & K \\ {\scriptstyle g_1}\big\downarrow & & \big\uparrow{\scriptstyle f_1''} \\ M_1'' & = & M_1''. \end{array}$$

This gives a Θ–isomorphism (over X) of $C_*(M_1'', N)$ with $C_*(M_1, M_2)$, hence a δ_2'–equivalence $C_*(M_1'', N) \to A_{2,*}$. Then we do surgeries in N on small $(2k)$–spheres corresponding to a basis of $B_{2,2k}$. The result is a normal map $g'' : N'' \to K$ which factors over M_1'', i.e., we have a commutative diagram of normal maps

$$\begin{array}{ccc} N'' & \xrightarrow{\ g''\ } & K \\ {\scriptstyle g_1''}\big\downarrow & & \big\uparrow{\scriptstyle f_1''} \\ M_1'' & = & M_1''. \end{array}$$

This turns the above δ_2'–equivalence into a $\widetilde{\delta}_2$–equivalence

$$C_*(M_1'', N'') \to B_{2,2k+1} \oplus B_{2,2k}.$$

Composing with $C_*(K, M_1'') \to B_{1,2k+1} \oplus B_{1,2k}$ yields a $(\widetilde{\delta}_1 + \widetilde{\delta}_2)$–chain equivalence

$$C_*(K, N'') \to B_{1,2k+1} \oplus B_{1,2k} \oplus B_{2,2k+1} \oplus B_{2,2k} = G_1 \oplus G_2.$$

Then we apply Proposition 2.4 of [6] to get a CW–pair (P', N'') which is δ_3''–homotopy equivalent to (K, N''), and has cells only in dimension $2k+1$. Here δ_3'' is small if $\widetilde{\delta'}_1$ and $\widetilde{\delta'}_2$ are small. Therefore, the pair $(G_1 \oplus G_2, \lambda_1 \oplus \lambda_2)$ is an ϵ_3–associated non–singular form of $f_1 \circ f_2 : M_2 \to K$ over X with $\epsilon_3 = 4\delta_3''$. □

The next theorem says that non–singular ϵ–forms (G, λ) can be realized as associated non–singular forms of normal maps (see Proposition 2.7 of [6]). We state it in a slightly different way which can be proved as Proposition 2.7 of [6].

Theorem 2.3. *Let X be a compact ANR–space, and let N_0^{4k-1} be a closed manifold. Suppose that a map $p : N_0 \to X$ is UV^1. Then, given a real number $\delta > 0$ and a δ–symmetric even non–singular form (G, λ) over X, there is a degree 1 normal map $F : V \to N_0 \times I$ with*

$$F|_{\partial_0 V} = \mathrm{Id} : \partial_0 V = N_0 \to N_0.$$

Moreover, if $\gamma > 0$ is given, then $F|_{\partial_1 V} : \partial_1 V \to N_0$ is a γ–homotopy equivalence if δ is sufficiently small.

Remark 2.2. (I) The modification we have made consider an arbitrary manifold N_0^{4k-1} instead of the boundary of a regular neighbourhood of X in $\mathbb{R}^{4k}$. This requires that we have to transform the geometric non–singular δ–form over X to one over N_0. If $\{a_i\}$ are the generators of the geometric module G corresponding to points $\{x_i\}$ in X, then $d(a_i, a_j) = d(x_i, x_j)$. For arbitrary a, $b \in G$ with $a = \sum_i \alpha_i a_i$ and $b = \sum_j \beta_j a_j$, the distance $d(a, b)$ is defined to be the minimum of $d(a_i, a_j)$ with $\alpha_i \neq 0$ and $\beta_j \neq 0$. If (G, λ) is a δ–form, then we have $\lambda(a, b) = 0$ whenever $d(a, b) \geq \delta$. Let $\{z_i\}$ be points in N_0 with $p(z_i) = y_i$ such that

$$d(y_i, y_j) \geq \delta \Rightarrow d(x_i, x_j) \geq \delta.$$

Then we may consider G as a geometric module over N_0, and λ a non–singular δ–form over N_0. Now the proof of Theorem 2.3 proceeds as in [6], replacing the boundary of a regular neighbourhood projection $\partial W \to X$ (of X in $\mathbb{R}^{4k}$) by the UV^1–map $p : N_0 \to X$. Connecting each y_i to x_i by a path defines a morphism of the geometric modules G over X with respect to $\{y_i\}$ and $\{x_i\}$. Different lifts $\{z_i'\}$ of $\{y_i\}$ define, up to homotopy, a unique isomorphism of G over $\{z_i'\}$ to G over $\{z_i\}$ because $p : N_0 \to X$ is UV^1. So the δ–form (G, λ) over N_0 is unique, up to the choice of $\{y_i\}$.

(II) If γ is sufficiently small, then the map $F|_{\partial_1 V} : \partial_1 V \to N_0$ is homotopic to a homeomorphism. The homotopy is controlled, that is, given $\alpha > 0$, then for sufficiently small γ (i.e., sufficiently small δ), the restriction $F|_{\partial_1 V}$ is α–homotopic to a homeomorphism (this is the theorem of Chapman and Ferry [2]). We have to made use also of parts (2), (3) and (4) of Proposition 2.7 of [6]. Recall that

$$H_{4k}(X, \mathbb{L}) = H_{4k}(X, \mathbb{Z}) \times H_{4k}(X, G/\mathrm{TOP}) \cong \mathbb{Z} \times H_{4k}(X, G/\mathrm{TOP}),$$

since X is a $4k$–dimensional compact Poincaré complex. Let $\sigma \in H_{4k}(X, \mathbb{L})$ be given. According to Proposition 2.7 (1) of [6] the $\mathbb{Z}$–component can be computed as follows. Choose a degree 1 normal map $\mathcal{X} \to \mathcal{M}$ over $p : \mathcal{M} \to X$, which can be assumed to be UV^1, representing σ. Let

$\{(H^\delta, \mu^\delta)\}$ be the family of associated non-singular δ–forms over X. For sufficiently small δ, the pairing (H^δ, μ^δ) can be realized as $16k$–dimensional closed simply connected surgery problem $f_\mu : P_\mu \to Q_\mu$. Then the $\mathbb{Z}$–component of σ is $1 + 8\sigma(f_\mu)$, where $\sigma(f_\mu)$ is the surgery obstruction of f_μ (for simplicity we have written μ for μ^δ). We will call $1 + 8\sigma(f_\mu)$ the *Quinn index* of (H, μ).

Corollary 2.4. *Let $f : X \to Y$ be a UV^1–map of $4k$–dimensional compact connected Poincaré spaces. Then the induced homomorphism*

$$f_{4k} : H_{4k}(X, \mathbb{L}) \to H_{4k}(Y, \mathbb{L})$$

is the identity on the $\mathbb{Z}$–factor.

Proof. This follows immediately from Proposition 2.7 (4) of [6]. Namely, $f_{4k}(\sigma) \in H_{4k}(Y, \mathbb{L})$ can be represented by

$$\mathcal{X} \longrightarrow \mathcal{M} \xrightarrow{p} X \xrightarrow{f} Y.$$

Since f is UV^1, we have the associated $\overline{\delta}$–forms $\{(\overline{H}^{\overline{\delta}}, \overline{\mu}^{\overline{\delta}})_{\overline{\delta}>0}\}$ with

$$(\overline{H}^{\overline{\delta}}, \overline{\mu}^{\overline{\delta}}) = (H^\delta, \mu^\delta)$$

measured over Y. Let $f_{\overline{\mu}} : P_{\overline{\mu}} \to Q_{\overline{\mu}} \to Y$ be the closed realization of a generic $(\overline{H}^{\overline{\delta}}, \overline{\mu}^{\overline{\delta}})$ as closed $16k$–dimensional surgery problem. By Proposition 2.7 (4) of [6] for a given $\epsilon > 0$ there is an $(\epsilon, 1)$–normal cobordism over Y between f_μ and $f_{\overline{\mu}}$, hence they have the same surgery obstruction. □

The proof shows also the following consequence which we spell out for later use.

Corollary 2.5. *Let $\sigma \in H_{4k}(X, \mathbb{L})$ be represented by the degree 1 normal map $\mathcal{X} \to \mathcal{M}$ over X and let $\{(H^\delta, \mu^\delta)_{\delta>0}\}$ be the associated forms. If $f : X \to Y$ is UV^1, then $\{(\overline{H}^{\overline{\delta}}, \overline{\mu}^{\overline{\delta}})_{\overline{\delta}>0}\}$ are the associated forms of $\mathcal{X} \to \mathcal{M}$ over Y, and $\overline{\delta} \to 0$ as $\delta \to 0$.*

We state a special case of Proposition 2.7 (3) of [6].

Corollary 2.6. *Let W be a compact ANR–space and $X \subset W$ a closed subspace. Let $\{(H^\delta, \mu^\delta)_{\delta>0}\}$ be a family of a symmetric even non-singular δ–forms over X, hence over W via the inclusion $X \subset W$. Then the Quinn index constructed over X coincides with the one constructed over W.*

We observe that the inclusion is not required to be UV^1.
Finally, we need the following special converse case of Theorem 2.1 in [6].

Lemma 2.7. *Let $f : M \to K$ be a $4k$–dimensional degree 1 normal map over the UV^1 map $p : K \to X$ with associated δ–forms $\{(G^\delta, \lambda^\delta)_{\delta>0}\}$. If f is an ϵ–equivalence over X, then for a certain $\delta = \delta(\epsilon)$ the form $(G^\delta, \lambda^\delta)$ is ϵ'–cobordant to the trivial one. Moreover, $\epsilon' \to 0$ and $\delta \to 0$ as $\epsilon \to 0$.*

Proof. Imitating the proof of Theorem 2.1 in [6] (see also the proof of Proposition 2.2 above), one obtains a ϵ_1–chain equivalence of the complexes $C_*(K, M) \to A_*$, where A_* is of the form $\partial_{2k+1} : B_{2k+1} \underset{\cong}{\to} B_{2k}$. By [8] (Proposition 2.4 and Section 9) one can assume that $\epsilon_1 = 3\epsilon$. Then one does surgeries on small $(2k)$–spheres corresponding to a bases of B_{2k} to get a normal map $M'' \to K$ (according to notation of [6]). Then $G = H_{2k+1}(K, M'')$ is by definition an associated module with $\lambda : G \times G \to \mathbb{Z}$ defined by setting $\lambda(x, y)$ equal to the intersection number of ∂x and ∂y in M''. Since $\partial_{2k+1} : B_{2k+1} \underset{\cong}{\to} B_{2k}$, the intersection pairing is standard. Moreover, if G is an ϵ_2–module, then λ is an $(4\epsilon_2)$–form. Now the small trivial surgeries on the bases B_{2k} are made on places according to to the ϵ_1–chain equivalence $C_{2k+1}(K, M) \to A_*$, hence ϵ_2 depends on ϵ_1, and $\epsilon_2 \to 0$ as $\epsilon_1 \to 0$. This proves the lemma. □

3. Some technical preliminaries.

In this section we summarize some technical preliminaries proved in Section 4 of [1]. We report also Proposition 4.7 of [1] though we shall not use it (see Proposition 3.4 below).

Theorem 3.1. *(Bestvina' s theorem, see* [1], *Proposition 4.3)*
Let $f : (M^n, \partial M^n) \to K$ be a map from a compact n–manifold to a polyhedron, where $n \geq 5$. If the homotopy fiber of f is simply connected, then f is homotopic to an UV^1-map. If $f|_{\partial M}$ is already UV^1, then the homotopy is relative to ∂M.

Remark 3.1. If $n \geq 5$, then the map f can be ϵ–approximated by UV^1–maps.

We need the following "controlled" gluing construction of compact manifolds (see [1], Proposition 4.5).

Theorem 3.2. *Given n and a finite complex B, there are real numbers $\epsilon_0 > 0$ and $T > 0$ such that if $0 < \epsilon < \epsilon_0$, $(M_i, \partial M_i)$, $i = 1, 2$, are orientable manifolds, $p_i : M_i \to B$, $i = 1, 2$, are UV^1–maps, and $h : \partial M_1 \to \partial M_2$ is an orientation–preserving ϵ–equivalence over B (this includes $d(p_1, p_2 \circ h) < \epsilon$), then $M_1 \cup_h M_2$ is a $T\epsilon$–Poincaré duality space over B.*

The proof of Theorem 3.2 uses the following lemma which explains the real numbers ϵ_0 and T.

Lemma 3.3. *Let B be a finite polyhedron. Then there are real numbers $\epsilon_0 > 0$ and $T > 0$ so that if $0 < \epsilon \leq \epsilon_0$, then for any space S and for any two maps $f, g : S \to B$ with $d(f,g) < \epsilon$ the maps f and g are $T\epsilon$–homotopic.*

Lemma 3.3 can be proved by embedding B into $\mathbb{R}^m$ and considering small regular neighbourhoods of $B \subset \mathbb{R}^m$.

Let us mention another technical proposition (see [1], Proposition 4.7) which will however not be used in our construction.

Proposition 3.4. *Given B and n as above, there is a real number $T > 0$ so that if $p_i : X_i \to B$, $i = 1, 2$, are ϵ–Poincaré spaces over B of the same formal dimension $\leq n$ with UV^1–control maps, and $f : X_1 \to X_2$ is a map satisfying $d(p_2 \circ f, p_1) < \epsilon$ such that the algebraic mapping cone of f is ϵ–acyclic through the middle dimension, then f is a $T\epsilon$–equivalence.*

We will use the following result (see [1], Proposition 4.10):

Proposition 3.5. *Suppose that X and Y are finite polyhedra, V is a regular neighbourhood of X with $\dim V \geq 2\dim Y + 1$, $p : V \to B$ is a map, $r : V \to X$ is a retraction, and $f : Y \to X$ is an ϵ–equivalence over B. Then we can choose an embedding $i : Y \to V$ so that there exists a retraction $s : V \to i(Y)$ with $d(p \circ r, p \circ s) < 2\epsilon$.*

There is another important theorem concerning controlled Poincaré spaces. In the definition of an ϵ–Poincaré structure of a locally compact ANR–pair $(K, \partial K)$, given by Quinn in [6], appears the following property:

There are a mapping cylinder neighbourhood $(U, \partial_0 U)$ of a proper embedding $(K, \partial K) \subset (\mathbb{R}^{n+k-1} \times [0, \infty[, \mathbb{R}^{n+k-1} \times 0)$ and a spherical fibration

$$\mathbb{S}^{k-1} \to \mathbb{S}(\xi) \to K$$

such that there is an ϵ–homotopy equivalence

$$(U, \partial_0 U, \partial_1 U) \to (D(\xi), D(\xi|_{\partial K}), \mathbb{S}(\xi))$$

over the control space (here $D(\xi)$ is the disc–fibration of $\mathbb{S}(\xi)$).

In other words, the canonical normal Spivak fibration of $(K, \partial K)$ has the ϵ–approximative lifting property. The definition of ϵ–Poincaré complexes given in [1] does not include the ϵ–approximative lifting property of the Spivak fibration. However, this property is a consequence of their definition (see [1], Proposition 4.5). We recall the statement of that result.

Proposition 3.6. *Given n and B, there are real numbers $\epsilon_0 > 0$ and $T > 0$ such that if $0 < \epsilon \leq \epsilon_0$ and X is an ϵ–Poincaré duality space of topological dimension $\leq n$ over B with UV^1 control map $p : X \to B$, then for any abstract regular neighbourhood N of X in which X has codimension at least 3, the restriction of the regular neighbourhood projection $\partial N \to X$ has the $T\epsilon$–lifting property.*

Remark 3.2. If M is a manifold with a PL structure, then M is an ϵ–Poincaré space for all $\epsilon > 0$ and for all proper control maps. This follows from the fact that Poincaré duality can be defined in terms of dual cells $\sigma^* = D(\sigma, M)$ of σ. If the triangulation of M is sufficiently fine, then we get ϵ–chain equivalences

$$\cap\xi : C^q(M) \to C_{n-q}(M)$$

for any $\epsilon > 0$. Thus, a necessary condition that a Poincaré complex is a manifold is the existence of arbitrary small ϵ–Poincaré duality equivalences.

4. A construction of $4k$–dimensional generalized manifolds.

Let M^{4k} be a triangulated closed simply connected manifold of dimension $4k$, where $k > 1$. We fix an element $\sigma \in H_{4k}(M, \mathbb{L}) \cong [M, \mathbb{Z} \times G/\mathrm{TOP}]$. Then σ determines a family of surgery problems

$$\{x(\tau) : \mathcal{X}_\tau \to \mathcal{M}_\tau \to D(\tau, M) : \tau \text{ simplex of } M\}.$$

They assemble to a normal map

$$\mathcal{X}^{4k} \longrightarrow \mathcal{M}^{4k} \longrightarrow M^{4k}$$

over M (as explained in Section 8 of [7]). We can assume that $\mathcal{M}^{4k}$ is simply connected, hence by Theorem 3.1 we may assume that the map $\mathcal{M}^{4k} \to M^{4k}$ is UV^1. Moreover, $\mathcal{M}^{4k}$ is a δ–Poincaré space for all $\delta > 0$ over M. By Theorem 2.1 there is a family $\{(G^\delta, \lambda^\delta)_{\delta>0}\}$ of non–singular symmetric even bilinear forms (over M) associated to $\mathcal{X}^{4k} \to \mathcal{M}^{4k} \to M$. We follow the idea of [1] to construct the spaces X_i. One decomposes $M = B \cup_D C$, where B is a regular neighbourhood of the 2–skeleton of M, C is the closure of the complement of B in M, and $D = \partial C = \partial B$. Observe that by Theorem 3.1 we can assume that $D \times I \to D \to M$ is UV^1 so the form $(G^\delta, \lambda^\delta)$ can be realized by a normal map $F_\sigma : V \to D \times I$ with $F_\sigma|_{\partial_0 V} = \mathrm{Id} : \partial_0 V = D \to D$ and $F_\sigma|_{\partial_1 V} = f_\sigma : \partial_1 V = D' \to D$ γ–equivalences over M, where $\gamma = \gamma(\delta)$ depends on δ.

We get for any $\delta > 0$ a normal map F_σ, but for simplicity we shall not mark F_σ with δ. Moreover, $\gamma(\delta) \to 0$ as $\delta \to 0$.

We construct the space X_0. For conveniency, we give two descriptions, $\widetilde{X}_0$ and $\widetilde{\widetilde{X}}_0$ say, homeomorphic to each other:

$$(1) \qquad \widetilde{X}_0 = B \cup_D V \cup_{f_\sigma} C$$

(identification of D' with $\partial C = D$ via $f_\sigma : D' \to D$)

$$(2) \qquad \widetilde{\widetilde{X}}_0 = B \cup_D V \cup_{f_\sigma} (D \times I) \cup_{\mathrm{Id}} C$$

(choosing a small collar of $D \subset M$ one easily describes a homeomorphism).

Let $p_0 : X_0 \equiv \widetilde{X}_0 \to B \cup_{\mathrm{Id}} (D \times I) \cup_{\mathrm{Id}} C \equiv M$ be given by $p_0|_B = \mathrm{Id}$, $p_0|_V = F_\sigma$, and $p_0|_C = \mathrm{Id}$. We can again assume that p_0 is UV^1. By Theorem 3.2, X_0 is a $T\gamma(\delta)$–Poincaré duality space for some $T > 0$. We define the manifold

$$M_0 = B \cup_D V \cup_{D'} (-V) \cup_D C$$

where $-V$ denotes the cobordism V "upside–down".

Let

$$g_0 : M_0 \to X_0 \equiv \widetilde{\widetilde{X}}_0 = B \cup_D V \cup_{f_\sigma} (D \times I) \cup_{\mathrm{Id}} C$$

be the map defined by

$$g_0|_{B \cup_D V} = \mathrm{Id}, \quad g_0|_{-V} = -F_\sigma, \quad g_0|_C = \mathrm{Id}$$

where $-F_\sigma$ means F_σ "upside–down". By Theorem 3.1 we may assume that g_0 is UV^1.

Lemma 4.1. *With the above notation, $g_0 : M_0 \to X_0$ is a normal map of degree 1 and $(G^\delta, -\lambda^\delta)$ is an associated non-singular symmetric δ–form over M.*

Proof. Following the proof of Theorem 2.1 (see also the proof of Proposition 2.2), it is obvious that the essential construction regards the map

$$-F_\sigma = g_0|_{-V} : -V \to D \times I$$

which realizes $(G^\delta, -\lambda^\delta)$. □

To summarize, we have constructed a Poincaré space X_0, a map $p_0 : X_0 \to M$, and a degree 1 normal map $g_0 : M_0 \to X_0$ which satisfy the following properties:

(i) X_0 is a $T\gamma(\delta)$–Poincaré space over M;

(ii) $p_0 : X_0 \to M$ is UV^1 (not a homotopy equivalence);

(iii) $(G^\delta, -\lambda^\delta)$ is an associated non-singular δ–form to $g_0 : M_0 \to X_0$ over M.

Before the next step we shall transform $(G^\delta, \lambda^\delta)$ in forms over X_0. Note that X_0 is not yet a metric space. We embed M into $\mathbb{R}^L$, for L large, and approximate $p_0 : X_0 \to M \subset \mathbb{R}^L$ by an embedding. Let $r_0 : W_0 \to X_0$ be the restriction of a cylindrical neighbourhood of $X_0 \subset \mathbb{R}^L$. We can assume that $M \subset W_0$. Then the δ–forms $(G^\delta, \lambda^\delta)$ over M become δ'–forms $(G^{\delta'}, \lambda^{\delta'})$ over X_0 by using $r_0|_M : M \to X_0$. Since by Theorem 3.1 we can assume that $r_0|_M$ is UV^1, Corollary 2.5 implies that there exist non–singular δ'–forms over X_0, which we denote by $\{(G^{\delta'}, \lambda^{\delta'})\}_{\delta'>0}$. Note that M_0 is an ϵ–Poincaré space over X_0 for any $\epsilon > 0$. We decompose $M_0 = B_1 \cup_{D_1} C_1$, where B_1 is a regular neighbourhood of the 2–skeleton of M_0 (for some fine triangulation yet to choose), and C_1 is the closure of its complement with $\partial B_1 = D_1 = \partial C_1$. We realize the form $(G^{\delta'}, \lambda^{\delta'})$ by the map

$$F_{1,\sigma} : V_1 \longrightarrow D_1 \times I \longrightarrow D_1 \xrightarrow{g_0} X_0$$

(over X_0) with

$$F_{1,\sigma}|_{\partial_0 V_1} = \mathrm{Id} : \partial_0 V_1 = D_1 \to D_1$$

and

$$f_{1,\sigma} = F_{1,\sigma}|_{\partial_1 V_1} = \partial_1 V_1 = D_1' \to D_1$$

a $\gamma_1 = \gamma_1(\delta')$–equivalences by Theorem 2.3 (here $\gamma_1(\delta')$ is small whenever δ' is small, that is, if δ is small).

Now let $X_1' = B_1 \cup_{D_1} V_1 \cup_{f_{1,\sigma}} C_1$, and let the map

$$f_1' : X_1' \to M_0 \equiv B_1 \cup_{\mathrm{Id}} (D_1 \times I) \cup_{\mathrm{Id}} C_1$$

be defined by

$$f_1'|_{B_1} = \mathrm{Id}, \quad f_1'|_{V_1} = F_{1,\sigma}, \quad f_1'|_{C_1} = \mathrm{Id}.$$

By Theorem 3.2, X_1' is a $T_1\gamma_1(\delta')$–Poincaré space over X_0 with respect to the map $g_0 \circ f_1'$, for some $T_1 > 0$. Furthermore, f_1' is a normal map of degree 1 outside the singular set, where the points $x \in D_1'$ are identified with the points $f_{1,\sigma}(x) \in D_1$. Finally, $(G^{\delta'}, \lambda^{\delta'})$ is associated to f_1'. By Proposition 2.2, $(G^{\delta'}, \lambda^{\delta'}) \oplus (G^\delta, -\lambda^\delta)$ is ϵ'–associated to the composition over M:

$$X_1' \xrightarrow{f_1'} M_0 \xrightarrow{g_0} X_0 \xrightarrow{p_0} M,$$

where ϵ' depends on δ and δ', and ϵ' is small if δ is sufficiently small. But it is trivial, so we can do surgery on X_1' (outside the singular set) to obtain an α_1-equivalence $p_1 : X_1 \to X_0$ applying Theorem 2.1 (2). The real number α_1 depends on δ', i.e., $\alpha_1 = \alpha_1(\delta')$, and $\alpha_1(\delta')$ is small if δ' is sufficiently small. Now we observe that X_1 is still a $T_1\gamma_1(\delta')$-Poincaré space over X_0. Because this fact is used many times, we formulate it as a lemma.

Lemma 4.2. *Let X be an ϵ-Poincaré complex over Y. Suppose that $X = X_1 \cup X_2$ and $\operatorname{int} X_2$ is an open manifold of the same dimension as X. Then surgeries on $\operatorname{int} X_2$ (on spheres which are contractible in Y) give an ϵ-Poincaré complex X' over Y.*

Proof. Let us suppose that we do surgeries only in the middle dimension $2k$ ($k > 1$) and $H_{2k}(X_1) \cong 0$ (this will be sufficient for our applications). Let X_2' be the result after the surgeries, i.e., $\partial X_2' = \partial X_2$ and $X' = X_1 \cup X_2'$. So a change in Poincaré duality regards only $C^{2k}(X') \to C_{2k}(X')$, i.e., $C^{2k}(X_2') \to C_{2k}(X_2')$, which can be made an arbitrary fine chain equivalence if we choose a fine triangulation of X_2'. □

To summarize, we have obtained a Poincaré space X_1 and a map $p_1 : X_1 \to X_0$ so that:

(1) p_1 is UV^1 (apply Theorem 3.1);

(2) X_1 is a $T_1\gamma_1(\delta')$-Poincaré space over X_0;

(3) p_1 is a $\alpha_1(\delta')$-homotopy equivalence;

(4) there exist an embedding $X_1 \to W_0$ and a retraction $r_1 : W_0 \to X_1$ such that $d(r_0, r_1) < 2\alpha_1(\delta)$.

The property (4) follows from Proposition 3.5. It will be convenient to restate the two steps in a generic form as follows.

Step (1). Given $\eta_0 > 0$, we have:

(i) there are a Poincaré complex X_0 and an UV^1-map $p_0 : X_0 \to M$;

(ii) X_0 is an η_0-Poincaré complex over M;

(iii) there is a degree 1 normal map $g_0 : M_0 \to X_0$ with associated δ-form $(G, -\lambda)$ (For this we choose δ so that $T\gamma(\delta) < \eta_0$).

Step (2). Given $\eta_1 > 0$ and $\zeta_1 > 0$, there are a Poincaré complex X_1 and a map $p_1 : X_1 \to X_0$ with the following properties:

(I) p_1 is UV^1;

(II) X_1 is an η_1-Poincaré complex over X_0;

(III) p_1 is a ζ_1–equivalence;

(IV) $d(r_0, r_1) < \zeta_1$.

For this we choose δ', i.e., δ, so small that $T_1\gamma_1(\delta') < \eta_1$, $2\alpha_1(\delta) < \zeta_1$, and $\alpha_1(\delta') < \zeta_1$.

In the *third step* we construct X_2, and then one proceeds by induction. What we need is a degree 1 normal map $g_1 : M_1 \to X_1$ which has an appropriate associated non–singular $\overline{\delta}$–form over X_0. First we show that there is an element $\overline{\sigma} \in H_n(X_0, \mathbb{L})$ with $p_*(\overline{\sigma}) = \sigma$. For this, we use $\mathbb{L}$–Poincaré duality for the manifolds M_0 and M (see [7]). The assertion follows from the following diagram

$$\begin{array}{ccccc} H_n(M_0, \mathbb{L}) & \xrightarrow{g_{0*}} & H_n(X_0, \mathbb{L}) & \xrightarrow{p_*} & H_n(M, \mathbb{L}) \\ \cong\uparrow & & & & \uparrow\cong \\ H^0(M_0, \mathbb{L}) & \xleftarrow[g_0^*]{} & H^0(X_0, \mathbb{L}) & \xleftarrow[p^*]{} & H^0(M, \mathbb{L}). \end{array}$$

This defines a normal map (over X_0)

$$\mathcal{X}_1 \longrightarrow \mathcal{M}_1 \longrightarrow X_0$$

which provides us with a family of associated δ–forms $\{(\overline{G}^{\delta}, \overline{\lambda}^{\delta})\}_{\delta>0}$. For this, we assume that the map $\mathcal{M}_1 \to X_0$ is UV^1 (see Theorem 3.1), and then we apply Theorem 2.1. Since $p_1 : X_1 \to X_0$ is a controlled UV^1–homotopy equivalence, for any small $\delta > 0$ there is a degree 1 normal map $g_1 : M_1 \to X_1$ which has $(\overline{G}^{\delta}, -\overline{\lambda}^{\delta})$ as associated non–singular symmetric form. This can be deduced from the following diagram

$$\begin{array}{ccc} [X_0, G/\mathrm{TOP}] & \longrightarrow & H_n(X_0, \mathbb{L}) \\ p_1^*\downarrow & & \uparrow p_{1*} \\ [X_1, G/\mathrm{TOP}] & \longrightarrow & H_n(X_1, \mathbb{L}) \end{array}$$

where the horizontal maps send a normal map to its (controlled) surgery obstruction. By using the map g_1, one proceeds as in Step 2 to construct for any $\eta_2 > 0$ and $\zeta_2 > 0$ a Poincaré space X_2 and a map $p_2 : X_2 \to X_1$ which satisfy the following properties:

(i) p_2 is UV^1;

(ii) X_2 is an η_2–Poincaré complex over X_1;

(iii) p_2 is a ζ_2–equivalence over X_0;

(iv) there are an embedding $X_2 \to W_0$ and a retraction $r_2 : W_0 \to X_2$ such that $d(r_1, r_2) < \zeta_2$.

We briefly describe the construction of X_2 (compare it with the construction of X_1). We decompose $M_1 = B_2 \cup_{D_2} C_2$, where B_2 is a regular neighbourhood of the 2–skeleton of M_1 in a sufficiently fine triangulation, and C_2 is the closure of the complement of B_2 in M_1. Hence we have $D_2 = \partial B_2 = \partial C_2$. We can assume that $g_1|_{D_1} : D_2 \to X_1$ is UV^1 by Theorem 3.1. Then we transform $\{(\overline{G}^{\delta}, \overline{\lambda}^{\delta})\}_{\delta>0}$ into a family of forms $\{(\overline{G}^{\delta'}, \overline{\lambda}^{\delta'})\}_{\delta'>0}$ over X_1 by using an embedding of X_1 into W_0 close to $p_1 : X_1 \to X_0 \subset W_0$ (compare the construction of $(G^{\delta}, \lambda^{\delta})$ over M with that of $(G^{\delta'}, \lambda^{\delta'})$ over X_0). By Theorem 2.3 we can realize $(\overline{G}^{\delta'}, \overline{\lambda}^{\delta'})$ as the associated form of a degree 1 normal map (over X_1)

$$F_{2,\overline{\sigma}} : V_2 \to D_2 \times I$$

with

$$F_{2,\overline{\sigma}}|_{\partial_0 V_2} = \mathrm{Id} : \partial_0 V_2 = D_2 \to D_2$$

and

$$f_{2,\overline{\sigma}} = F_{2,\overline{\sigma}}|_{\partial_1 V_2} : \partial_1 V_2 = D_2' \to D_2$$

a controlled homotopy equivalence. Then let $X_2' = B_2 \cup_{D_2} V_2 \cup_{f_{2,\overline{\sigma}}} C_2$. This space is a controlled Poincaré complex. Let the map

$$f_2' : X_2' \to M_1 = B_2 \cup (D_2 \times I) \cup C_2$$

be defined by

$$f_2'|_{B_2} = \mathrm{Id}, \quad f_2'|_{V_2} = F_{2,\overline{\sigma}} \quad \text{and} \quad f_2'|_{C_2} = \mathrm{Id}\,.$$

By using Proposition 2.2 we can do surgery on the composition

$$X_2' \xrightarrow{f_2'} M_1 \xrightarrow{g_1} X_1$$

since the associated form $(\overline{G}^{\delta}, -\overline{\lambda}^{\delta}) \oplus (\overline{G}^{\delta'}, \overline{\lambda}^{\delta'})$ is trivial. The result is a controlled homotopy equivalence $p_2 : X_2 \to X_1$.

For convenience, we use from now on the following notations:

$$(G^{\delta}, \lambda^{\delta}) = (G_1^{\delta}, \lambda_1^{\delta}) \qquad \text{and} \qquad (\overline{G}^{\delta}, \overline{\lambda}^{\delta}) = (G_2^{\delta}, \lambda_2^{\delta}).$$

Remark 4.1. We emphasize the important fact that for any given η_1–Poincaré complex X_1 (over X_0) we can construct an η_2–Poincaré complex X_2 over X_1.

Putting all together we have proved the following result:

Theorem 4.3. *Suppose to be given the sequences of positive real numbers $\{\eta_i\}$ and $\{\zeta_i\}$ (all sufficiently small). Then there is a sequence of 4k-dimensional Poincaré complexes (over M) and maps*

$$\cdots \longrightarrow X_m \xrightarrow{p_m} X_{m-1} \longrightarrow \cdots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0 \xrightarrow{p_0} M$$

such that:

(1) p_m is UV^1 for any $m \geq 0$;

(2) X_m is an η_m-Poincaré complex over X_{m-1};

(3) $p_m : X_m \to X_{m-1}$ is a ζ_m-homotopy equivalence over X_{m-2} for any $m \geq 1$, where $X_{-1} = M$;

(4) there is a regular neighbourhood W_0 of X_0 in $\mathbb{R}^L$, L sufficiently large, and there are embeddings $X_m \to W_0$ and retractions $r_m : W_0 \to X_m$ so that $d(r_m, r_{m-1}) < \zeta_m$ for any $m \geq 1$.

As explained in Section 1 we choose regular neighbourhoods W_m of $X_m \subset W_0$ so that $W_{m+1} \subset \operatorname{int} W_m$ and $W_m \setminus \operatorname{int} W_{m+1}$ are (ζ_{m+1}, h)-cobordism for any $m \geq 1$. Then $X = \cap_m W_m$ is a generalized ANR-manifold. Our construction comes with a sequence of normal maps

$$\mathcal{X}_m \longrightarrow \mathcal{M}_m \longrightarrow X_{m-1}$$

defined by elements in $H_{4k}(X_{m-1}, \mathbb{L}) \cong \mathbb{Z} \times [X_{m-1}, G/\mathrm{TOP}]$ which have the same $\mathbb{Z}$-component as $\sigma \in H_{4k}(M, \mathbb{L})$. Then we have realized the associated forms $(G_m^\delta, -\lambda_m^\delta)$ of $\mathcal{X}_m \to \mathcal{M}_n$ over X_{m-1} by a degree 1 normal map $g_m : M_m \to X_m$ over X_{m-1}. By Proposition 2.7 (2) of [6] (see Remark 2.3), they have the same Quinn invariant, i.e., $g_m : M_m \to X_m$ defines an element in $H_{4k}(X_{m-1}, \mathbb{L}) \cong \mathbb{Z} \times [X_{m-1}, G/\mathrm{TOP}]$ which belongs to the $\mathbb{Z}$-component as $-\overline{\sigma}$ (which is the same as the one of $-\sigma \in H_{4k}(M, \mathbb{L})$).

It remains to prove that X has Quinn index $i(X) \neq 1$. In fact, we prove that it coincides with the component of $-\sigma$. For this, we consider $g_m : M_m \to X_m$ as a degree 1 normal map over W_0 in two different ways:

$$\text{(1)} \qquad M_m \xrightarrow{g_m} X_m \xrightarrow{p_m} X_{m-1} \xrightarrow{i_{m-1}} W_0$$

and

$$\text{(2)} \qquad M_m \xrightarrow{g_m} X_m \xrightarrow{\mathrm{Id}} X_m \xrightarrow{i_m} W_0$$

Here $i_k : X_k \to W_0$ are UV^{1}-approximations of the inclusions $X_k \subset W_0$. Choosing m sufficiently large, we can assume that for any $x \in X_m$ there is a straight segment in W_0 which connects $i_m(x)$ and $i_{m-1} \circ p_m(x)$. This defines a homotopy $h_m : X_m \times I \to W_0$ which we may assume to be UV^1. Then the map

$$g_m \times \mathrm{Id} : M_m \times I \to X_m \times I$$

is a normal cobordism over the UV^1-map h_m. Let us assume that X is a manifold. Then $\rho : W_0 \to X$ is a fibration, hence it is UV^1. Therefore, the composition $\rho \circ h_m$ is UV^1. Hence the associated non-singular δ-forms over X of both problems are ϵ-cobordant by Theorem 2.1 (3), so they have the same Quinn index. By Corollary 2.4 the first problem has the Quinn index as $-\sigma$. Since the composite map

$$X_m \xrightarrow{i_m} W_0 \xrightarrow{\rho} X$$

is UV^1, it follows from Corollary 2.4 that the surgery problems

$$M_m \xrightarrow{g_m} X_m \xrightarrow{\mathrm{Id}} X_m$$

and

$$M_m \xrightarrow{g_m} X_m \xrightarrow{\rho \circ i_m} X$$

have the same Quinn index, which is equal to the $\mathbb{Z}$-component of $-\sigma \in H_{4k}(M, \mathbb{L})$. Because $\rho \circ i_m$ is a homotopy equivalence, we have an obvious normal map

$$f_m = \rho \circ i_m \circ g_m : M_m \to X.$$

We shall consider it as a degree 1 normal map over $\mathrm{Id}_X : X \to X$, and we show that its Quinn index coincides with the $\mathbb{Z}$-component of $-\sigma$. If it is not 1, then X cannot be a manifold. This completes the existence proof.

Let $f'_m : X \to X_m$ be a controlled UV^1 homotopy inverse of $\rho \circ i_m$ (take for instance the composition $X \subset W_m \stackrel{\pi_m}{\to} X_m$, and then approximate it by an UV^1-map). Let us consider the normal map $f'_m : X \to X_m$ over $\mathrm{Id} : X_m \to X_m$. If m is large, i.e., f'_m is an ϵ-homotopy equivalence, then its associated $\delta = \delta(\epsilon)$-form is ϵ-cobordant to the zero form (see Lemma 4.7).

By Proposition 2.2, the associated form of the composition

$$M_m \xrightarrow{f_m} X \xrightarrow{f'_m} X_m \xrightarrow{\mathrm{Id}} X_m$$

is therefore ϵ'-cobordant to the associated form of $f_m : M_m \to X$ over $f'_m : X \to X_m$. If ϵ' is sufficiently small, then their Quinn indexes coincide (see Proposition 2.7.2 of [6], or the proof of Corollary 2.4). Now the claim follows from the following two observations:

(a) By Corollary 2.4 the Quinn index of $f_m : M_m \to X$ over $\mathrm{Id} : X \to X$ coincides with the one of $f_m : M_m \to X$ over $f'_m : X \to X_m$ since f'_m is UV^1.

(b) The composition

$$M_m \xrightarrow{f_m} X \xrightarrow{f'_m} X_m$$

is homotopic to $g_m : M_m \to X_m$, since

$$f'_m \circ \rho \circ i_m : X_m \to X_m$$

is ϵ–homotopic to the identity for m sufficiently large. The homotopy

$$\phi_t : X_m \to X_m$$

is a homotopy equivalence, hence $\phi_t \circ g_m : M_m \to X_m$ is a normal map over $\mathrm{Id} : X_m \to X_m$. Therefore, the map

$$\overline{\phi} \circ (g_m \times \mathrm{Id}) : M_m \times I \to X_m \times I$$

is a normal cobordism between $f'_m \circ f_m$ and g_m over the first projection

$$X_m \times I \to X_m.$$

Here the map $\overline{\phi} : X_m \times I \to X_m \times I$ is given by $\overline{\phi}(x,t) = (\phi_t(x), t)$. By Theorem 2.1 (3) we obtain that the Quinn index of the composite map

$$M_m \xrightarrow{f_m} X \xrightarrow{f'_m} X_m$$

over $\mathrm{Id} : X_m \to X_m$ is the $\mathbb{Z}$–component of $-\sigma$. From (a) and (b) we obtain our main result.

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